Math 396. Topology on projective space

Let V be a finite-dimensional vector space over $\mathbf{R}$ with dimension $n+1 \geq 2$. Let $\mathbf{P}(V)$ denote the set of hyperplanes in V (or lines in $V^\vee$). In class we saw how to put a topology on this set upon choosing an ordered basis $\mathbf{e} = \{e_0, \dots, e_n\}$ of V: we covered $\mathbf{P}(V)$ by the subsets $U_{i,\mathbf{e}}$ (consisting of hyperplanes not containing e_i) equipped with bijections $\phi_{i,\mathbf{e}} : U_{i,\mathbf{e}} \to \mathbf{R}^n$, and we checked that when using these bijections to topologize the $U_{i,\mathbf{e}}$'s the criterion from the handout on gluing topologies is satisfied. It may appear, however, that this topology might depend on $\mathbf{e}$ and so is not intrinsic to V. We want to give two alternative descriptions of the topology that avoid the mention of the basis, and so ensure that the topology is independent of the basis. The second of our two descriptions below will recover the topologists' method for constructing projective spaces. (The method used in class is the algebraic geometer's method, and it is better in certain respects because it generalizes to other topological fields in the role of $\mathbf{R}$ whereas the topologists' method only works for $\mathbf{R}$; of course, the topologists' method is important for the study of the topology of $\mathbf{P}^n(\mathbf{R})$!)

1. A dual-space description

Consider the map of sets $\pi : V^\vee - \{0\} \to \mathbf{P}(V)$ that sends a nonzero linear functional $\ell \in V^\vee$ to its hyperplane kernel $\ker \ell$ considered as a point in $\mathbf{P}(V)$. This is a surjective map: for any hyperplane H in V, we may choose a basis of the 1-dimensional space V/H and so we get a linear functional $\ell : V \twoheadrightarrow V/H \simeq \mathbf{R}$ whose kernel is H. Note that the definition of π is intrinsic to V and does not mention bases. Thus, the following result shows that the topology we put on $\mathbf{P}(V)$ via a choice of ordered basis $\mathbf{e}$ of V is in fact independent of that choice:

Theorem 1.1. *Let $\mathbf{e}$ be an ordered basis of V, and give $\mathbf{P}(V)$ the resulting topology from the $U_{i,\mathbf{e}}$'s and $\phi_{i,\mathbf{e}}$'s as in class. A subset $S \subseteq \mathbf{P}(V)$ is open for this topology if and only if its preimage $\pi^{-1}(S) \subseteq V^\vee - \{0\}$ is an open subset of $V^\vee$ with respect to the usual topology on the finite-dimensional $\mathbf{R}$-vector space $V^\vee$.*

Of course, we recall that finite-dimensional vector spaces over $\mathbf{R}$ have a topology that is independent of bases, given by using any norm or any choice of linear isomorphism with a Euclidean space over $\mathbf{R}$.

Proof. Since $S = \pi(\pi^{-1}(S))$ for any subset $S \subseteq \mathbf{P}(V)$ (due to surjectivity of π), the theorem says exactly that π is a continuous open mapping. By definition, $U_{i,\mathbf{e}}$ is the set of hyperplanes H not containing e_i, so the preimage $\widetilde{U}_{i,\mathbf{e}} = \pi^{-1}(U_{i,\mathbf{e}})$ consists of those nonzero functionals $\ell \in V^\vee$ such that e_i is not contained in the hyperplane $\pi(\ell) = \ker \ell$, which is to say $\ell(e_i) \neq 0$. Note that the set of such ℓ's in $V^\vee$ is open: using dual basis coordinates $\{e_j^*\}$ to identify $V^\vee$ with a Euclidean space $\mathbf{R}^{n+1}$, a functional $\ell = \sum a_j e_j^*$ is given Euclidean coordinates $(a_0, \dots, a_n)$ and $\ell(e_i) \neq 0$ says exactly $a_i \neq 0$.

