Math 396. Maxwell's equations

We wish to consider the reformulation of Maxwell's equations in terms of the Hodge-star and d-operator on differential forms in flat Minkowski space-time. We will see that there are two equations in differential forms that encode the four classical equations, and conversation of charge ($\partial_t \rho + \nabla \cdot \mathbf{j} = 0$) is easily deduced from these much like in the classical setup with vector calculus.

The reformation via differential forms has several advantages: it clarifies the close link between topology and the theory of electromagnetic potential fields, it gives a straightforward explanation of the relationship between Maxwell's equations and special relativity (i.e., general relativity in the absence of matter), and by replacing the pseudo-Riemannian tangent bundle with a more general structure (vector bundle with connection over the spacetime manifold) the Maxwell equations in terms of differential forms become a special case of the Yang–Mills equations (a fact that is difficult to perceive when the equations are expressed in coordinatized form instead of in terms of the common mathematical language of vector bundles over manifolds). We will not discuss Yang–Mills theory any further here.

Of course, to really understand the physical relevance of the equations and where they come from one must approach these matters through action principles and other ideas from physics (Lagrangians, and so forth). Our purpose here is not to derive physical laws from first principles, but rather to explain how to put some known laws in a convenient mathematical form that enables us to better understand some of their mathematical properties.

It is assumed that the reader has skimmed over the handout on Stokes' theorem on Riemannian manifolds, so as to see how the apparatus of classical vector calculus on $\mathbf{R}^3$ is transported to oriented Riemannian 3-manifolds via the d-operator and Hodge stars.

1. Classical setup and orientations

We shall fix a 3-dimensional *flat* Riemannian manifold with corners S ("space") and let $X = S \times \mathbf{R}$ be endowed with the Lorentzian "product metric" that comes from the metric tensor on S and the negative-definite flat metric on $\mathbf{R}$ induced by the quadratic form $-t^2$. Thus, for all $x = (s,t) \in X$ the vector space decomposition $\mathrm{T}_x(X) = \mathrm{T}_s(S) \oplus \mathrm{T}_t(\mathbf{R})$ is orthogonal. The *classical case* is $S = \mathbf{R}^3$ with its standard inner product (and associated flat metric). One puzzling feature of the classical case is that there should be no preferred point in space and so in particular no meaningful "linear structure" on space. It is therefore a bit peculiar to say that classically S is a vector space. In classical physics what happens is that at the beginning of every physical problem one chooses an origin and somehow this choice never affects the answer. It would be better to have a framework in which there is no need to be choosing random origins, but we will not discuss the matter any further here; our space S is a smooth manifold (with corners) and so the issue of an origin and linear structure on S is eliminated (but we retains the information of tangent spaces at points, which is what really matters).

It will be convenient to sometimes assume S is connected and oriented (as in the classical case), but in the end we will get equations that do not require an orientation on S or even that S is orientable. Note that since $X = S \times \mathbf{R}$ with $\mathbf{R}$ oriented in the canonical way, orientability for S is equivalent to that for X. An orientation on either of S or X uniquely determines an orientation on the other so that X has a product orientation. We shall speak in the language of orientations for S. The Lorentzian manifold with corners X is called *spacetime*. The flatness of the metric tensor on S will be essential for everything that we do. The role of flatness is to permit us to carry out local calculations in *flat* coordinate systems, in terms of which the metric tensor acquires the same simple form as in the classical case. The case of non-flat metrics is the framework for General Relativity (as opposed to special relativity, which is essentially the context in which we are working).

Our goal is to write down "Maxwell's equations" in the language of differential forms on X in the case when there is no magnetic or polarized material present and the units are chosen to trivialize natural constants: $c = \varepsilon_0 = \mu_0 = 1$ (so Coloumb's constant k acquires the value $1/4\pi$). We first make an important definition that encodes the fact that the force fields of classical physics exhibit time dependence in their evolution but they do not point in a "time direction".

Definition 1.1. A smooth vector field $\vec{v}$ over an open set U in $X = S \times \mathbf{R}$ is *spacelike* if for each $u = (s,t) \in U$ the tangent vector $\vec{v}(u) \in \mathrm{T}_u(X) \simeq \mathrm{T}_s(S) \oplus \mathrm{T}_t(\mathbf{R})$ lies in the hyperplane $\mathrm{T}_s(S)$.

In other words, if we choose local coordinates $\{x, y, z\}$ on S and write a vector field locally as a smooth linear combination of ∂_x, ∂_y, ∂_z, and ∂_t then the spacelike condition on $\vec{v}$ says that the ∂_t-component of $\vec{v}$ vanishes. More globally, if $p : X \to S$ is the natural projection then $p^*(TS)$ is naturally a subbundle (even direct summand) of TX and the sections of this subbundle are the spacelike vector fields.

