

MATH 396. PRODUCT TOPOLOGY

The aim of this handout is to address two points: metrizable of finite products of metric spaces, and the abstract characterization of the product topology in terms of universal mapping properties among topological spaces. This latter issue is related to explaining why the definition of the product topology is not merely *ad hoc* but in a sense the “right” definition. In particular, when you study topology more systematically and encounter the problem of topologizing infinite products of topological spaces, if you think in terms of the universal property to be discussed below then you will inexorably be led to the right definition of the product topology for a product of infinitely many topological spaces (it is not what one would naively expect it to be, based on experience with the case of finite products).

1. METRICS ON FINITE PRODUCTS

Let X_1, \dots, X_d be metrizable topological spaces. The product set

$$X = X_1 \times \cdots \times X_d$$

admits a natural product topology, as discussed in class. It is natural to ask if, upon choosing metrics ρ_j inducing the given topology on each X_j , we can define a metric ρ on X in terms of the ρ_j 's such that ρ induces the product topology on X . The basic idea is to find a metric which describes the idea of “coordinate-wise closeness”, but several natural candidates leap out, none of which are evidently better than any others:

$$\begin{aligned} \rho^{\max}((x_1, \dots, x_d), (x'_1, \dots, x'_d)) &= \max_{1 \leq j \leq d} \rho_j(x_j, x'_j) \\ \rho^{\text{Euc}}((x_1, \dots, x_d), (x'_1, \dots, x'_d)) &= \sqrt{\sum_{j=1}^d \rho_j(x_j, x'_j)^2} \\ \rho_1((x_1, \dots, x_d), (x'_1, \dots, x'_d)) &= \sum_{j=1}^d \rho_j(x_j, x'_j) \\ \rho_p((x_1, \dots, x_d), (x'_1, \dots, x'_d)) &= \left(\sum_{j=1}^d \rho_j(x_j, x'_j)^p \right)^{1/p}, \quad p \geq 1 \end{aligned}$$

When $X_j = \mathbf{R}$ for all j , with ρ_j the usual absolute value metric, these recover the various concrete norms we've seen on $X = \mathbf{R}^d$. Our first aim will be to show that all of these rather different-looking metrics are at least bounded above and below by a positive multiple of each other (which is the best we can expect, since they sure aren't literally the same), and so in particular they all define the same topology. In fact, we will see that the common topology they define is the product topology.

We first axiomatize the preceding examples. Let $N : \mathbf{R}^d \rightarrow \mathbf{R}$ be any norm which satisfies the property that on the orthant $[0, \infty)^d$ with non-negative coordinates it is a monotonically increasing function in each individual coordinate when all others are held fixed. Examples of such N 's include our old friends

$$\|\cdot\|_{\max}, \quad \|\cdot\|_{\text{Euc}}, \quad \|\cdot\|_1, \quad \|\cdot\|_p \text{ (for } p \geq 1)$$

where we recall that

$$\|(a_1, \dots, a_n)\|_p = \left(\sum_{j=1}^d |a_j|^p \right)^{1/p}.$$

Here is the general theorem which shows that many metrics (including all those mentioned above) on a product space are bounded above and below by a positive multiple of each other and hence determine the same theory of open sets, closed sets, and convergence of sequences.

Theorem 1.1. *Let $N : \mathbf{R}^d \rightarrow \mathbf{R}$ be any norm as considered above. Then for metric spaces (X_j, ρ_j) for $1 \leq j \leq d$, with product space $X = X_1 \times \cdots \times X_d$, the function $\rho_N : X \times X \rightarrow \mathbf{R}$ defined by*

$$\rho_N((x_1, \dots, x_d), (x'_1, \dots, x'_d)) = N(\rho_1(x_1, x'_1), \dots, \rho_d(x_d, x'_d))$$

is a metric on X , and all such ρ_N 's are bounded above and below by a positive multiple of each other.

Proof. Let's first check that each ρ_N really is a metric. Since N is a non-negative function, clearly ρ_N is always non-negative. Also, if

$$\rho_N((x_1, \dots, x_d), (x'_1, \dots, x'_d)) = 0$$

then by the very definition of ρ_N and the fact that $N : \mathbf{R}^d \rightarrow \mathbf{R}$ only vanishes on the zero vector (as N is a norm), it follows that the d -tuple

$$(\rho_1(x_1, x'_1), \dots, \rho_d(x_d, x'_d)) \in \mathbf{R}^d$$

must be the zero vector. Thus, each $\rho_j(x_j, x'_j) = 0$, whence $x_j = x'_j$ for all j (since each ρ_j is a metric). That is, if $\rho_N(x, x') = 0$ for $x, x' \in X$, then x and x' have the same "coordinates" x_j and x'_j for each $1 \leq j \leq d$, and hence $x = x'$. This shows that ρ_N satisfies the first basic requirement to be a metric (it is "positive definite"). The symmetry property $\rho_N(x, x') = \rho_N(x', x)$ for $x, x' \in X$ is immediate from the definition of ρ_N and the fact that each ρ_j is a symmetric function on $X_j \times X_j$ (as ρ_j is a metric).

