Math 396. Quotients by group actions

Many important manifolds are constructed as quotients by actions of groups on other manifolds, and this often provides a useful way to understand spaces that may have been constructed by other means. As a basic example, the Klein bottle will be defined as a quotient of $S^1 \times S^1$ by the action of a group of order 2. (We will also see how this is related to the usual picture of the Klein bottle, to "justify" our definition.) Also, projective n-space as we defined it earlier will turn out to be the quotient of the standard n-sphere by the action of a group of order 2. The circle as defined concretely in $\mathbf{R}^2$ is isomorphic (in a sense to be made precise) to the the quotient of $\mathbf{R}$ by additive translation by $\mathbf{Z}$ (or $2\pi\mathbf{Z}$; it all comes to the same thing, as we shall see).

We first work out some definitions and examples in a set-theoretic setting, then we introduce topologies. In the geometric (as opposed to set-theoretic) setup, we will restrict our attention to very nice cases where the group has essentially no geometric structure. In practice one must go beyond this to allow the group that acts to be a positive-dimensional manifold, but that would introduce a host of complications that we cannot adequately treat without more knowledge of the theory of manifolds. Nonetheless, even within the restricted setup that we consider, we will see many interesting examples.

1. Set-theoretic case

Definition 1.1. Let X be a set and G a group. A *right action* of G on X is a map of sets $X \times G \to X$, denoted $(x, g) \mapsto x.g$, such that $x.1 = x$ for all $x \in X$ and $(x.g).g' = x.(gg')$ for all $x \in X$ and $g, g' \in G$.

The content in the conditions is that the composition of the operations on X by elements of G interact well with the group law on G, but there are two reasonable possibilities to demand: the action of $gg' \in G$ on X is equal to "first act by g, then by g'" or "first act by g', then by g". The first is most succinctly described with the notation $x.(gg') = (x.g).g'$ whereas the second is most succinctly described with the notation $(gg').x = g.(g'.x)$. Hence, this second possibility is called a *left action*. Of course, nothing prevents us from using the notation $g.x$ for right actions, but the action-formula then takes the form "$(gg').x = g'.(g.x)$" which does not look nice (as the order of appearance of g and g' on the two sides is not the same).

Since we require $1 \in G$ to act as the identity on X, it follows that for any action of G on X the elements $g, g^{-1} \in G$ act on X by mutually inverse self-maps of X and so in particular all $g \in G$ act on X by bijective maps. Hence, we can restate the notions of left-action and right-action as follows. Let $\mathrm{Aut}(X)$ be the group of bijections from X to itself (made into a group using composition, with the identity map of X serving as the identity of this group). A left action of G on X is a homomorphism of groups $\lambda : G \to \mathrm{Aut}(X)$ (as the condition $\lambda(gg') = \lambda(g) \circ \lambda(g')$ says exactly "$(gg').x = g.(g'.x)$"), and a right action of G on X is an antihomorphism $\rho : G \to \mathrm{Aut}(X)$ in the sense that $\rho(gg') = \rho(g') \circ \rho(g)$ for all $g, g' \in G$.

We will be most interested in the case when G is a commutative group, so $gg' = g'g$ for all $g, g' \in G$, in which case the notions of left-action and right-action coincide.

Example 1.2. Let $X = S^n$ be the unit sphere in $\mathbf{R}^{n+1}$ (using the usual inner product) with $n \geq 1$. The map $x \mapsto -x$ sending each point to its antipode (the unique other point where the line $\mathbf{R}x \subseteq \mathbf{R}^{n+1}$ meets the sphere) is called the *antipodal map* and applying it twice gives the identity. Thus, this is an action on X by the order-2 group of integers mod 2, where 0 mod 2 acts as the identity and 1 mod 2 acts as the antipodal map. (Check the action axioms!)

Example 1.3. Let V be a finite-dimensional $\mathbf{R}$-vector space, and let L be the $\mathbf{Z}$-linear span of a basis $\{e_1, \ldots, e_n\}$ of V, by which we mean $L = \{\sum a_i e_i \mid a_i \in \mathbf{Z}\}$. Such an L is called a *lattice* in V.

In $\{e_i\}$-coordinates, $V = \mathbf{R}^n$ and $L = \mathbf{Z}^n$. Each $\lambda \in L$ acts on V by the translation $[\lambda] : v \mapsto v + \lambda$. One readily checks the action axioms, since $[0]$ is the identity on V and

$$(v + \lambda) + \lambda' = v + (\lambda + \lambda')$$

for all $\lambda, \lambda' \in L$ and $v \in V$.

Example 1.4. Let n be a positive integer and let $X = \mathbf{R}^2$. Let G be the group of integers modulo n, considered as a group with respect to addition. Let $a \bmod n$ act on X via counterclockwise rotation by an angle of $2\pi a/n$; note this does indeed only depend on $a \bmod n \in G$ and not on the representative $a \in \mathbf{Z}$, and it really is an action (since rotating by $2\pi a/n$ and then $2\pi a'/n$ is the same as rotation by $2\pi(a + a')/n$).

Definition 1.5. If a group G is given a right action on a set X, the *G-orbit* of $x \in X$ is the set of points $x.g$ for $g \in G$. For a subset $S \subseteq X$ and an element $g \in G$, the *g-translate* $S.g$ is the set of points $x \in X$ with the form $x = s.g$ for some $s \in S$ and (not necessarily unique!) $g \in G$. The *quotient* X/G is the set of G-orbits, and the map $\pi : X \to X/G$ sending $x \in X$ to its G-orbit is the *quotient map*.

