Math 396. Products of premanifolds

Let $X_1, \dots, X^n$ be C^p premanifolds, with $0 \le p \le \infty$. The topological product $\prod X_i$ ought to admit a natural C^p premanifold structure, using as local C^p-charts the maps

$$\phi_1 \times \cdots \times \phi_n : U = U_1 \times \cdots \times U_n \to V_1 \oplus \cdots \oplus V_n$$

for C^p-charts $\phi_i : U_i \to V_i$ on X_i. Concretely, local coordinates on the product are concatenations of local coordinates on the factors. To be a bit more precise, if we take $V_i = \mathbf{R}^{m_i}$ and let $\{x_1^{(i)}, \dots, x_{m_i}^{(i)}\}$ denote the component functions $U_i \to \mathbf{R}$ given by ϕ_i (so these are the local C^p coordinates on U_i encoded in ϕ_i), then local C^p coordinates on the open subset $U = \prod U_i \subseteq \prod X_i$ are given by the collection of functions $x_j^{(i)} \circ \pi_i : U \to \mathbf{R}$ for $1 \le i \le n$ and $1 \le j \le m_i$, where $\pi_i : U \to U_i$ is the projection to the ith factor.

In this handout we do three things: we confirm that this procedure works, we verify that it is not merely *ad hoc* but in fact satisfies a "universal mapping property" as any good notion of product should (now in the category of C^p premanifolds, of course!), and we apply it to verify an "obvious" interpretation of maps to $\mathbf{R}^n$ in the category of C^p premanifolds. The Hausdorff and second countability conditions will also be tracked.

1. Construction of products

Recalling how products of topological spaces were characterized by a mapping property (roughly of the shape "to map continuously to the product is the same as to map continuously to the factors"), we are led to make a definition that may sound a bit abstract at first.

Definition 1.1. Let X_1 and X_2 be two C^p premanifolds. A C^p *premanifold product* is a triple (P, p_1, p_2) consisting of a C^p premanifold P equipped with C^p-maps $p_i : P \to X_i$ such that for any pair of C^p maps $f_1 : Z \to X_1$ and $f_2 : Z \to X_2$ from a C^p premanifold Z, there exists a unique C^p map $f : Z \to P$ such that $p_i \circ f = f_i$.

The exact same argument as in the topological case shows that if there exist two products (P, p_1, p_2) and (P', p'_1, p'_2) of X_1 and X_2 as C^p premanifolds then there is a unique C^p-isomorphism $\xi : P \simeq P'$ compatible with the p_i's and p'_i's (i.e., $p'_i \circ \xi = p_i$ for $i = 1, 2$). The existence aspect for C^p premanifold products provides no surprises: it should be a natural C^p-structure on the topological product. The construction is simple to describe in the language of atlases, as follows. Let $P = X_1 \times X_2$ as a topological space (with the product topology), and let $p_i : P \to X_i$ be the usual continuous projections. If there is to be a C^p-structure $\mathscr{O}$ on P making it a C^p product via the p_i's, then *a priori* $\mathscr{O}$ is uniquely determined: if $\mathscr{O}'$ is a second such structure then by the universal property in Definition 1.1 applied to $(P, \mathscr{O})$ and $(P, \mathscr{O}')$ we get a unique C^p isomorphism of C^p products

$$\xi : (P, \mathscr{O}) \simeq (P, \mathscr{O}')$$

respecting p_1 and p_2 on both the source and target. Thus, since $P = X_1 \times X_2$ topologically (with the usual p_i's), ξ is forced to be the identity on underlying topological spaces. Hence, the C^p inverse to ξ must also be the identity on underlying topological spaces. The condition that the identity map in both directions respects the C^p-structures $\mathscr{O}$ and $\mathscr{O}'$ says *exactly* that $\mathscr{O}(U) = \mathscr{O}'(U)$ inside the set of all $\mathbf{R}$-valued functions on U for every open set $U \subseteq \mathbf{R}$. In other words, $\mathscr{O}' = \mathscr{O}$ as desired.