Now (as always) we come to the mildly interesting part, which is the verification of the triangle inequality. It is here that we need the hypothesis that N on

$$[0, \infty)^d \subseteq \mathbf{R}^d$$

is monotonically increasing in each individual variable with all others held fixed. If $x, x', x'' \in X$ are three points, we want

$$\rho_N(x, x'') \leq \rho_N(x, x') + \rho_N(x', x'').$$

Since each ρ_j is a metric, so

$$\rho_j(x_j, x''_j) \leq \rho_j(x_j, x'_j) + \rho_j(x'_j, x''_j)$$

for all $1 \leq j \leq d$, we deduce from our hypothesis on N that

$$N(\rho_1(x_1, x''_1), \dots, \rho_d(x_d, x''_d)) \leq N(\rho_1(x_1, x'_1) + \rho_1(x'_1, x''_1), \dots, \rho_d(x_d, x'_d) + \rho_d(x'_d, x''_d)).$$

Thus, we compute

$$\begin{aligned} \rho_N(x, x'') &= N(\rho_1(x_1, x''_1), \dots, \rho_d(x_d, x''_d)) \\ &\leq N(\rho_1(x_1, x'_1) + \rho_1(x'_1, x''_1), \dots, \rho_d(x_d, x'_d) + \rho_d(x'_d, x''_d)) \\ &= N((\rho_1(x_1, x'_1), \dots, \rho_d(x_d, x'_d)) + (\rho_1(x'_1, x''_1), \dots, \rho_d(x'_d, x''_d))) \\ &\leq N(\rho_1(x_1, x'_1), \dots, \rho_d(x_d, x'_d)) + N(\rho_1(x'_1, x''_1), \dots, \rho_d(x'_d, x''_d)) \\ &= \rho_N(x, x') + \rho_N(x', x'') \end{aligned}$$

where the second inequality uses the triangle inequality for the norm N applied to the vectors

$$(\rho_1(x_1, x'_1), \dots, \rho_d(x_d, x'_d)), (\rho_1(x'_1, x''_1), \dots, \rho_d(x'_d, x''_d)) \in \mathbf{R}^d.$$

This completes the proof that ρ_N is a metric for any norm N on \mathbf{R}^d satisfying our monotonicity hypothesis on $[0, \infty)^d$.

It remains to show that if N and N' are any two norms on \mathbf{R}^d which satisfy our basic monotonicity hypothesis then ρ_N and $\rho_{N'}$ are each bounded above and below by a positive multiple of the other. But all norms on \mathbf{R}^d are equivalent! Thus, there is an inequality

$$aN \leq N' \leq AN$$

as functions on \mathbf{R}^d for some $a, A > 0$ (depending on the particular N and N'), and so we immediately deduce from evaluation on a vector

$$(\rho_1(x_1, x'_1), \dots, \rho_d(x_d, x'_d)) \in \mathbf{R}^d$$

that

$$a\rho_N(x, x') \leq \rho_{N'}(x, x') \leq A\rho_N(x, x')$$

for all $x, x' \in X$. This gives the desired boundedness result. ■

A consequence of this theorem is:

Corollary 1.2. *With notation as in the theorem, the common topology induced on X by any of the ρ_N 's is the product topology. Moreover, a sequence $\{\xi_1, \xi_2, \dots\}$ in X given by*

$$\xi_m = (\xi_{m,1}, \dots, \xi_{m,d})$$

is convergent with limit

$$\ell = (\ell_1, \dots, \ell_d) \in X = X_1 \times \dots \times X_d$$

if and only if

$$\lim_{m \rightarrow \infty} \xi_{m,j} \rightarrow \ell_j$$

for all $1 \leq j \leq d$.

Proof. It suffices to pick *one* N of the type we're considering and to show for that N that ρ_N defines the product topology and induced a theory of convergence for sequences which is exactly "coordinate-wise convergence".