Remark 1.6. Points $x, x' \in X$ lie in the same G-orbit if and only if $x' = x.g$ for some $g \in G$. Indeed, suppose x and x' lie in the G-orbit of a point $x_0 \in X$, so $x = x_0.\gamma$ and $x' = x_0.\gamma'$ for $\gamma, \gamma' \in G$. For $g = \gamma^{-1}\gamma'$ we then have

$$x.g = (x.\gamma^{-1}).\gamma' = ((x_0.\gamma).\gamma^{-1}).\gamma' = (x_0.(\gamma\gamma^{-1})).\gamma' = (x_0.1).\gamma' = x_0.\gamma' = x'.$$

Example 1.7. In Example 1.2, the orbits are the pairs of antipodal points. In Example 1.3, the orbits are the lattice translates $v + L$, so in particular the orbit of the origin is L. For example, if $V = \mathbf{R}$ and $L = \mathbf{Z}$, then a $\mathbf{Z}$-orbit is the same thing as the set of real numbers with a fixed "fractional part". In Example 1.4, the orbit of a *nonzero* point $x \in \mathbf{R}^2$ consists of n points equally spaces about the circle of radius $\|x\|$ (Euclidean norm) centered at the origin, whereas the orbit of the origin is a one-point set. This latter type of orbit is "bad", as we shall see.

Visually, you should think of the formation of X/G as collapsing all G-orbits into points. As a basic example, if we consider $\mathbf{Z}$ acting on $\mathbf{R}$ by additive translations, the quotient $\mathbf{R}/\mathbf{Z}$ is seen as $[0, 1)$ set-theoretically, since every real number differs by an integer from a unique number in $[0, 1)$ (the fractional part of the number). Set-theoretically, we could also use $[-1/2, 0) \cup (1, 3/2]$ instead of $[0, 1)$. However, in terms of later notions these both give the "wrong" topological description for the quotient.

Example 1.8. Again consider the translation action on $\mathbf{R}$ by $\mathbf{Z}$. The quotient $\mathbf{R}/\mathbf{Z}$ is identified with the unit circle $S^1 \subseteq \mathbf{R}^2$ via trigonometry: for $t \in \mathbf{R}$ we associate the point $(\cos(2\pi t), \sin(2\pi t))$, and this image point depends on exactly the $\mathbf{Z}$-orbit of t (i.e., $t, t' \in \mathbf{R}$ have the same image in the plane if and only they lie in the same $\mathbf{Z}$-orbit). Hence, we get a well-defined bijection of $\mathbf{R}/\mathbf{Z}$ onto S^1. This will soon be enhanced to more than a set-theoretic bijection (giving the "right" topology on $\mathbf{R}/\mathbf{Z}$). It is more traditional to work with $\mathbf{R}/2\pi\mathbf{Z}$ and with the map $t \mapsto (\cos t, \sin t)$.

Definition 1.9. Let X' and X be two sets endowed with right actions by G. A map $f : X' \to X$ is *G-equivariant* if $f(x'.g) = f(x').g$ for all $x' \in X'$ and $g \in G$.

Any G-equivariant map $f : X' \to X$ must carry the G-orbit of $x' \in X'$ into the G-orbit of $f(x') \in X$, so there is a well-defined map $\overline{f} : X'/G \to X/G$ sending the orbit of x' to the orbit of $f(x')$; the maps f and $\overline{f}$ are compatible with the projections $\pi' : X' \to X'/G$ and $\pi : X \to X/G$ in the sense that $\overline{f} \circ \pi' = \pi \circ f$. We call $\overline{f}$ the map *induced by* f.

Example 1.10. Let $X = \mathbf{R}$ and $G = 2\pi\mathbf{Z}$ acting by additive translation. The map $f : X \to X$ given by $f(x) = x + c$ for some $c \in \mathbf{R}$ is G-equivariant ($(x + c) + g = (x + g) + c$ for all $x \in \mathbf{R}$ and $g \in 2\pi\mathbf{Z}$), and the induced map $\overline{f} : S^1 \to S^1$ is rotation through an angle of c.

2. Topological case

We are not interested in group actions on bare sets X, but rather those actions which interact well with topology (and later, differentiable structure) on the set X. We will only consider the case when the group G is itself discrete, which is to say that all of the interesting geometric structure is on X.

Definition 2.1. Let X be a topological space and G a discrete group. A right action of G on X is *continuous* if for each $g \in G$ the action map $X \to X$ defined by $x \mapsto x.g$ is continuous (and hence a homeomorphism, as the action of g^{-1} gives an inverse); this just says that the map $X \times G \to X$ is continuous when using the product topology on $X \times G$ with G given the discrete topology.

The action is *free* if for each $x \in X$ the stabilizer subgroup $\{g \in G \mid x.g = x\}$ of $g \in G$ fixing x is the trivial subgroup $\{1\}$.

The action is *properly discontinuous* when it is continuous for the discrete topology on G and each $x \in X$ admits an open neighborhood U_x so that the G-translate $U_x.g$ meets U_x for only finitely many $g \in G$.

Remark 2.2. I think that the terminology "properly discontinuous" is due to the fact that G is given the discrete topology, and such a topology is somehow the opposite extreme from the idea of a "continuous group" (connected topological group manifold with positive dimension), whence it is "discontinuous".

Let us analyze free and properly discontinuous actions a bit more closely in order that we may visualize their significance.

Example 2.3. Suppose that X is a locally Hausdorff space, and that G acts on X on the right via a properly discontinuous action. For each $x \in X$, we get an open subset U_x such that U_x meets $U_x.g$ for only finitely many $g \in G$. This property is unaffected by replacing U_x with an open subset around x, so by the locally Hausdorff property we can assume (by replacing U_x with its intersection with an open Hausdorff set around x) that U_x is Hausdorff. The key is that we can do better: there exists an open set $U'_x \subseteq U_x$ such that U'_x meets $U'_x.g$ if and only if $x = x.g$. Thus, if the action is also free then U'_x is disjoint from $U'_x.g$ for all $g \in G$ with $g \neq 1$.