Definition 1.2. For an open set $U \subseteq X$ and a vector field $\vec{v} \in \mathrm{Vec}_X(U)$, let $\omega_{\vec{v}} \in \Omega^1_X(U)$ be the 1-form that is dual to $\vec{v}$ under the Lorentz metric. That is, for all $u \in U$, the linear functional $\omega_{\vec{v}}(u)$ on $\mathrm{T}_u(X)$ given by $\langle \cdot, \vec{v}(u) \rangle_u$.

Note in particular that since we give $X = S \times \mathbf{R}$ a product metric, if the vector field $\vec{v}$ is spacelike then for any $u_0 = (s_0, t_0) \in U$ the functional $\omega_{\vec{v}}(u_0)$ kills the line $\mathrm{T}_{t_0}(\mathbf{R})$ in $\mathrm{T}_{u_0}(X)$ and hence

$$\omega_{\vec{v}}(u_0) \in \mathrm{T}_{s_0}(S)^\vee \subseteq \mathrm{T}_{s_0}(S)^\vee \oplus \mathrm{T}_{t_0}(\mathbf{R})^\vee = \mathrm{T}_{u_0}(X)^\vee.$$

Explicitly, if we choose a local flat coordinate system $\{x, y, z\}$ on $U_0 \subseteq S$ then for a smooth vector field $\vec{v} = f_1\partial_x + f_2\partial_y + f_3\partial_z + f_4\partial_t$ over an open set $U \subseteq U_0 \times \mathbf{R}$ (with $f_i \in C^\infty(U)$) we compute pointwise that

$$\omega_{\vec{v}} = f_1 \mathrm{d}x + f_2 \mathrm{d}y + f_3 \mathrm{d}z - f_4 \mathrm{d}t \in \Omega^1_X(U).$$

(The correctness of this calculation rests crucially on the fact that $\{\partial_x, \partial_y, \partial_z\}$ is an orthonormal frame for $TS|_{U_0}$ and that $\langle \partial_t, \partial_t \rangle = -1$.)

The classical operators of divergence, gradient, and curl for vector fields over open subsets of $\mathbf{R}^3$ were generalized to vector fields over open subsets of 3-dimensional oriented Riemannian manifolds in the handout on Stokes' theorem on Riemannian manifolds. For spacelike vector fields on the 4-dimensional Lorentzian X, we have analogues of these operators by working in the 3-dimensional "time-slices":

Definition 1.3. Let S be an oriented 3-dimensional Riemannian manifold with corners. Let $X = S \times \mathbf{R}$ and let $p : X \to S$ be the standard projection. For an open set $U \subseteq X$ and a smooth U-section $\vec{v}$ of the subbundle $p^*(TS) \subseteq TX$ of spacelike vector fields, the *spacelike curl* $\nabla_S \times \vec{v} \in \mathrm{Vex}_X(U)$ is the spacelike vector field given on each time slice $U_t = U \cap (S \times \{t\})$ by the ordinary curl applied to the smooth vector field $\vec{v}|_{U_t} \in \mathrm{Vec}_S(U_t)$. The *spacelike divergence* $\nabla_S \cdot \vec{v} \in C^\infty(U)$ is defined similarly, as is the *spacelike gradient* $\nabla_S f \in (p^*(TS))(U)$ of a smooth function $f \in C^\infty(U)$.

Explicitly, if $\{x, y, z\}$ are local oriented flat coordinates on S then in the local coordinate system $\{x, y, z, t\}$ the above three spacelike operators are given by the habitual formulas in each time slice (using the differential operators $\partial_x, \partial_y, \partial_z$, and *not* ∂_t). Hence, we see that smoothness is preserved by these operations. Note that the spacelike divergence and spacelike gradient are independent of the orientation (as this is true on each time slice), and so they make sense without any orientability hypotheses on S (by globalizing from the local orientable case). In contrast, the spacelike curl only makes sense in the orientable case and negating the orientation cause it to change by a sign.

Beware that the spacelike divergence on spacelike vector fields $\vec{v}$ is rather different from the "usual" generalized divergence $\star_4 \mathrm{d} \star_1 \omega_{\vec{v}} \in C^\infty(U)$ that one would get through the global pseudo-Riemannian structure on X in the sense that the generalized divergence involves a t-derivative of the (usually t-dependent) local coefficient functions in local oriented flat coordinates $\{x, y, z, t\}$.