We now make a brief topological digression on the local nature of continuity and openness. Rather generally, if $f : X \to Y$ is a set-theoretic map between two topological spaces and if Y is covered by open subsets Y_i such that $X_i = f^{-1}(Y_i)$ is *open* in X for all i then f is continuous if and only if the restricted maps $f_i : X_i \to Y_i$ are continuous for all i, and (assuming continuity) likewise for the property of being an open map. Indeed, if f is continuous then certainly all maps f_i are continuous, and conversely if each f_i is continuous and $U \subseteq Y$ is an open set then $f^{-1}(U) \cap X_i = f_i^{-1}(U \cap Y_i)$ is open in X_i for all i – and hence open in X, as each X_i is assumed to be *open* in X – so the union $f^{-1}(U)$ of the open subsets $f^{-1}(U) \cap X_i$ in X is also open in X. This takes care of the continuity aspect, and for openness if f is open then for any open set $U \subseteq X_i$ we see that $f_i(U) = f(U)$ is open in Y since U must be open in X (as X_i is open in X) and so $f_i(U)$ is also open in Y_i. Conversely, if all maps f_i are open then for each open set U in X the overlap $f(U) \cap Y_i = f_i(U \cap X_i)$ is open in Y_i for all i and hence is open in Y for all i, so the union $f(U)$ of the $f(U) \cap Y_i$'s is open.

For the map $\pi : V^\vee - \{0\} \to \mathbf{P}(V)$ and the open covering of $\mathbf{P}(V)$ by the $U_{i,\mathbf{e}}$'s we have checked above that $\widetilde{U}_{i,\mathbf{e}} = \pi^{-1}(U_{i,\mathbf{e}})$ is an open subset $V^\vee - \{0\}$ (or equivalently, in $V^\vee$) for all i. Thus, we may apply the preceding paragraph to conclude that continuity and openness for π is *equivalent* to continuity and openness for the restricted maps $\pi_i : \widetilde{U}_{i,\mathbf{e}} \to U_{i,\mathbf{e}}$ for all i. We shall show that there is a natural homeomorphism $h_i : \widetilde{U}_{i,\mathbf{e}} \simeq U_{i,\mathbf{e}} \times \mathbf{R}^\times$ carrying π_i over to the standard projection $p_i : U_{i,\mathbf{e}} \times \mathbf{R}^\times \to U_{i,\mathbf{e}}$ (i.e., $\pi_i \circ h_i^{-1} = p_i$), so continuity and openness for π_i follow from continuity and openness for the projection map p_i.

How do we topologically relate the open set $\widetilde{U}_{i,\mathbf{e}} \subseteq V^\vee$ consisting of linear functionals ℓ satisfying $\ell(e_i) \neq 0$ and the product space $U_{i,\mathbf{e}} \times \mathbf{R}^\times$ consisting of pairs (H, c) where H is a hyperplane not containing e_i and c is a nonzero real number? Set-theoretically, we proceed as follows: since $e_i \notin H$ we can find a unique linear form $\sum a_j e_j^*$ with kernel H and $a_i = 1$, but likewise for *any* nonzero $c \in \mathbf{R}$ we can uniquely scale to find a linear form $\ell_{H,c} = \sum b_j e_j^*$ such that $b_i = c$ (i.e., $\ell_{H,c}(e_i) = c$) and $\ker \ell_{H,c} = H$. Conversely, if ℓ is a linear form on V satisfying $\ell(e_i) \neq 0$, then from ℓ we get both a hyperplane $H = \ker \ell$ and a nonzero number $c = \ell(H)$. These two procedures are inverse to each other: given H and c we make $\ell_{H,c}$ that is rigged to have kernel H and to satisfy $\ell_{H,c}(e_i) = c$, and given ℓ we make $H = \ker \ell$ and $c = \ell(e_i) \neq 0$ so that ℓ satisfies the two properties that uniquely characterize $\ell_{H,c}$ (its kernel is H and its value on e_i is c).