To find U'_x, let $g_1, \dots, g_n \in G$ be an enumeration of the finite set of elements $g \in G$ such that U_x meets $U_x.g$. For any open subset $U \subseteq U_x$ we can only have $U \cap U.g \neq \emptyset$ for g equal to one of the g_i's, so it suffices to show that for each i with $x.g_i \in U_x - \{x\}$ there is an open subset $U_i \subseteq U_x$ such that $U_i \cap (U_i).g_i = \emptyset$ (and then we may take U'_x to be the intersection of the U_i's over the finitely many i such that $x.g_i \neq x$). By the Hausdorff property of U_x, when $x.g_i \in U_x - \{x\}$ there exist disjoint opens $V_i, V'_i \subseteq U_x$ around x and $x.g_i$ respectively. By continuity of the action on X by $g_i \in G$ there is an open $W_i \subseteq X$ around x such that $(W_i).g_i \subseteq V'_i$. Thus, $U_i = W_i \cap V_i$ is disjoint from V'_i yet satisfies $(U_i).g_i \subseteq V'_i$, so $U_i \cap (U_i).g_i = \emptyset$. This completes the construction of U'_x.

The interest in free and properly discontinuous actions is that for such actions in the locally Hausdorff case we may find an open U_x around each $x \in X$ such that U_x is disjoint from $U_x.g$ whenever $g \neq 1$. (Conversely, when this holds we see that the action is certainly free and properly discontinuous.) Thus, for such actions we may say that in X/G we are identifying points in the same G-orbit with this identification process not "crushing" the space X by identifying points of X that are arbitrarily close to each other. An example where things go horribly wrong is the action of

$G = \mathbf{Q}$ on $\mathbf{R}$ via additive translations (with $\mathbf{Q}$ given the discrete topology, so as to fit into the above framework). This is a continuous action, but the quotient $\mathbf{R}/\mathbf{Q}$ is very bad: any two $\mathbf{Q}$-orbits in $\mathbf{R}$ contain arbitrarily close points! However, there are more subtle examples where things go wrong:

Example 2.4. Consider Example 1.4. For any nonzero $x \in \mathbf{R}^2$, its orbit consists of n equally space points on the circle C_x of radius $\|x\|$ centered at the origin. It is geometrically obvious (and you should be able to give a rigorous proof) that for a Euclidean open ball B_x centered at x with sufficiently small radius (on the order of $\|x\|\pi/n$), B_x is disjoint from its translates by nontrivial elements of the group; that is, rotating B_x about the origin by angles $2\pi a/n$ with $a \in \mathbf{Z}$ gives sets disjoint from B_x except for when $n|a$. Hence, on $\mathbf{R}^2 - \{0\}$ the action is properly discontinuous and free. However, the story at the origin is very different (when $n > 1$): the origin is fixed by the entire group, and so every neighborhood of the origin meets its translate by any element of the group (e.g., they meet at the origin). Thus, the action on all of X is not free (though it is properly discontinuous) because of the difficulties at the origin.

Here are some more examples of free and properly discontinuous actions.

Example 2.5. The antipodal map on S^n viewed as an action of the integers mod 2 is free and properly discontinuous: freeness is clear, as is continuity, and for any $x \in S^n$ the points near x all have their antipodes quite far away! Also, the additive translation on a finite-dimensional vector space V by a lattice $L \subseteq V$ is free and properly discontinuous. Indeed, the continuity of the action (relative to the discrete topology on L) is clear, and in terms of linear coordinates the freeness and proper discontinuity follow upon noting that for each $x \in \mathbf{R}^n$ a small neighborhoord of x is disjoint from its translates by nonzero elements of $\mathbf{Z}^n$; if $x = (t_1, \dots, t_n)$ then the open neighborhood

$$(t_1 - 1/2, t_1 + 1/2) \times \cdots \times (t_n - 1/2, t_n + 1/2) \subseteq \mathbf{R}^n$$

around x does the job.

Example 2.6. Let $X = S^1 \times S^1$ be a product of two circles, where the circle

$$S^1 = \{z \in \mathbf{C} \,|\, |z| = 1\}$$

is viewed as a topological group (using multiplication in $\mathbf{C}$, so both the group law and inversion $z \mapsto 1/z = \overline{z}$ on S^1 are continuous). The visibly continuous map $(z, w) \mapsto (1/z, -w) = (\overline{z}, -w)$ reflects through the x-axis in the first circle and rotates 180-degrees in the second circle, and is its own inverse (applying it twice gives the identity). Thus, this gives an action by the order-2 group G of integers mod 2 (as in the case of the antipodal map on the sphere). It is free and properly discontinuous because a small neighborhood of a point (z_0, w_0) in S^1 moves quite far away from itself under application of this action by the unique non-trivial element of G, due to the fact that w_0 is moved 180-degrees away (even though for exceptional points $z_0 = \pm 1$ we have $\overline{z}_0 = z_0$). The associated quotient X/G will be called the (set-theoretic) *Klein bottle*. In Remark 2.15 we will explain how this definition is related to the traditional visualization of the Klein bottle.

Example 2.7. Let X_0 be a topological space and let $X = X_0 \times G$ with G given the discrete topology. In other words, X is a disjoint union of copies of X_0 indexed by elements of G; we call $X_0 \times \{g\}$ the *gth copy* of X_0 in X. We make G act on X through right multiplication on indices: $(x_0, g_0).g = (x_0, g_0 g)$. This is easily seen to be a continuous action that is free and properly discontinuous. In this case the quotient is identified with X_0 via the standard projection $X = X_0 \times G \to X_0$. In general, we say that a continuous right G-action on a topological space Y is a *split* action if it arises by this example: there is a topological space Y_0 and a homeomorphism $Y \simeq Y_0 \times G$ carrying the G-action on Y over to the one on $Y_0 \times G$ through right multiplication on indices. Intrinsically, this just says that Y contains an open set Y_0 such that the open sets $Y_0.g$ for varying $g \in G$ are pairwise disjoint and cover Y.