Remark 1.4. If $f \in C^\infty(U)$ is a smooth function, then a simple calculation using local flat coordinates on S yields the identity

$$\mathrm{d}f = \omega_{\nabla_S f} + (\partial_t f)\mathrm{d}t$$

as the unique decomposition of $\mathrm{d}f \in \Omega^1_X(U)$ in accordance with Exercise 3(*iii*) in Homework 8. Indeed, if $\{x, y, z\}$ is a local *flat* coordinate system then this identity is just the expansion

$$\mathrm{d}f = (\partial_x f)\mathrm{d}x + (\partial_y f)\mathrm{d}y + (\partial_z f)\mathrm{d}z + (\partial_t f)\mathrm{d}t$$

that one has for an arbitrary local smooth coordinate system on X. This global decomposition identity for $\mathrm{d}f$ will come up later in our considerations of potential functions for the electric field and vector potential for the magnetic field.

The *classical Maxwell equations* on open sets U in $X = S \times \mathbf{R}$ are as follows: for spacelike vector fields $\mathbf{E}$ and $\mathbf{B}$ on U expressing the electric and magnetic fields as functions of position and time (so $\mathbf{B}$ is sign-dependent on the choice of orientation on S),

$$\nabla_S \cdot \mathbf{B} = 0, \;\; \nabla_S \times \mathbf{E} + \partial_t \mathbf{B} = 0,$$
$$\nabla_S \times \mathbf{B} = \mathbf{j} + \partial_t \mathbf{E}, \;\; \nabla_S \cdot \mathbf{E} = \rho$$

where $\rho : U \to \mathbf{R}$ is called the electric charge density and $\mathbf{j}$ is a spacelike vector field that is called the current density over U. These equations are respectively called *non-existence of magnetic monopoles* (or Gauss' law for magnetism), *Faraday's law of induction*, *Ampère's law*, and *Gauss' law for electricity*. In the classical case, these are the traditional equations of Maxwell's theory.

The supplementary law of *conservation of charge*, $\partial_t \rho = -\nabla_S \cdot \mathbf{j}$, is an immediate consequence of taking the t-partial of Gauss' law for electricity and the divergence of Ampère's law (which kills the curl term): the two sides of the identity for conservation of charge are simply two different ways to compute $\partial_t(\nabla_S \cdot \mathbf{E}) = \nabla_S \cdot (\partial_t \mathbf{E})$ (the commutativity of differential operators being equality of mixed partials).

Remark 1.5. There is much more to classical electrostatics than Maxwell's equations, such as Coloumb's law and the action principles that construct potential fields *a priori*.

Observe that just as the definition of $\mathbf{B}$ is sign-dependent on a choice of orientation for S, the spacelike curl also has such sign dependence. This is good, because one sees by inspection that all four classical Maxwell equations are thereby *independent* of the choice of orientation: the left side of the second equation is sign-dependent but the property that it equal 0 is thereby unaffected, and the third equation has no orientation intervention on the right side but has it intervening twice (in the formation of $\nabla_S \times \mathbf{B}$) and thereby cancelling out on the left side.

We conclude that the classical Maxwell equations only require the flat Riemannian structure and orientability of S; they do not depend on the *choice* of orientation. Since equations in physics should not be coordinate-dependent, the above coordinate-free equations clarify the underlying geometrical aspects of the classical Maxwell theory. However, there are some defects. First of all, the equations involve the input $\mathbf{B}$ whose definition necessitates a choice of global orientation. Since there does not seem to be a natural orientation in the real world, it is preferable if we can formulate the equations without such a choice (even if the real world is orientable). Also, we would like to understand how the theory of electromagnetic potential is controlled by geometry in spacetime, and the classical formulation is not well-suited for such questions. We shall recast the equations in terms of differential forms on the abstract flat Lorentzian manifold $X = S \times \mathbf{R}$ for any abstract flat Riemannian 3-dimensional manifold with corners S, and in so doing we will be able to eliminate orientation conditions and the topological input will become clearer.

2. Some identities

Let us temporarily assume S is orientable and connected. Choose an orientation (there are two), whence we get a product orientation on $X = S \times \mathbf{R}$. It will be seen that the equations we get in the end will not depend on this choice, so the equations will globalize to the case of possibly non-orientable S. The choice of orientation gives rise to the Hodge-star bundle isomorphisms $\star_r : \Omega_X^r \simeq \Omega_X^{4-r}$ that satisfy $\star_{4-r} \circ \star_r = (-1)^{r+1}$. Upon choosing an orthonormal positive basis of a tangent space $\mathrm{T}_s(S)$, the Hodge star is given explicitly by formulas in Example 2.3 in the old handout on the Hodge star (taking $c = 1$ there); letting Vol denote the volume form on X arising from the Lorentz structure and the orientation, in local oriented flat coordinates $\{x, y, z\}$ over an open $U_0 \subseteq S$ we have $\mathrm{Vol} = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t$ over $U_0 \times \mathbf{R}$. Of course, as always we have $\mathrm{Vol} = \star_0(1)$ and so we will usually write "$\star(1)$" rather than "Vol". In particular, if we change the orientation of S then Vol is negated since the only other orientation on S is the opposite one (as S is connected).