We consider $N : \mathbf{R}^d \rightarrow \mathbf{R}$ to be the max norm. We claim that for the result metric, which is just ρ^{\max} as at the top of this handout, we have $\xi_m \rightarrow \ell \in X$ relative to ρ^{\max} if and only if $\xi_{m,j} \rightarrow \ell_j$ in X_j relative to ρ_j for all $1 \leq j \leq d$. That is, we must show that

$$\rho^{\max}(\xi_m, \ell) \rightarrow 0$$

if and only if

$$\rho_j(\xi_{m,j}, \ell_j) \rightarrow 0$$

for all $1 \leq j \leq d$ (all limits as $m \rightarrow \infty$). If we have ρ^{\max} -convergence, then since

$$\rho_j(\xi_{m,j}, \ell_j) \leq \rho^{\max}(\xi_m, \ell)$$

for each j (by definition of ρ^{\max}), we get $\rho_j(\xi_{m,j}, \ell_j) \rightarrow 0$ as $m \rightarrow \infty$ (for each j) by a squeezing argument. Conversely, if for each $1 \leq j \leq d$ we have $\rho_j(\xi_{m,j}, \ell_j) \rightarrow 0$ as $m \rightarrow \infty$, then for any $\varepsilon > 0$ we have $\rho_j(\xi_{m,j}, \ell_j) < \varepsilon$ for $m > M_{\varepsilon,j}$, whence for

$$m > M_\varepsilon \stackrel{\text{def}}{=} \max_{1 \leq j \leq d} M_{\varepsilon,j}$$

we get $\rho_j(\xi_{m,j}, \ell_j) < \varepsilon$ for all j and hence

$$\rho^{\max}(\xi_m, \ell) < \varepsilon$$

for any $m > M_\varepsilon$. This says exactly that $\xi_m \rightarrow \ell$ in X relative to the metric ρ^{\max} .

We now check that the topology induced by ρ^{\max} on X is the product topology. First let $U_j \subseteq X_j$ be open (and hence ρ_j -open), and we want to prove that $\prod U_j \subseteq X$ is ρ^{\max} -open. For $u = (u_1, \dots, u_d) \in \prod U_j$ there exists $\varepsilon_j > 0$ such that $B_{\varepsilon_j}(u_j) \subseteq U_j$. Hence, for $\varepsilon = \min \varepsilon_j > 0$ we have that the open ρ^{\max} -ball of radius ε centered at u is contained in U ; this establishes that U is ρ^{\max} -open, and so by definition of the product topology we conclude that every open set in X for the product topology is ρ^{\max} -open. Conversely, for a subset $W \subseteq X$ that is ρ^{\max} -open we want to prove that W is open for the product topology. We choose $w = (w_1, \dots, w_d) \in W$ and we see to find open subsets $W_j \subseteq X_j$ around w_j such that $\prod W_j \subseteq W$. By definition, for some $\varepsilon > 0$ the ρ^{\max} -ball $B_\varepsilon(w) \subseteq W$ is contained in W . Let W_j be the ρ_j -ball $B_\varepsilon(w_j) \subseteq X_j$ that is open in X_j . By definition of ρ^{\max} we have $\prod W_j \subseteq B_\varepsilon(w)$, so $\prod W_j \subseteq W$ as desired. ■

The metric ρ^{\max} is usually called the *box metric* on X , due to the picture of hypercubes when we apply this construction for $X_j = \mathbf{R}$ for all j (with the usual metric on each factor). Clearly we have enormous flexibility in the choice of metric to describe the product topology on X , and so it is important to recognize that the fundamental structure is the product topology and not the specific metric used to encode it; of course, it is also important to recognize that a finite product of metric spaces has a product topology that is metrizable via many different choices of metric. Depending on the particular situation, one of these “product metrics” may be more convenient than others.

2. UNIVERSAL PROPERTY OF THE PRODUCT TOPOLOGY

Here is a basic lemma concerning product spaces.

Lemma 2.1. *Let X_1, \dots, X_d be finitely many topological spaces, and let X denote the product space, equipped with the product topology. The projection maps $\pi_j : X \rightarrow X_j$ are continuous.*

Proof. Fix $1 \leq j \leq d$ and choose an open set V in X_j . We must show that $\pi_j^{-1}(V)$ is an open set in the product space X . By the definition of π_j , clearly

$$\pi_j^{-1}(V) = X_1 \times \cdots \times V \times \cdots \times X_d$$

where V is in the j th slot and for all $i \neq j$ we have the whole space X_i in the i th slot. But this is a product of opens in each factor space (as X_i is trivially open in X_i for all $i \neq j$ and V is open in X_j by hypothesis), so by definition of the product topology this is an open set in X . ■

We can now give the following fundamental theorem.