This gives a bijection $\xi_i : \widetilde{U}_{i,\mathbf{e}} \to U_{i,\mathbf{e}} \times \mathbf{R}^\times$, and we now describe it more concretely so that the topological aspects of the bijection may be seen. The dual basis $\{e_j^*\}$ on $V^\vee$ linearly (and topologically!) identifies $V^\vee$ with a Euclidean space $\mathbf{R}^{n+1}$ such that $\widetilde{U}_{i,\mathbf{e}}$ goes over to the open subset of points $(a_0, \dots, a_n)$ such that $a_i \neq 0$. By construction, $U_{i,\mathbf{e}}$ is topologized by means of the bijection $\phi_{i,\mathbf{e}} : U_{i,\mathbf{e}} \simeq \mathbf{R}^n$ that assigns to each hyperplane H not containing e_i the coefficients (aside from the ith) of the *unique* linear form $\sum a_j e_j^*$ with kernel H and $a_i = 1$. More specifically, the topology on $U_{i,\mathbf{e}}$ is rigged to make $\phi_{i,\mathbf{e}}$ a homeomorphism (though we still had to compute "transition maps" to check agreement of topologies on overlaps $U_{i,\mathbf{e}} \cap U_{i',\mathbf{e}}$ in order that these "local" topologies all arose from a unique global topology on $\mathbf{P}(V)$). Composing with homeomorphisms has no impact on whether or not a set-theoretic map between topological spaces is a homeomorphism, so the homeomorphism problem for our bijection $\widetilde{U}_{i,\mathbf{e}} \to U_{i,\mathbf{e}} \times \mathbf{R}^\times$ is equivalent to that for the composite

$$\{(a_0, \dots, a_n) \in \mathbf{R}^{n+1} \,|\, a_i \neq 0\} \simeq \widetilde{U}_{i,\mathbf{e}} \to U_{i,\mathbf{e}} \times \mathbf{R}^\times \simeq \mathbf{R}^n \times \mathbf{R}^\times.$$

This composite map is rather concrete:

$$(a_0, \dots, a_n) \mapsto ((a_j/a_i)_{j \neq i}, a_i)$$

because the linear form $\ell = \sum a_j e_j^*$ has the same kernel as $e_i^* + \sum_{j \neq i} (a_j/a_i) e_j^*$ and it satisfies $\ell(e_i) = a_i$. This composite map is obviously continuous by inspection, and its inverse is also continuous by inspection: it is given by the formula

$$((\alpha_j)_{j \neq i}, c) \mapsto (c\alpha_0, \dots c\alpha_{i-1}, c, c\alpha_{i+1}, \dots, c\alpha_n).$$

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2. A sphere description

Fix a positive-definite inner product on V. Let $S \subseteq V - \{0\}$ be the unit sphere, so using the identification $V \simeq V^\vee$ via the inner product gives a continuous open map

$$V - \{0\} \simeq V^\vee - \{0\} \xrightarrow{\pi} \mathbf{P}(V)$$

via $v \mapsto \ker(\langle v, \cdot\rangle)$.

Theorem 2.1. *The composite map $S \to \mathbf{P}(V)$ is a continuous* open *surjection that is a local homeomorphism (i.e., for $x \in S$ a sufficiently small open $U \subseteq S$ around x maps isomorphically onto an open set in $\mathbf{P}(V)$) and its fibers consist of pairs of antipodal points on S.*

This theorem expresses $\mathbf{P}(V)$ as the set obtained by "identifying" antipodal points on the sphere S, using a topology that "locally" comes from that on the sphere. This is exactly the way projective space is usually constructed by topologists, usually in the presence of coordinates: $V = \mathbf{R}^{n+1}$ with the standard inner product, so $S = S^n$ is the standard unit sphere.) In terms of homogenous coordinates $[a_0, \dots, a_n]$ for a point x in projective space $\mathbf{P}^n(\mathbf{R}) = \mathbf{P}(\mathbf{R}^{n+1})$, the set-theoretic picture is as follows. We scale through by $1/\sqrt{\sum a_j^2}$ to arrive at homogenous coordinates $[b_0, \dots, b_n]$ for the same point x but with $\sum b_j^2 = 1$. The only remaining scaling that preserves this condition is scaling by c satisfying $c^2 = 1$, which is to say that the point $(b_0, \dots, b_n) \in S^n$ is well-defined (in terms of x) up to negation. This gives a bijection between the set $\mathbf{P}^n(\mathbf{R})$ and the "quotient set" of S^n modulo identification of antipodal points, and the real issue is to make sure that this procedure is topologically well-behaved (with respect to both the topology that we already have on projective space and the topology on the sphere as a compact subset of $\mathbf{R}^{n+1}$).