Theorem 2.8. *Let X be a locally Hausdorff topological space equipped with a free and properly discontinuous action by a group G. There is a unique topology on X/G such that the quotient map $\pi : X \to X/G$ is a continuous map that is a local homeomorphism (i.e., each $x \in X$ admits a neighborhood mapping homeomorphically onto an open subset of X/G). Moreover, the quotient map is open.*

A subset $S \subseteq X/G$ is open if and only if its preimage in X is open, and if $U \subseteq X$ is an open set that is disjoint from $U.g$ for all non-trivial $g \in G$ then the map $U \to X/G$ is a homeomorphism onto its open image $\overline{U}$ and the natural map $U \times G \to \pi^{-1}(\overline{U})$ over $\overline{U}$ given by $(u, g) \mapsto u.g$ is a homeomorphism when G is given the discrete topology; that is, the G-action on $\pi^{-1}(\overline{U})$ is split in the sense of Example 2.7*.*

The topology in this theorem is called the *quotient topology*, and it is locally Hausdorff since $X \to X/G$ is a local homeomorphism. The most important thing to keep in mind is that it is the hypotheses of freeness and proper discontinuity for the action of G on X that ensure we can topologize X/G such that $X \to X/G$ is a local homeomorphism. In particular, two points of X/G are close if and only if the corresponding G-orbits in X contain "nearby" points, and for any small opens U in X with $U \cap (U.g) = \emptyset$ for all $g \in G - \{1\}$ (there are many such U!) the open U "represents" its open image in X/G with the same topology via the bijective projection from U onto its open image in X/G. This is how you should visualize the topology. The final part of the theorem is extremely important. We will see some explicit examples shortly.

Proof. We first show the uniqueness, granting existence, for such a topology τ. Suppose that X is given a topology such that $X \to X/G$ is a continuous map and is a local homeomorphism. For any open set $\overline{U} \subseteq X/G$, continuity implies that its preimage $U \subseteq X$ must be τ-open. More important is that we claim the converse must be true: if $\overline{U} \subseteq X/G$ is a subset whose preimage $U \subseteq X$ is an open subset of X then $\overline{U}$ must be open in X/G. By surjectivity of the quotient map, $\overline{U}$ is the image of its preimage U. Hence, it suffices to show that necessarily for any open $U \subseteq X$ its image in X/G must be open with respect to τ. For each $x \in U$ the local homeomorphism property provides an open neighborhood $U_x \subseteq X$ around x such that $U_x \to X/G$ is a homeomorphism onto an open image. The same therefore holds for any open subset of U_x, such as $U_x \cap U$, so we may suppose $U_x \subseteq U$. Hence, U is a union of open subsets U_x that have open image in X/G, so the image of U is also open (as it is the union of the open images of the U_x's). We have therefore given a characterization of the τ-open sets in X/G in terms of the topology on X: these are the subsets whose preimage in X is open. This proves the uniqueness aspect for the topology τ.

Now we prove the existence aspect, and we see that there is no choice: we simply declare a subset $S \subseteq X/G$ to be *open* if its preimage in X is open. One checks from the definition of a topology that this is indeed a topology on X/G, and we have to prove that this really "works": it makes $X \to X/G$ a continuous local homeomorphism and an open map. The necessity of being an open map was explained above once we have the local homeomorphism condition. Also, continuity is trivial: open sets in X/G have open preimages by definition. Now pick $x \in X$, and we seek an open U_x around x in X such that U_x maps homeomorphically onto an open image in X/G. By the assumption that the action is properly discontinuous and free, we can find an open U_x around x that is disjoint from the g-translate $U_x.g$ of U_x for all non-identity $g \in G$, so in particular U_x injects into X/G. (If distinct points $u, u' \in U_x$ have the same image in X/G then they lie in a common G-orbit and hence $u = u'.g$ for some $g \in G$ that must be non-trivial, as $u \neq u'$ but $u'.1 = u'$. This implies U_x meets $U_x.g$, a contradiction since $g \neq 1$.)

To show that U_x maps homeomorphically onto an open image, it remains to check that for any open $U \subseteq U_x$, the image of U in X/G is open. This says that the preimage of the image is open in X, and this preimage is the union of subsets $U.g$ for $g \in G$. Since the action $x \mapsto x.g$ is a homeomorphism from X to itself for each $g \in G$ (by the continuity condition on the action for G on X), it follows that all subsets $U.g$ are open, whence their union is open.

For an open $U \subseteq X$ such that the translates $U.g$ for varying $g \in G$ are pairwise disjoint, it remains to show that the natural action map $U \times G \to \pi^{-1}(\pi(U))$ defined by $(u, g) \mapsto u.g$ is a homeomorphism (where the left side is the product with G given the discrete topology. This map is certainly surjective, since $x \in \pi^{-1}(\pi(U))$ if and only if $\pi(x) = \pi(u)$ for some $u \in U$, which is to say that x and u have the same G-orbit, or equivalently $x = u.g$ for some $g \in G$. Injectivity holds because if $u.g = u'.g'$ then $u = u'.(g'g^{-1})$ and so U meets $U.(g'g^{-1})$; this forces $g'g^{-1} = 1$ (i.e., $g' = g$) and hence $u' = u$ (since $u'.1 = u'$). For the homeomorphism aspect, we note that by assumption the open translates $U.g$ in X are pairwise disjoint, and their union does cover $\pi^{-1}(\pi(U))$. Thus, $\pi^{-1}(\pi(U))$ as a topological space is a disjoint union of open subsets $U.g$ and by the continuity of the actions of g and g^{-1} we see that the action mapping $U \to U.g$ given by $u \mapsto u.g$ is a homeomorphism (with inverse given by the action of g^{-1}). Hence, the bijective map $U \times G \to \pi^{-1}(\pi(U))$ carries the open subset $U \times \{g\}$ (endowed with the topology of U) over to the open subset $U.g$ via the homemorphism $(u, g) \mapsto u.g$, so we have a bijective map between topological spaces that restricts to homeomorphisms between collections of opens that each cover the respective spaces with no overlaps. Such a map is obviously a homeomorphism. ■

We want to revisit some of our earlier examples and see how the topology works out, but first we need some further properties of this quotient topology, especially a criterion for the quotient topology to be Hausdorff. This is a very nice application of the diagonal criterion for being Hausdorff.