Recall that $\star_r$ is characterized by the local identities $\omega \wedge \star_r \eta = \langle \omega, \eta \rangle_r \mathrm{Vol}$ (with $\langle \cdot, \cdot \rangle_r$ denoting the induced pseudo-Riemannian metric on Ω_X^r, so for example in local oriented flat coordinates as above we have $\langle \mathrm{d}x, \mathrm{d}x \rangle_1 = 1$ but $\langle \mathrm{d}t, \mathrm{d}t \rangle_1 = -1$). Hence, if we negate the orientation on S then each $\star_r$ is also negated. Thus, as we know from Homework 11, the operator $\mathrm{d}^\dagger = \star_{5-r} \circ \mathrm{d} \circ \star_r : \Omega_X^r \to \Omega_X^{r-1}$ for $1 \le r \le 4$ is independent of the orientation and so globalizes to the case when there are no orientability (or connectivity) hypotheses on S.

The following lemma is just a translation of the pointwise Hodge-star formula from Example 2.3 in the handout on Hodge-star (again, setting c there to be 1). We record it for ease of reference in case the reader wishes to verify any of the calculations that follow.

Lemma 2.1. *Fixing an orientation on S and using the induced product orientation on $X = S \times \mathbf{R}$, if $\{x, y, z\}$ is a local oriented flat coordinate system on an open $U_0 \subseteq S$ then*

$$\star_1(\mathrm{d}x) = \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t,\ \ \star_1(\mathrm{d}y) = -\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}t,\ \ \star_1(\mathrm{d}z) = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t,\ \ \star_1(\mathrm{d}t) = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

in $\Omega_X^3(U_0 \times \mathbf{R})$ and

$$\star_2(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}z \wedge \mathrm{d}t,\ \ \star_2(\mathrm{d}x \wedge \mathrm{d}z) = -\mathrm{d}y \wedge \mathrm{d}t,\ \ \star_2(\mathrm{d}x \wedge \mathrm{d}t) = -\mathrm{d}y \wedge \mathrm{d}z,$$
$$\star_2(\mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}x \wedge \mathrm{d}t,\ \ \star_2(\mathrm{d}y \wedge \mathrm{d}t) = \mathrm{d}x \wedge \mathrm{d}z,\ \ \star_2(\mathrm{d}z \wedge \mathrm{d}t) = -\mathrm{d}x \wedge \mathrm{d}y$$

in $\Omega_X^2(U_0 \times \mathbf{R})$.

Moreover, $\star_2^2 = -1$, $\star_1 \circ \star_3 = 1$, and $\star_3 \circ \star_1 = 1$.

We next need some global identities that relate the spacelike divergence and curl (for spacelike vector fields) with the d and Hodge-star operators. We first have to define the time-derivative of a spacelike vector field. This goes as follows.

If $\vec{v} \in \mathrm{Vec}_X(U)$ is an arbitrary smooth spacelike vector field, then by Exercise 3(*iii*) in Homework 8 there is a unique identity of the form

$$\mathrm{d}(\omega_{\vec{v}}) = -\theta \wedge \mathrm{d}t + \eta$$

where the 1-form θ is a section of $p^*(\Omega_S^2)$ and the 2-form η is a section of $p^*(\Omega_S^1)$ (i.e., they "involve no dt's"). By the duality between 1-forms and vector fields provided by the Lorentz metric, we can therefore uniquely write $\theta = \omega_{\partial_t \vec{v}}$ for a unique smooth *spacelike* vector field $\partial_t \vec{v}$ over U. Explicitly, for local coordinates $\{x, y, z\}$ on S (unrelated to the orientation and Riemannian structure) we can uniquely write $\vec{v} = f_1\partial_x + f_2\partial_y + f_3\partial_z$ with $f_i \in C^\infty(U)$ and it is easy to check directly that

$$\partial_t\vec{v} = (\partial_t f_1)\partial_x + (\partial_t f_2)\partial_y + (\partial_t f_3)\partial_z \in \mathrm{Vec}_X(U).$$

Lemma 2.2. *Assume S is orientable and fix an orientation. Let $U \subseteq X = S \times \mathbf{R}$ be an open subset and $\vec{v} \in \mathrm{Vec}_X(U)$ a spacelike smooth vector field. The following hold:*

(1) $\mathrm{d}^\dagger(\omega_{\vec{v}}) = -\nabla_S \cdot \vec{v}$,
(2) $(\mathrm{d}\omega_{\vec{v}}) \wedge \mathrm{d}t = \star_1(\omega_{\nabla_S \times \vec{v}})$,
(3) $\mathrm{d}^\dagger(\omega_{\vec{v}} \wedge \mathrm{d}t) = -(\nabla_S \cdot \vec{v})\mathrm{d}t - \omega_{\partial_t\vec{v}}$,
(4) $\mathrm{d}\omega_{\vec{v}} = \star_2(\omega_{\nabla_S \times \vec{v}} \wedge \mathrm{d}t) - \omega_{\partial_t\vec{v}} \wedge \mathrm{d}t$.