Having settled uniqueness for $\mathscr{O}$ on P, we now treat the existence problem. For local C^p-charts (ϕ_1, U_1) on X_1 and (ϕ_2, U_2) on X_2 with $\phi_i : U_i \to V_i$ a homeomorphism onto an open set $\phi_i(U_i) \subseteq V_i$ for finite-dimensional $\mathbf{R}$-vector spaces V_i, the map

$$\phi_1 \times \phi_2 : U_1 \times U_2 \to V_1 \oplus V_2$$

is a homeomorphism onto the open subset $\phi_1(U_1) \times \phi_2(U_2)$ in the finite-dimensional $\mathbf{R}$-vector space $V_1 \oplus V_2$. By the definition of "atlas", the U_1's and U_2's cover X_1 and X_2 respectively, so the opens $U_1 \times U_2$ cover $X_1 \times X_2$. The content of the existence problem is:

Theorem 1.2. *As we let (ϕ_1, U_1) and (ϕ_2, U_2) range through elements of C^p-atlases for X_1 and X_2 respectively, the data $(\phi_1 \times \phi_2, U_1 \times U_2)$ form a C^p-atlas on $X_1 \times X_2$. With respect to the associated C^p premanifold structure, the projections $p_i : X_1 \times X_2 \to X_i$ are C^p and satisfy the universal property to be a product; in particular, this C^p-structure on the topological space $X_1 \times X_2$ is independent of the choices of C^p-atlases covering X_1 and X_2.*

In concrete terms, the local C^p-coordinates on the product are just concatentations of local C^p-coordinates on each of the factors (on products of small opens from the factors). Note also that when X_1 and X_2 are C^p manifolds, then so is $X_1 \times X_2$ (as the Hausdorff and second-countability conditions are topological and are preserved under the formation of products). Whenever we speak of a product of C^p premanifolds, it is always understood that we use the C^p-structure on the topological product as in this theorem, but do not forget the key point: this is not merely some nice-looking C^p structure on the topological product but it satisfies the right universal mappping property to deserve being called a product in the category of C^p premanifolds (and in particular it is *independent* of the C^p-atlases chosen on the factors).

Proof. Let (ϕ_i, U_i) and (ϕ'_i, U'_i) be two C^p-charts on X_i (with ϕ_i and ϕ'_i taking values in finite-dimensional $\mathbf{R}$-vector spaces V_i and V'_i), and suppose $U_1 \times U_2$ meets $U'_1 \times U'_2$ inside of $X_1 \times X_2$. Since

$$(U_1 \times U_2) \cap (U'_1 \times U'_2) = (U_1 \cap U'_1) \times (U_2 \times U'_2)$$

inside of $X_1 \times X_2$, we have to check that the composite homeomorphism

$$(\phi'_1 \times \phi'_2) \circ (\phi_1 \times \phi_2)^{-1} : (\phi_1 \times \phi_2)((U_1 \cap U'_1) \times (U_2 \cap U'_2)) \to (\phi'_1 \times \phi'_2)((U_1 \cap U'_1) \times (U_2 \cap U'_2))$$

between respective opens in $V_1 \oplus V_2$ and $V'_1 \oplus V'_2$ is a C^p-isomorphism. But this map is the product of the homeomorphisms

$$\phi'_1 \circ \phi_1^{-1} : \phi_1(U_1 \cap U'_1) \simeq \phi'_1(U_1 \cap U'_1), \ \ \phi'_2 \circ \phi_2^{-1} : \phi_2(U_2 \cap U'_2) \simeq \phi'_2(U_2 \times U'_2)$$

between opens in V_1 and V'_1, and in V_2 and V'_2. Both of these latter maps are C^p-isomorphisms because we are comparing local C^p-charts from a common C^p-atlas on each of X_1 and X_2, and hence we just have to note that in the setting of multivariable calculus on vector spaces a product of C^p maps is again C^p (as the C^p property may be checked using component functions with respect to a choice of linear coordinates on the target). This completes the verification that we have a C^p-atlas on $X_1 \times X_2$.

By construction, the projections $X_1 \times X_2 \to X_i$ are C^p: working with the local C^p-charts in the above C^p-atlas reduces this to the classical fact that if $U \subseteq V$ and $U' \subseteq V'$ are open domains in finite-dimensional $\mathbf{R}$-vector spaces then upon viewing $U \times U'$ as an open domain in the vector space $V \oplus V'$ the projection $U \times U' \to U$ is C^p; it is even C^∞ since it is the restriction of the linear projection $V \oplus V' \to V$. We therefore have a C^p-structre on $X_1 \times X_2$, and it remains to show it "works".