Proof. Pick an orthonormal basis $\{e_0, \dots, e_n\}$ of V and for each i let $U_i \subseteq \mathbf{P}(V)$ be the subset of hyperplanes H not containing e_i. Such an H admits a unique linear equation of the form

$$e_i^* + \sum_{j \neq i} c_j e_j^* = 0,$$

and the topology of $\mathbf{P}(V)$ makes U_i an open set that is homeomorphically identified with $\mathbf{R}^n$ via $H \mapsto \phi_i(H) = (c_j)_{j \neq i} \in \mathbf{R}^n$. Since continuity is a local property, if we let $\widetilde{U}_i \subseteq V^\vee - \{0\}$ be the open preimage of U_i then for the local homeomorphism aspect of the theorem it suffices to show that $S \cap \widetilde{U}_i \to U_i$ is a local homeomorphism for each i (where we embed S into $V^\vee$ via $v \mapsto \langle v, \cdot\rangle$). By relabelling, we may restrict attention to the case $i = 0$.

We saw in the previous proof that $\widetilde{U}_0$ consists of nonzero linear functionals ℓ on V whose dual basis expansion has nonzero e_0^*-coefficient, say $\ell = \sum a_j e_j^*$ with $a_0 \neq 0$. We also saw that the map $\widetilde{U}_0 \to U_0$ is identified with the natural map $\mathbf{R}^n \times \mathbf{R}^\times \to \mathbf{R}^n$ using $\phi_0 : U_0 \simeq \mathbf{R}^n$ and the isomorphism $\xi_0 : \widetilde{U}_0 \simeq \mathbf{R}^n \times \mathbf{R}^\times$ given by $\ell = \sum a_j e_j^* \mapsto (a_1/a_0, \dots, a_n/a_0; a_0)$. The real problem is to describe $S \cap \widetilde{U}_0$ in terms of this coordinatized description of $\widetilde{U}_0$. Since we are using an orthonormal basis of V, in terms of e_j^*-coordinates S goes over to the set of points $\sum a_j e_j^*$ with $\sum a_j^2 = 1$.

It follows that in $\widetilde{U}_0 = \mathbf{R}^n \times \mathbf{R}^\times$ the points from S are those points $(b_1, \dots, b_n; b)$ such that $b^2 + (bb_1)^2 + \cdots + (bb_n)^2 = 1$ (as we see by setting $a_0 = b$ and $a_j = a_0 b_j$ and computing $\sum a_j^2$ in terms of the b's). In other words, the projection $S \cap \widetilde{U}_0 \to U_0$ is identified with the projection

$$\left\{ (b_1, \dots, b_n, b) \in \mathbf{R}^{n+1} \,|\, b^2(1 + \sum_j b_j^2) = 1 \right\} \to \mathbf{R}^n$$

defined by $(b_1, \dots, b_n, b) \mapsto (b_1, \dots, b_n)$ (with the source having the subspace topology in $\mathbf{R}^{n+1}$ because the product $\mathbf{R}^n \times \mathbf{R}^\times$ has topology agreeing with its subspace topology from $\mathbf{R}^{n+1}$). Fixing a point $x = (b_1, \dots, b_n, b)$ in the source, there is a unique sign $\varepsilon = \pm 1$ such that $b = \varepsilon/\sqrt{1 + \sum_j b_j^2}$, and for all $x' = (b_1', \dots, b_n', b')$ near x in the source we must have that the sign of $b' \neq 0$ is the same as that of b, and hence $b' = \varepsilon/\sqrt{1+\sum_j {b'_j}^2}$. In other words, the maps

$$(b'_1,\ldots,b'_n) \mapsto \left(b'_1,\ldots,b'_n,\varepsilon/\sqrt{1+\sum_j {b'_j}^2}\right),\ \ (b'_1,\ldots,b'_n,b') \mapsto (b'_1,\ldots,b'_n)$$

are mutually inverse continuous maps between an open neighborhood of $x \in \widetilde{U}_0$ and an open neighborhood of its image in U_0. This calculation also shows that there are exactly two points over $(b_1,\ldots,b_n) \in U_0$ in $S \cap \widetilde{U}_0$, having the form $(b_1,\ldots,b_n;\pm b)$ with $b = 1/\sqrt{1+\sum b_j^2}$. For $a_0 = \pm b$ and $a_j = a_0 b_j$ the two resulting points $\sum a_j e_j^* \in V^\vee - \{0\}$ lie in S and are negative to each other (i.e., antipodal). ■