Recall that $\mathrm{d}^\dagger$ is locally defined as $\star\mathrm{d}\star$ via a local orientation (as in Homework 11, Exercise 3(*ii*)). Note the important consistency check that both sides of (1) and (3) are *independent* of the orientation on S. Also, the left sides of (2) and (4) are independent of the orientation on S, and so are the right sides because they involve a single Hodge star but also a single spacelike curl operator.

Proof. It is possible to deduce these identities by pure thought using (i) the definitions of the spacelike divergence and curl in terms of the 3-dimensional counterparts on time slices, and (ii) the relations between the classical divergence and curl with d and Hodge star in the classical 3-dimensional case on $\mathbf{R}^3$. However, it is a notational pain to give such an "intrinsic" proof. Hence, we instead leave it to the reader to carry out the pleasant exercise of verifying the identities by coordinate calculation upon picking local oriented *flat* coordinates on S. This is essentially a mechanical exercise once one has available the identities in Lemma 2.1 (and one knows how the 3-dimensional curl and divergence work out in such local coordinates). ■

There remains one final lemma:

Lemma 2.3. *Fix an orientation on S. For any open set $U \subseteq X = S \times \mathbf{R}$ and smooth differential forms $F \in \Omega^2_X(U)$ and $J \in \Omega^1_X(U)$ there exists a unique smooth function $\rho \in C^\infty(U)$ and unique spacelike smooth vector fields $\mathbf{E}, \mathbf{B}, \mathbf{j} \in \mathrm{Vec}_X(U)$ such that*

$$F = \star_2(\omega_{\mathbf{B}} \wedge \mathrm{d}t) - \omega_{\mathbf{E}} \wedge \mathrm{d}t, \quad J = \rho\mathrm{d}t - \omega_{\mathbf{j}}.$$

The vector fields $\mathbf{E}$ and $\mathbf{j}$ are orientation-independent, as is the function ρ, but $\mathbf{B}$ changes by a sign if we negate the orientation on S.

Proof. As with the proof of Lemma 2.2, one can give a proof without mentioning any local coordinates but we take the quick way out. Pick local oriented flat coordinates on S and write out the "general form" of the right side in terms of ρ and coefficient functions of the vector fields. One sees by inspection (check!) that these are just an encoding of the coefficient functions of the differential forms F and J. This gives the asserted existence/uniqueness results locally, and due to the local uniqueness it follows that the local solutions agree on overlaps and hence globalize. As for the sign-dependence on the orientation of S, this is immediate from the fact that the only ingredient in the "shape" of the formulas that depends on the orientation is the Hodge star $\star_2$. This changes by a sign if we change the orientation of S, and so by uniqueness it follows that $\mathbf{B}$ must also exhibit the same sign dependence in order for the effect to cancel out and give the initial choice of F. ■

3. The modern equations

We now drop all orientability and connectivity hypotheses on S. Choose an open set $U \subseteq X$ and pick smooth differential forms $F \in \Omega^2_X(U)$ and $J \in \Omega^1_X(U)$. We call these the *electromagnetic* *form* and the *current density form* respectively. The *abstract Maxwell equations* are

$$\mathrm{d}F = 0,\ \ \mathrm{d}^\dagger F = J, \tag{3.1}$$

where $\mathrm{d}^\dagger = \star\mathrm{d}\star$ over orientable open subsets (for any choice of orientation). The first equation encodes the non-existence of magnetic monopoles and Faraday's law of induction, and the second encodes Gauss' law for electricity and Ampère's law, as we shall show below. It is the second equation ($\mathrm{d}^\dagger F = J$) that encodes the serious physical information, in the sense that the first equation ($\mathrm{d}F = 0$) is a physical triviality: action principles that logically precede the electromagnetic theory provide an electromagnetic potential and it will be seen below that in terms of differential forms this leads to the condition $F = \mathrm{d}A$ as *a-priori input* in the theory from a physical point of view. This makes the first equation physically uninteresting because of the general identity $\mathrm{d}^2 = 0$. On the other hand, the second equation takes on the form $\mathrm{d}^\dagger \mathrm{d}A = J$ and this turns out to be of enormous physical significance.