To check the universal property, let Y be a C^p premanifold and let $f_i : Y \to X_i$ be C^p maps. There is a unique continuous map $f : Y \to X_1 \times X_2$ with component maps f_1 and f_2 along the factors, and our problem is to prove that this f is necessarily a C^p map. Such a property for a *continous* map between C^p premanifolds may be checked working locally on the source and target, so it suffices to separately study the maps $f^{-1}(U_1 \times U_2) \to U_1 \times U_2$ for local C^p-charts (ϕ_1, U_1) and (ϕ_2, U_2) in the chosen atlases on X_1 and X_2. Thus, we may assume that X_1 and X_2 admit global C^p-charts, and hence we may suppose X_i is open in a finite-dimensional $\mathbf{R}$-vector space V_i. We may work locally on Y, and so we can assume Y admits a global C^p-chart as well. Thus, we can assume Y is open in a finite-dimensional $\mathbf{R}$-vector space V. Hence, we are in the following situation: we are given an open set $Y \subseteq V$ and a pair of C^p maps $Y \to V_i$ (landing in open subsets $X_i \subseteq V_i$), and we want the product map $Y \to V_1 \oplus V_2$ (which factors through the product map to the open subset $X_1 \times X_2$) to be a C^p map; but this is obvious from the "component function" criterion in multivariable calculus for a map to a finite-dimensional vector space to be C^p. ■

Remark 1.3. Of course, the same construction can be iterated for finite products in the evident manner. We leave this to the reader's imagination.

2. Mapping property for $\mathbf{R}^n$

Consider $\mathbf{R}^n$. This is always taken to be a C^p manifold via the usual $\mathbf{R}$-space structure of C^p-functions on opens as in calculus. As a topological space, $\mathbf{R}^n$ is a product of copies of $\mathbf{R}$. Does the product manifold structure as in Remark 1.3 using that on each of the factors $\mathbf{R}$ recover the *usual* C^p manifold structure on $\mathbf{R}^n$? Indeed it does; this follows from inspecting the construction using the identity map on $\mathbf{R}$ as a global chart on each copy of $\mathbf{R}$. The same argument shows that for finite-dimensional vector spaces V and W, the product C^p-manifold structure on $V \times W$ coincides with the usual one via the vector space structure on the direct sum $V \oplus W$. Of course, when working with finite-dimensional vector spaces as manifolds we often implicitly consider them in the C^∞ category. The preceding example has the following pleasing consequence:

Corollary 2.1. *Let X be a C^p premanifold, and $f_1, \dots, f_n \in \mathscr{O}(X)$. The map of sets $f : X \to \mathbf{R}^n$ given by $x \mapsto (f_i(x))$ is a C^p mapping. Conversely, any C^p map $X \to \mathbf{R}^n$ arises in this way.*

This is the sort of statement that one wants to use without thinking twice, and fortunately we can. (If we couldn't, there would be something seriously wrong with the definitions.)

Proof. This is largely a matter of unwinding definitions, ultimately reducing to the fact that in the very definition of a premanifold we require at the level of local C^p-charts that the distinguished functions on an open set correspond to the usual C^p functions on the image open set in the target vector space for the local C^p-chart. Also, once we settle the case $n = 1$, the general case follows by the universal property of C^p products and the preceding verification that the usual C^p manifold structure on $\mathbf{R}^n$ is the product structure coming from its n factor spaces $\mathbf{R}$ (with its usual C^p-structure).

In the case $n = 1$, the content is this: for a C^p premanifold X, a set-theoretic function $f : X \to \mathbf{R}$ lies in $\mathscr{O}(X)$ if and only if f is a C^p map when $\mathbf{R}$ is itself viewed as a C^p premanifold in the usual manner. To verify this "obvious" claim, we may work locally on X (as both sides of the implication can be checked locally on X; e.g., $f \in \mathscr{O}(X)$ if and only if $f|_{X_i} \in \mathscr{O}(X_i)$ for opens X_i that cover X), and so we may assume that there is a global C^p-chart $\phi : X \simeq \phi(X)$ onto an open subset of a finite-dimensional $\mathbf{R}$-vector space V. Hence, by *definition* of C^p-charts on a C^p premanifold, $f \in \mathscr{O}(X)$ if and only if $f \circ \phi : U \to \mathbf{R}$ is a C^p map in the traditional sense. However, since ϕ is a C^p isomorphism when the open subset $\phi(X) \subseteq V$ is given the induced C^p-structure from the usual manifold structure on V as in calculus on vector spaces (this follows from the very definition of a C^p-chart!), and since $f = (f \circ \phi) \circ \phi^{-1}$, it follows that f is a C^p map if and only if $f \circ \phi$ is a C^p map. Hence, we may replace f with $f \circ \phi$ so as to reduce to the case when X is an open subset of V with the induced C^p-structure from the usual one on V. But the C^p-structure on V is *exactly* to declare the distinguished functions on an open subset to be the C^p-functions in the usual sense of calculus on vector spaces, so we are done. ■