Lemma 2.9. *Let G have a properly discontinuous and free right action on a locally Hausdorff topological space X. The quotient X/G is Hausdorff if and only if the image of the "action mapping"*

$$X \times G \to X \times X$$

given by $(x, g) \mapsto (x.g, x)$ is closed in $X \times X$.

In the special case $G = \{1\}$, the quotient map $X \to X/G$ is a homeomorphism and this lemma recovers the diagonal criterion for X to be Hausdorff. The lemma does not give a new proof of the criterion because its proof uses the criterion.

Proof. The space X/G is Hausdorff if and only if its diagonal map into $(X/G) \times (X/G)$ has closed image. Since the continuous surjective map $\pi : X \to X/G$ is open, so is the map

$$\pi \times \pi : X \times X \to (X/G) \times (X/G)$$

(due to the definition of product topologies). But if $f : Y \to Y'$ is a continuous surjective open map between topological spaces then a subset $Z' \subseteq Y'$ is closed if and only if $f^{-1}(Z') \subseteq Y$ is closed. Indeed, by passing to complements we get an equivalent assertion for openness, and this follows from the fact that f is a continuous surjective open map. Hence, we apply this with $f = \pi \times \pi$ to conclude that X/G has closed diagonal image if and only if the preimage of this diagonal in $X \times X$ is a closed subset of $X \times X$. But a point $(x, x') \in X \times X$ lands in the diagonal of X/G precisely when x and x' have the same image in X/G, which is to say $x = x'.g$ for some $g \in G$. Hence, this preimage is the image of the map $(x', g) \mapsto (x'.g, x')$ from $X \times G$ to $X \times X$. ■

To work out examples, we require one further result. The next lemma answers the question of why the quotient topology as defined above on X/G is the "right" one: it satisfies a convenient mapping property.

Lemma 2.10. *Let X be a locally Hausdorff topological space with a free and properly discontinuous right action by a group G. Let $\pi : X \to X/G$ be the continuous open surjective quotient map.*

Let $f : X \to Y$ be a continuous map to a topological space Y, and assume that f is G-invariant in the sense that $f(x.g) = f(x)$ for all $x \in X$ and $g \in G$. There is a unique set-theoretic map $\overline{f} : X/G \to Y$ satisfying $\overline{f}(\pi(x)) = f(x)$ for all $x \in X$, and it is continuous.

Proof. Since π is surjective, the uniqueness of $\overline{f}$ is clear. As for existence, we have no choice but to define $\overline{f}$ to send the G-orbit $\pi(x)$ of $x \in X$ to $f(x)$, and we have to check that this is well-defined and continuous. The G-invariance assumption says exactly that $f(x) = f(x')$ when $x, x' \in X$ are in the same G-orbit, so indeed $\overline{f}$ is well-defined. To check continuity, for open $U \subseteq Y$ we want $\overline{f}^{-1}(U)$ to be open in X/G. By definition of the quotient topology, this says that $\pi^{-1}(\overline{f}^{-1}(U))$ is open in X, and since this is $(\overline{f} \circ \pi)^{-1}(U)$ with $\overline{f} \circ \pi = f : X \to Y$ a map that was assumed to be continuous, we are done. ∎

Example 2.11. Consider the action of the group of order 2 on the standard n-sphere $S^n \subseteq \mathbf{R}^{n+1}$ via the antipodal map. In this case, the quotient is identified with projective n-space $\mathbf{P}^n(\mathbf{R})$. This is addressed in the homework, and recovers the standard topological construction of projective spaces.

Example 2.12. Let us revisit the example of $\mathbf{Z}$ acting on $\mathbf{R}$ by additive translations. This action has been checked to be continuous for the discrete topology on $\mathbf{Z}$ and to be free and properly discontinuous. To verify that the quotient $\mathbf{R}/\mathbf{Z}$ is Hausdorff, we use the criterion for Hausdorffness of properly discontinuous group-action quotients: is the image of the map $\mathbf{Z} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ defined by $(n, x) \mapsto (n + x, x)$ closed? This is is the set of points $(x, y) \in \mathbf{R}^2$ such that $y - x \in \mathbf{Z}$. In other words, it consists of the infinite collection of lines $y = x + n$ ($n \in \mathbf{Z}$) parallel to the line $y = x$. This is clearly closed. Since every point in $\mathbf{R}$ differs from some element of $[0, 1)$ by an element of $\mathbf{Z}$, the compact subset $[0, 1] \subseteq \mathbf{R}$ surjects onto $\mathbf{R}/\mathbf{Z}$. Since the maps $[0, 1] \to \mathbf{R}$ and $\mathbf{R} \to \mathbf{R}/\mathbf{Z}$ are continuous, it follows that $[0, 1]$ continuously surjects onto $\mathbf{R}/\mathbf{Z}$, whence $\mathbf{R}/\mathbf{Z}$ is a compact Hausdorff space.

Geometrically, the situation is rather simple: we take $[0, 1]$ and we "identify" the ends and assign the "circle" topology. To make this precise, consider the map $\mathbf{R} \to S^1 \subseteq \mathbf{R}^2$ given by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$. This is a continuous map onto S^1 that is invariant under the $\mathbf{Z}$-action on $\mathbf{R}$, so by the mapping property for quotients it follows that this map uniquely factors through a continuous surjective map $\mathbf{R}/\mathbf{Z} \to S^1$. However, this latter map is also injective since the points of $\mathbf{R}$ with a common image in S^1 are exactly the $\mathbf{Z}$-orbits. Hence, $\mathbf{R}/\mathbf{Z} \to S^1$ is a continuous bijection between compact Hausdorff spaces. It is therefore a homeomorphism! (It is instructive to "see" this homeomorphism property directly from the definition of $\mathbf{R}/\mathbf{Z}$ as a topological space, without appealing to compactness; draw pictures.) Trigonometry thereby lets us always visualize $\mathbf{R}/\mathbf{Z}$ as a circle. We consider $\mathbf{R}/\mathbf{Z}$ to be an "abstract circle" with no specified embedding into the plane (or into anything else).