The local calculation

$$\mathrm{d}\star_1(J) = \mathrm{d}(\star_1 \circ \star_4 \circ \mathrm{d} \circ \star_2(F)) = \mathrm{d}^2(\star_2 F) = 0 \tag{3.2}$$

resting on the second equation in (3.1) will turn out to be a repackaging of the identity for conservation of charge. The calculation (3.2) is a disguised version of the classical deduction of this conservation law from Maxwell's equations. Finally, the reason for the names of F and J is that they will turn out to encode precisely the information of the electromagnetic fields and the charge/current densities respectively.

How do we make the translation from the classical equations in the oriented case? By Lemma 2.3, if we assume S is orientable and we pick an orientation of S then we get a unique $\rho \in C^\infty(U)$ and unique smooth spacelike vector fields $\mathbf{E}, \mathbf{B}, \mathbf{j} \in \mathrm{Vec}_X(U)$ such that they recover F and J via the formulas in Lemma 2.3. In particular, ρ, $\mathbf{E}$, and $\mathbf{j}$ are orientation-independent but $\mathbf{B}$ changes by a sign on a connected component of U if we change the orientation on that component. Upon fixing an orientation to get a definite $\mathbf{B}$ and to be able to write $\mathrm{d}^\dagger = \star\mathrm{d}\star$ we now use Lemma 2.2 to compute

$$\begin{aligned}
\mathrm{d}F &= \mathrm{d}\star(\omega_{\mathbf{B}} \wedge \mathrm{d}t) - \mathrm{d}(\omega_{\mathbf{E}} \wedge \mathrm{d}t) \\
&= \star(\star\mathrm{d}\star(\omega_{\mathbf{B}} \wedge \mathrm{d}t)) - \mathrm{d}(\omega_{\mathbf{E}} \wedge \mathrm{d}t) \\
&= \star(-(\nabla_S \cdot \mathbf{B})\mathrm{d}t - \omega_{\partial_t \mathbf{B}}) - \star\omega_{\nabla\times\mathbf{E}} \\
&= -(\nabla_S \cdot \mathbf{B})(\star\mathrm{d}t) - \star(\omega_{(\partial_t\mathbf{B} + \nabla_S\times\mathbf{E})}).
\end{aligned}$$

Thus, the condition $\mathrm{d}F = 0$ says exactly $\nabla_S \cdot \mathbf{B} = 0$ and $\partial_t \mathbf{B} + \nabla_S \times \mathbf{E} = 0$. These are exactly the non-existence of magnetic monopoles and Faraday's law, as promised.

Since $\star_2^2 = -1$, we similarly compute via Lemma 2.2 that

$$\begin{aligned}
\mathrm{d}^\dagger F &= (\star \circ \mathrm{d} \circ \star)(F) \\
&= -\star\mathrm{d}(\omega_{\mathbf{B}} \wedge \mathrm{d}t) - (\star\mathrm{d}\star)(\omega_{\mathbf{E}} \wedge \mathrm{d}t) \\
&= -\star(\star\omega_{\nabla_S\times\mathbf{B}}) + ((\nabla_S \cdot \mathbf{E})\mathrm{d}t + \omega_{\partial_t\mathbf{E}}) \\
&= \omega_{(-\nabla_S\times\mathbf{B} + \partial_t\mathbf{E})} + (\nabla_S \cdot \mathbf{E})\mathrm{d}t,
\end{aligned}$$

so the condition $\mathrm{d}^\dagger F = J \overset{\mathrm{def}}{=} \rho\mathrm{d}t - \omega_{\mathbf{j}} = \rho\mathrm{d}t + \omega_{-\mathbf{j}}$ says $\nabla_S \cdot \mathbf{E} = \rho$ and $\nabla_S \times \mathbf{B} = \partial_t\mathbf{E} + \mathbf{j}$. These two identities are respectively Gauss' law for electricity and Ampère's law, as promised.

Finally, since $\langle \mathrm{d}t, \mathrm{d}t \rangle = -1$ we have $\mathrm{d}t \wedge \star(\mathrm{d}t) = -\star(1)$ and hence $\mathrm{d}(f \cdot (\star\mathrm{d}t)) = \mathrm{d}f \wedge \star(\mathrm{d}t) = \partial_t f \cdot \star(1)$ for any $f \in C^\infty(U)$. Taking $f = \rho$, one computes (using $\star_0 \circ \star_4 = -1$) that

$$\mathrm{d}\star J = \mathrm{d}(\rho \cdot \star(\mathrm{d}t)) - \star\omega_{\mathbf{j}} = \mathrm{d}\rho \wedge \star(\mathrm{d}t) + \star(\star\mathrm{d}\star\omega_{\mathbf{j}}) = -(\partial_t\rho + \nabla_S \cdot \mathbf{j}) \cdot (\star(1)),$$