Theorem 2.2. *Let X_1, \dots, X_d, Y be topological spaces, and let $f : Y \rightarrow X = X_1 \times \cdots \times X_d$ be a set-theoretic map, described by*

$$f(x) = (f_1(x), \dots, f_d(x))$$

for all $x \in X$, with $f_j = \pi_j \circ f : Y \rightarrow X_j$ the “coordinate maps”. For any point $y_0 \in Y$, the map $f : Y \rightarrow X$ is continuous at y_0 if and only if $f_j : Y \rightarrow X_j$ is continuous at y_0 for all $1 \leq j \leq d$. In particular, $f : Y \rightarrow X$ is continuous if and only if $f_j : Y \rightarrow X_j$ is continuous for all $1 \leq j \leq d$.

Proof. The statement at the end concerning continuity of maps on spaces is immediate from the pointwise version (if we just vary y_0 over all of Y). Thus, we just have to prove the pointwise version for a fixed point $y_0 \in Y$. Also, when f is continuous at y_0 then it is immediate that f_j is continuous at y_0 for all j because $f_j = \pi_j \circ f$ with π_j the projection

$$\pi_j : X_1 \times \cdots \times X_d \rightarrow X_d$$

which is continuous (by Lemma 2.1). Thus, we just have to show that when f_j is continuous at y_0 for all j , then f is continuous at y_0 .

Let N be a neighborhood of $f(y_0) = (x_1, \dots, x_d) \in X$. We must show that $f^{-1}(N)$ is a neighborhood of y_0 to deduce that f is continuous at y_0 . From the definition of “neighborhood” and the definition of the product topology, N contains a subset of the form $N_1 \times \dots \times N_d$ with $N_j \subseteq X_j$ a neighborhood of x_j (the N_j ’s may even be chosen to be open in X_j for every j). Since

$$f^{-1}(N) \supseteq f^{-1}(N_1 \times \dots \times N_d) = \bigcap_{j=1}^d f_j^{-1}(N_j)$$

(since $f(x) \in N_1 \times \dots \times N_d$ is equivalent to $f_j(x) \in N_j$ for all $1 \leq j \leq d$) and $f_j^{-1}(N_j)$ is a neighborhood of y_0 for every j (since each f_j is continuous at y_0 !), we conclude that $f^{-1}(N)$ contains a finite intersection of neighborhoods $f_j^{-1}(N_j)$. But a finite intersection of neighborhoods is again a neighborhood, and a superset of a neighborhood is a neighborhood. Thus, $f^{-1}(N)$ is a neighborhood of y_0 , as desired. (The reader is urged to come up with alternative proofs of this theorem in the case of metrizable X_j ’s and the associated box metric on X by using ε ’s and δ ’s, and by using sequences.) ■

We conclude with a reformulation of the theorem in terms of what is called a “universal property”. The crux is the recognition that we should not think of the product $X = \prod X_j$ as a bare topological space, but rather as equipped with the continuous maps $\pi_j : X \rightarrow X_j$. Here is the interest in this viewpoint:

Corollary 2.3. *The maps $\pi_j : X \rightarrow X_j$ are continuous, and for any collection of continuous maps $f_j : Y \rightarrow X_j$ from a topological space Y there exists a unique continuous map $f : Y \rightarrow X$ such that $f_j = \pi_j \circ f$ for all j .*

Proof. From the description of X and the π_j ’s on the set-theoretic level, it is clear that f is uniquely determined set-theoretically as the map $y \mapsto (f_1(y), \dots, f_d(y))$, and so the only issue is whether or not this map f is continuous. Since the f_j ’s are continuous, the continuity of f follows from the preceding theorem. ■

Observe that the statement of the corollary does not mention points, nor even the specific nature of the underlying set of X . We claim that the property of the data $(X, \pi_j : X \rightarrow X_j)$ as given in this corollary characterizes it *up to unique isomorphism*. To make this precise, let us make a temporary definition:

Definition 2.4. An *abstract product* of the X_j ’s in the category of topological spaces is the data consisting of a topological space P equipped with continuous maps $p_j : P \rightarrow X_j$ such that the following universal mapping property holds: for any topological space Y equipped with continuous maps $f_j : Y \rightarrow X_j$, there exists a unique continuous map $f : Y \rightarrow P$ such that $p_j \circ f = f_j$ for all j .