Example 2.13. The preceding example can be pushed further. Let V be a finite-dimensional $\mathbf{R}$-vector space and let $L \subseteq V$ be a lattice (i.e., the $\mathbf{Z}$-linear span of an $\mathbf{R}$-basis of V). We have seen that the action of L on V (via additive translations) is continuous for the discrete topology on L and is free and properly discontinuous as well. Thus, we get a topological quotient V/L. There is an evident continuous map $\mathbf{R}^n \to (\mathbf{R}/\mathbf{Z})^n$ that is invariant with respect to additive translation by $\mathbf{Z}^n$ on the source, so it induces a continuous map $V/L = \mathbf{R}^n/\mathbf{Z}^n \to (\mathbf{R}/\mathbf{Z})^n$ that is clearly bijective. Thus, V/L is always compact Hausdorff (since $\mathbf{R}/\mathbf{Z}$ is)! It is homemorphic to a product of n circles.

A good example is $V = \mathbf{R}^2$ and L spanned over $\mathbf{Z}$ by $\lambda_1 = (1,0)$ and $\lambda_2 = (a,b)$ with $b > 0$. To visualize V/L in this case, consider the compact parallelogram

$$P = \{t_1\lambda_1 + t_2\lambda_2 \,|\, t_1, t_2 \in [0,1]\} \subseteq \mathbf{R}^2.$$

(Why is this compact? Express it as a continuous image of $[0,1] \times [0,1]$.) The map $P \to V/L$ is surjective and it identifies opposite sides of P "in the same direction." Upon picking linear coordinates dual to the $\mathbf{R}$-basis $\{\lambda_1, \lambda_2\}$ for V, we get a description as $\mathbf{R}^2/\mathbf{Z}^2$, so the quotient is homemorphic to $S^1 \times S^1$. Such a quotient is called a *torus* (or rather, a 2-*torus*).

Example 2.14. Consider the Klein bottle again. We defined it earlier (in the set-theoretic sense) to be the quotient of $S^1 \times S^1$ for the action by the group of order 2 for which the non-trivial element acts by $(z, w) \mapsto (1/z, -w)$. In terms of viewing S^1 as $\mathbf{R}/\mathbf{Z}$ (via $t \mapsto (\cos(2\pi t), \sin(2\pi t))$), this is induced by the self-map $(x, y) \mapsto (-x, 1/2+y)$ on $\mathbf{R}^2$ (which does not give the identity on $\mathbf{R}^2$ when applied twice, but does so when working modulo the additive translation action of $\mathbf{Z}^2$ on $\mathbf{R}^2$). We saw above that this action is free and properly discontinuous, and so we get a *topological* structure on the quotient, called the *topological Klein bottle* K.

Since the compact $X = S^1 \times S^1$ continuously surjects onto K, it follows that K must be compact. Of course, we also want to check that K is Hausdorff. We again use the quotient criterion: we must check that the subset of $X \times X$ given by pairs $((z, w), (z', w'))$ with $(z', w') = (z, w)$ or $(z', w') = (1/z, -w)$ is closed. This is clear by using the sequential criterion for closedness in $X \times X = (S^1)^4$. Later in the course we will construct a nice embedding of the Klein bottle into $\mathbf{R}^4$ (perhaps thereby making it seems less abstract, much as the concrete model $S^1 \subseteq \mathbf{R}^2$ makes the quotient $\mathbf{R}/\mathbf{Z}$ seems less abstract).

Remark 2.15. Some words of explanation are now in order concerning how the preceding definition of the topological Klein bottle K is related to other definitions or pictures that the reader may have seen. The preceding definition presents K as a quotient of $\mathbf{R}^2/\mathbf{Z}^2$ by the action $\iota : (x, y) \bmod \mathbf{Z}^2 \mapsto (-x, 1/2 + y) \bmod \mathbf{Z}^2$. Consider the rectangle $R = [-1/2, 1/2] \times [0, 1/2]$ with its product topology. The continuous incluson of R into $\mathbf{R}^2$ induces a continuous map $R \to K$ by composition with the continuous maps $\mathbf{R}^2 \to \mathbf{R}^2/\mathbf{Z}^2 \to K$. This continuous map $R \to K$ between compact Hausdorff spaces is *surjective*. Indeed, any point (x, y) in $\mathbf{R}^2$ can be modified modulo $\mathbf{Z}^2$-translation to lie in $[-1/2, 1/2] \times [-1/2, 1/2]$, and if the second coordinate of such a translation lies in $[-1/2, 0)$ then applying $(x, y) \mapsto (-x, 1/2 + y)$ carries it into R. This establishes the surjectivity.

Which points of R get mapped to a common point of K? The only way a pair of distinct points in R can be related through translation by a nonzero element of $\mathbf{Z}^2$ is for them to be related through translation by $(\pm 1, 0)$ (due to the lengths of the sides of R), which is to say the points have the form $(-1/2, y_0)$ and $(1/2, y_0)$ for some $y_0 \in [0, 1/2]$. The only way a pair of distinct points in R with distinct images in $\mathbf{R}^2/\mathbf{Z}^2$ can map to the same point in K is for their images in $\mathbf{R}^2/\mathbf{Z}^2$ to be related through the involution ι. That is, the points must have the form $(x_0, 0)$ and $(-x_0, 1/2)$ for some $x_0 \in [-1/2, 1/2]$. Hence, set-theoretically the way we get K from the rectangle R is to "identify" the left and right edges in the "same direction" via $(-1/2, y) \leftrightarrow (1/2, y)$ for $y \in [0, 1/2]$ and to "identify" the bottom and top edges in the "opposite direction" via $(x, 0) \leftrightarrow (-x, 1/2)$ for all $x \in [-1/2, 1/2]$. That is, if we define an equivalence relation $\sim$ on the set R via $(-1/2, y) \sim (1/2, y)$ for all $y \in [0, 1/2]$, $(x, 0) \sim (-x, 1/2)$ for all $x \in [-1/2, 1/2]$, $(x, y) \sim (x, y)$ for all $(x, y) \in R$, and $(-1/2, 0) \sim (1/2, 1/2)$ and $(-1/2, 1/2) \sim (1/2, 0)$ (stare at a picture to check that this really is an equivalence relation; the actual content is that the final two conditions identifying "opposite corners" are necessary for transitivity!), then $K = R/\sim$ is the set of equivalence classes. Moreover, the topology is determined by that of R: a subset $U \subseteq K$ is open if and only if its preimage in R is open, and likewise for closedness (in fancier language, K inherits the *quotient topology* from R).