where the last step again uses Lemma 2.2. Thus, the "abstract" calculation in (3.2) that $\mathrm{d}(\star J) = 0$ says precisely that $\partial_t \rho + \nabla_S \cdot \mathbf{j} = 0$, and this latter identity is conservation of charge. The vanishing of d^2 as used in (3.2) is just a repackaging of the vanishing of $\nabla_S \cdot (\nabla_S \times (\cdot))$ on spacelike vector fields, and this latter vanishing (applied on time slices) is the content of the classical deduction of conservation of charge from Maxwell's equations.

Remark 3.1. Suppose S is oriented. On open subsets $U \subseteq X$ on which the current density J vanishes, the 2-form $\star F \in \Omega^2_X(U)$ is closed and so defines a cohomology class in $\mathrm{H}^2_{\mathrm{dR}}(U)$ that is supposed to have physical significance.

4. Topological consequences: potential fields

So far, the above has just been clever linguistic repackaging. But does the rephasing in terms of differential forms tell us anything interesting? The starting point is:

Theorem 4.1. *Let $U \subseteq X = S \times \mathbf{R}$ be an open subset that is smoothly contractible to a point. If $F \in \Omega^2_X(U)$ is a closed 2-form then it can be expressed as $F = \mathrm{d}A$ for a smooth 1-form $A \in \Omega^1_X(U)$ that is unique up to an additive term $\mathrm{d}f$ for $f \in C^\infty(U)$.*

Theorem 4.1 is an immediate consequence of the Poincaré lemma. Indeed, since U is smoothly contractible we know that $\mathrm{H}^i_{\mathrm{dR}}(U) = 0$ for all $i > 0$, and so the closed 2-form F must be exact. Upon writing $F = \mathrm{d}A$, the extent to which $A \in \Omega^1_X(U)$ is non-unique is adding a closed 1-form to A. But $\mathrm{H}^1_{\mathrm{dR}}(U) = 0$ as well, so closed 1-forms on U are exact, which is to say that they have the form $\mathrm{d}f$ for $f \in C^\infty$.

Example 4.2. The most important instance of the theorem is when $U = U_0 \times I$ for an open subset $U_0 \subseteq V$ that is smoothly contractible to a point and $I \subseteq \mathbf{R}$ a non-empty open interval.

What is the meaning of A in the classical theory? I claim it encodes the theory of electromagnetic potential. Since $X = S \times \mathbf{R}$, by Exercise 3(*iii*) in Homework 8 we can unique write $A = \eta + \phi \mathrm{d}t$ with $\phi \in C^\infty(U)$ and $\eta \in \Omega^1_X(U)$ a 1-form such that $\eta(u) \in \mathrm{T}_u(X)^\vee \simeq V^\vee \oplus \mathbf{R}^\vee$ has vanishing "$(\mathrm{d}t)(u)$-component" for all $u \in U$ (i.e., as a functional on $\mathrm{T}_u(X)$ it kills the tangent line along the time direction). We may write $\eta = \omega_{\mathbf{A}}$ for a unique smooth spacelike vector field $\mathbf{A} \in \mathrm{Vec}_X(U)$. Since $\mathrm{d}f$ likewise has the spacetime decomposition $\mathrm{d}f = \omega_{\nabla_S f} + (\partial_t f)\mathrm{d}t$ (see Remark 1.4), replacing A with $A + \mathrm{d}f$ corresponds to the change $(\mathbf{A}, \phi) \mapsto (\mathbf{A} + \nabla_S f, \phi + \partial_t f)$. The spacelike vector field $\mathbf{A}$ and the smooth function $\phi : U \to \mathbf{R}$ are orientation-independent and are together unique up to a linked change in terms of $f \in C^\infty(U)$.

The physical meaning is seen as follows. Taking S to be oriented now, we have

$$\star(\omega_{\mathbf{B}} \wedge \mathrm{d}t) - \omega_{\mathbf{E}} \wedge \mathrm{d}t = F = \mathrm{d}A = \mathrm{d}(\omega_{\mathbf{A}} + \phi \mathrm{d}t) = \mathrm{d}\omega_{\mathbf{A}} + \mathrm{d}\phi \wedge \mathrm{d}t.$$

Since $\mathrm{d}\phi = \omega_{\nabla_S \phi} + (\partial_t \phi)\mathrm{d}t$, clearly $\mathrm{d}\phi \wedge \mathrm{d}t = \omega_{\nabla_S \phi} \wedge \mathrm{d}t$. Also, the final identity in Lemma 2.2 gives

$$\mathrm{d}\omega_{\mathbf{A}} = \star_2(\omega_{\nabla_S \times \mathbf{A}} \wedge \mathrm{d}t) - \omega_{\partial_t \mathbf{A}} \wedge \mathrm{d}t.$$