Note that in this definition we do *not* require that the underlying set of P be $\prod X_j$; as far as the definition is concerned, P is unknown. By Corollary 2.3, one example of an abstract product is $X = \prod X_j$ with its product topology and its evident continuous projections $\pi_j : X \rightarrow X_j$. The remarkable fact is that this is the “only” example up to “unique” isomorphism. That is, if $(P, \{p_j\})$ and $(P', \{p'_j\})$ are two abstract products of the X_j ’s then we claim that there exist unique continuous maps $f : P \rightarrow P'$ and $f' : P' \rightarrow P$ satisfying $p_j = p'_j \circ f$ for all j and $p'_j = p_j \circ f'$ for all j , and that moreover f and f' are inverse to each other! Indeed, the existence and uniqueness of the continuous f satisfying $p_j = p'_j \circ f$ for all j follows from the fact that $(P', \{p'_j\})$ is an abstract product (and P is simply a topological space), and similarly the existence and uniqueness of the continuous f' satisfying $p'_j = p_j \circ f'$ for all j follows from the fact that $(P, \{p_j\})$ is an abstract

product (and P' is simply a topological space). Now for the beautiful part: since $f' \circ f : P \rightarrow P$ satisfies

$$p_j \circ (f' \circ f) = (p_j \circ f') \circ f = p'_j \circ f = p_j$$

for all j and clearly $p_j \circ \text{id}_P = p_j$ for all j , by the *uniqueness* aspect of the property of $(P, \{p_j\})$ it follows that $f' \circ f = \text{id}_P$! Likewise, with the roles reversed we get $f \circ f' = \text{id}_{P'}$ since $(P', \{p'_j\})$ have an analogous uniqueness aspect in its nature as an abstract product. This makes precise the sense in which the *entire structure* of an abstract product, $(P, \{p_j\})$ (and not the bare space P) is unique up to *unique* isomorphism within the category of topological spaces.

Remark 2.5. The exact same argument works in the framework of sets, and shows that if one considers the product set *equipped with its projection maps* then the resulting structure is unique up to unique isomorphism in the category of sets (where “isomorphism” means “bijection”). More remarkably, note that the above “abstract nonsense” argument does not use that there are only finitely many X_j ’s! Indeed, it proves that for an arbitrary collection of X_j ’s there is (up to unique isomorphism) at most one abstract product. What the argument does not do is show that the theory is non-vacuous: it does not *construct* an abstract product. It is the existence aspect of the story for which we have to exhibit a specific model, and in the case of infinitely many X_j ’s we have not done so. You may wish to see if you can make one on your own. (or look in a topology book to see what to do)

The preceding argument is of a very formal type that will come up again in other contexts later; whereas the product topology is rather concrete and can be “understood” and thought about by using its definition, there are other constructions in mathematics (such as tensor products) which become thoroughly confusing if one tries to think in terms of their explicit constructions. The viewpoint of characterizing mathematical structures up to unique isomorphism by means of mapping properties and not explicit definitions and constructions is one of the most important ideas in 20th century mathematics.

We conclude with an amusing calculation. Since we have shown that up to unique isomorphism there is only one abstract product of the X_j ’s, and yet we have exhibited an explicit model with extra features (such as its underlying set being the product set, which is *not* information that is part of the definition of an abstract product of topological spaces), it is natural to ask if we can derive these features of the explicit model from the universal mapping property alone. Indeed we can show that *if* an abstract product $(P, \{p_j\})$ exists, then the underlying set of P must (in a natural way) be identified with $\prod X_j$. To see how this works, we first note that the only way to get our hands on P is via its universal property, and so we need to encode the underlying set of a topological space in terms of the data of continuous maps.

Let Z be the one-point topological space with its unique topology. Observe that for *any* topological space Y , the “underlying set” $|Y|$ of Y is naturally identified with the set $\text{Hom}(Z, Y)$ of continuous maps from Z to Y . Aha, so our problem of determining $|P|$ is the same as that of determining $\text{Hom}(Z, P)$. Here is where the defining property of $(P, \{p_j\})$ enters: the data of a continuous map $f : Z \rightarrow P$ is the same as the data of continuous maps $f_j : Z \rightarrow X_j$ for all j ! Hence, by the nature of products in the category of sets, we have natural set-theoretic bijections

$$|P| = \text{Hom}(Z, P) = \prod_j \text{Hom}(Z, X_j) = \prod_j |X_j|,$$

showing that indeed the underlying set of P is naturally identified with the product of the underlying sets of the X_j ’s. We leave it as an exercise to check that this set-theoretic identification is exactly what is obtained from the unique homeomorphism $P \simeq \prod X_j$ via the “abstract product” structure on both sides.