Indeed, it suffices to check closedness (why?), and the "only if" implication is just the continuity of the map $R \to K$ whereas the "if" implication is just the fact that $R \to K$ is a surjective map carrying closed sets to closed sets (as it is a continuous surjection between compact Hausdorff spaces). In this way, we may say that K is obtained *topologically* from a rectangle by identification of pairs of opposite sides using the same orientation for one pair and opposite orientations for the other pair. This is the traditional way that the Klein bottle is defined by topologists, and the relation with the traditional drawing is obtained by proceeding in two steps: first identify the pair of sides with the same orientation to get the compact Hausdorff cylinder $S^1 \times [0, 1/2]$ and then identify the boundary circles via the antipodal map. (It is a theorem in topology that K cannot be homeomorphically embedded into $\mathbf{R}^3$; the usual picture shows the antipodal map identification by the artifice of the cylinder pass through itself, so to speak, but this "self-intersection" exposes the fact that the usual picture is really showing the image of a non-injective continuous map of K to $\mathbf{R}^3$.)

Example 2.16. Consider the case of a split action, so Y contains an open set Y_0 such that the open sets $Y_0.g$ for varying $g \in G$ are pairwise disjoint and cover Y. In other words, the natural action map $Y_0 \times G \to Y$ that respects the G-actions on both sides is a homeomorphism. In this case the continuous standard projection $Y \to Y_0$ is an open map that is G-invariant, and the induced continuous mapping $Y/G \to Y_0$ is certainly bijective and open (as opens in Y/G are images of opens in Y), so it is a homeomorphism.

We emphasize that, by the end of Theorem 2.8, every quotient situation $X \to X/G$ looks (in the topological sense) like the split example *locally over* X/G.

Example 2.17. We now consider a new example. Choose $a > 0$. Let $X = (-a, a) \times S^1$, and let the group of order 2 act on it with the non-trivial element acting by $(t, w) \mapsto (-t, -w)$. This is easily checked to be a continuous action for the discrete topology of the group of order 2, and it is free and properly discontinuous (why?). The quotient M_a is the *Möbius strip* of height $2a$. Draw a picture and convince yourself that this quotient really deserves to be called a Möbius strip. Note that for $0 < a < 1/2$ this is a "piece" of the Klein bottle upon viewing $(-a, a)$ as an open subset of $\mathbf{R}/\mathbf{Z} = S^1$ (the crux is that for $z = e^{2\pi i t}$ we have $1/z = e^{2\pi i(-t)}$).

To check that the Möbius strip M_a is Hausdorff, we again use the quotient criterion: the set of points in $X \times X$ with the form $((t, w), (t', w'))$ with $(t', w') = (t, w)$ or $(t', w') = (-t, 1/w)$ is checked to be closed by using the sequential criterion in $X \times X$.

We can do the same using $X = \mathbf{R} \times S^1$, and then resulting Hausdorff quotient M_∞ is called the *Möbius strip with infinite height*.

Example 2.18. Let us revisit Example 1.4. We remove the origin, and so we consider $X^* = \mathbf{R}^2 - \{0\}$ with the rotation action through angles that are integer multiples of $2\pi/n$. In this case the action is free and properly discontinuous and so we can ask to describe the quotient with its topological structure. This quotient is again the punctured plane! To see this, let us view X^* as $\mathbf{C}^\times$ (via $(u, v) \mapsto u + iv$ for a fixed choice of $i = \sqrt{-1} \in \mathbf{C}$), and consider the map $X^* \to X^*$ corresponding to $z \mapsto z^n$ on $\mathbf{C}^\times$. In terms of standard coordinates (u, v) in the plane, this self-map of the punctured plane X^* is given by the real and imaginary components of $(u + iv)^n$; it is a "polynomial map" of total degree n in u and v.

The formula for local extraction of nth roots on $\mathbf{C}^\times$ in terms of polar coordinates (which are C^∞ in terms of the standard (u, v) coordinates and *vice-versa*) shows that this "nth-power map" on $\mathbf{C}^\times$ is a surjective local C^∞ isomorphism, and in particular is a surjective local homeomorphism. It is also clear that this self-map of $h : X^* \to X^*$ is invariant under rotation by an angle of $2\pi/n$ (which corresponds to $z \mapsto \zeta z$ with $\zeta = e^{2\pi i/n}$ that satisfies $\zeta^n = 1$), and so by the mapping property of the topological quotient map $\pi : X^* \to X^*/G$ it follows that h uniquely factors through π via a continuous map $\overline{h} : X^*/G \to X^*$ (sending the G-orbit $\{\zeta z\}_{\zeta^n=1}$ of $z \in X^* = \mathbf{C}^\times$ to $z^n \in \mathbf{C}^\times = X^*$). This map $\overline{h}$ is clearly bijective, and it is a local homeomorphism since $\overline{h} \circ \pi = h$ with π and h local homeomorphisms. Thus, $\overline{h}$ is a homeomorphism. This identifies X^*/G with X^*, under which the projection map to the quotient is identified with the nth-power map on $X^* = \mathbf{C}^\times$.