Putting this together, we get the identity

$$\star(\omega_{\mathbf{B}} \wedge \mathrm{d}t) - \omega_{\mathbf{E}} \wedge \mathrm{d}t = \star(\omega_{\nabla_S \times \mathbf{A}} \wedge \mathrm{d}t) - \omega_{(-\nabla_S \phi + \partial_t \mathbf{A})} \wedge \mathrm{d}t.$$

Comparing both sides, this encodes precisely the pair of identities

$$\mathbf{B} = \nabla_S \times \mathbf{A}, \;\; \mathbf{E} = -\nabla_S \phi + \partial_t \mathbf{A}$$

with $\mathbf{A}$ a spacelike vector field on U and $\phi \in C^\infty(U)$ a function such that the pair $(\mathbf{A}, \phi)$ is uniquely determined up to adding $\nabla_S f$ to $\mathbf{A}$ and $\partial_t f$ to ϕ for some $f \in C^\infty(U)$.

This is *exactly* the classical theory of electromagetric potential: the vector field $\mathbf{A}$ is called the *vector potential* for the magnetic field and ϕ is called the electrostatic potential function. In the absence of magnetic fields ($\mathbf{B} = 0$) we may take the vector potential $\mathbf{A}$ to vanish and so the electrostatic potential function ϕ is uniquely determined up to adding a function $\partial_t f$ such that the *spacelike gradient* $\nabla_S f$ vanishes (to retain the condition of vanishing vector potential). But the condition of vanishing for the spacelike gradient says precisely that f is locally "independent of the space variables", and so in the special case that $U = U_0 \times I$ for an open set $U_0 \subseteq S$ it follows that f is a smooth function of time and hence the electrostatic potential ϕ is unique up to adding an arbitrary function of time. In this case we may fix the value of ϕ to be a specific constant at one point $u_0 \in U_0$ for all time $t \in I$, and this eliminates all of the ambiguity. This is precisely the classical device of uniquely determining the electrostatic potential (in the absence of magnetic forces) by requiring it to be zero at some point of U_0.

Of course, since there are other physical laws such as Coloumb's law and action principles, it is always possible to infer the existence of an electrostatic potential function even when the de Rham cohomology is nonzero. That is, the physics tells us a lot more than what is mathematically deducible from Maxwell's theory alone. In particular, the electromagnetic potential is much more than just a device for extracting the electromagnetic field, and so the mathematics is not the whole story.

Example 4.3. In the case of a time-dependent magnetric field complementary to a line in space, there is a vector potential (since the relevant de Rham cohomology is an H^2 that vanishes) but its non-uniqueness is controlled by an H^1 that is *nonzero* and so it is possible to change the choice of the vector potential by more than just vector fields that are spacelike gradients. That is, in such cases there are spacelike vector fields $\vec{v}$ on the domain that are not gradients and yet have vanishing curl, so we can add such a $\vec{v}$ to the vector potential without affecting its property of having curl equal to the magnetic field. (By taking $\vec{v}$ to be constant in time, so $\partial_t \vec{v} = 0$, this modification of the vector potential does not force any changes in the choice of electrostatic potential function.) But is this physically relevant? After all, presumably the potential is chosen according to principles that go beyond just Maxwell's theory, and so the additive modification by a curl-free non-gradient field $\vec{v}$ as suggested above may well be unreasonable on physical grounds. I am not technically qualified to pass judgement on these non-mathematical matters.

The main point is this: the existence of electromagnetic potential can be understood in many situations purely based on topological properties of the domain under consideration, and when the region is topologically complicated (i.e., has nonvanishing higher de Rham cohomology) then there can be rather intricate ways in which the non-uniqueness of the solution to the equation $F = \mathrm{d}A$ manifests itself. (That is, non-uniqueness can occur by more operations that naive ones that are available on contractible domains.) However, it appears that A is more fundamental than F, and so "solving for A given F" may be physically unsound. One last point worth noting is that if we take the 1-form J and the 1-form A as the primary objects of study (as seems to be the case in physics) then the only interesting Maxwell equation is $\mathrm{d}^\dagger \mathrm{d}A = J$ yet this turns out to be exactly the Euler–Lagrange equation arising from the action principle. Hence, in a sense the Maxwell equations are consequences of more fundamental physical principles applied to the current form and potential form, coupled with mathematical trivialities such as $\mathrm{d}^2 = 0$. Lemma 2.3 provides the link between these abstractions and the classical formulation of the theory (with oriented S).