We conclude the topological aspects by seeing how passage to the quotient interacts with open and closed subsets. Let $S \subseteq X$ be a subset and assume that it is G-stable in the sense that $s.g \in S$ for all $s \in S$ and $g \in G$. Since $x = (x.g^{-1}).g$ for all $x \in X$ and $g \in G$, it follows that for such S we have $S = S.g$ for all $g \in G$. We consider S as a set with a right G-action via $s \mapsto s.g$, and (check!) this is free and properly discontinuous when S is given the subspace topology.

Theorem 2.19. *If $S \subseteq X$ is a G-stable subset then the map of quotient sets $S/G \to X/G$ is injective and when S is given the subspace topology it is a homeomorphism onto its image. The image is an open* (*resp. closed, resp. locally closed*) *subset if and only if S is open* (*resp. closed, resp. locally closed*) *in X. Moreover, $S \mapsto S/G$ is a bijection between the set of G-stable subsets of X and set of subsets of X/G.*

As a special case, for G-stable locally closed subsets $U \subseteq X$ with image $\overline{U} \subseteq X/G$ that is necessarily locally closed, the situation $U \to \overline{U}$ is "the same" (respecting topologies and G-actions) as the more abstract-looking situation $U \to U/G$. This is very useful for "localizing" general problems with such quotients by working locally on X/G.

Proof. The injectivity of the map of quotients $S/G \to X/G$ says that if $s, s' \in S$ and they are in the same G-orbit in X then they are in the same G-orbit in S; this property is obvious. If $x \in X$ is in the G-orbit of some $s \in S$ then $x \in S$ because of the assumption of G-stability for S. Thus, if $\pi : X \to X/G$ is the surjective quotient map then $S = \pi^{-1}(S/G)$. This shows that $S \mapsto S/G$ is an injection from the set of G-stable subsets of X into the set of subsets of X/G, and conversely if $\overline{S} \subseteq X/G$ is any subset then $S = \pi^{-1}(\overline{S})$ is a G-stable subset (because $\pi(x.g) = \pi(x)$ for all $x \in X$ and $g \in G$) with $S/G = \overline{S}$ inside of X/G, so we get the desired bijectivity result.

We next show that S/G is open (resp. closed) in X/G if and only if the same holds for S in X. Since $S = \pi^{-1}(S/G)$ and π is continuous, if S/G is open (resp. closed) in X/G then the same certainly holds for S in X. The same even holds for the property of being locally closed, for if S/G is an intersection of an open set and a closed set in X/G, then $S = \pi^{-1}(S/G)$ is the intersection of the preimages of these two sets. Conversely, since π is open, if S is open in X then $S/G = \pi(S)$ is open in X/G. Also, if S is closed then by G-stability of S we see that the open set $X - S$ in X is G-stable with $S/G = X/G - \pi(X - S)$, so S/G is closed in X/G. This takes care of the properties of being open and closed, and so now we can deduce that the map $S/G \to X/G$ is a homeomorphism onto its image as follows. It is continuous because composition with the local homeomorphism $S \to S/G$ yields the map $S \to X/G$ that is continuous (as it factors as the composite of $S \to X$ and $X \to X/G$). The homeomorphism property onto the image says that every open set $\overline{U}$ in S/G is the preimage of an open set in X/G. But $\overline{U}$ is the image of a G-stable open set $U \subseteq S$, and by the definition of the subspace topology we have $U = S \cap U'$ for an open set in X. Hence, $\overline{U}$ is the part of S/G that meets the open set $\pi(U')$ in X/G. This settles the result that $S/G \to X/G$ is a homeomorphism onto its image.

Finally, assume S is a G-stable locally closed set in X. We need to prove that S/G is locally closed in X/G. Here the subtle point is that if we write $S = U \cap C$ for an open set U and a closed set C in X then neither U nor C needs to be G-stable and hence they do not interact well with π. However, note that $\overline{S} \subseteq C$, so $S = U \cap \overline{S}$. That is, S is open in its closure $\overline{S}$. This is more promising because the closed set $\overline{S}$ *is* G-stable (any $g \in G$ acts by a homeomorphism on X and carries S bijectively back to itself, so it must likewise do the same for the minimal closed set $\overline{S}$ containing S).

The bijectivity between sets in X/G and G-stable sets in X, and likewise for closed sets in each, implies that the closed set $\overline{S}/G = \pi(\overline{S})$ in X/G is the minimal closed set containing S/G, so it is the closure of S/G in X/G. Since S is open in $\overline{S}$, by applying the preceding considerations to the topological space $\overline{S}$ with its free and properly discontinuous right G-action we get that S/G is open in the topological space $\overline{S}/G$. However, $\overline{S}/G$ maps homeomorphically onto its image in X/G (!), so $S/G \subseteq X/G$ is an open subset of a closed set (with its subspace topology from X/G), so S/G is a locally closed set in X/G. ■

3. Premanifold case

To conclude our discussion, we need to bring in the differentiable structures. For example, we want to consider the Klein bottle and the Möbius strip not merely as topological spaces, but as C^∞ manifolds. Such enhanced structure on topological quotients requires that the group action also respect the differentiable structure:

Definition 3.1. Let X be a C^p premanifold with corners, $0 \le p \le \infty$, and let G be a group equipped with a right action on X. This action is a C^p *action* (for the discrete structure on G) if, for each $g \in G$, the map $x \mapsto x.g$ from X to X is a C^p map (in which case it is a C^p isomorphism, as $x \mapsto x.g^{-1}$ is a C^p map that is its inverse). Equivalently, the action map $X \times G \to X$ is a C^p map when G is given the structure of a discrete 0-dimensional C^p premanifold.

In the setup of the definition, when the G-action is free and properly discontinuous we can make the topological quotient X/G (as the topological space X is locally Hausdorff) and it is certainly a topological premanifold with corners (as the projection $X \to X/G$ is a surjective local homeomorphism, and X is locally homeomorphic to an open set in a sector in a finite-dimensional $\mathbf{R}$-vector space). We can often do much better:

Theorem 3.2. *Let X be a C^p premanifold with corners equipped with a C^p right-action by a discrete group G. Assume that this action is free and properly discontinuous. There exists a unique structure of C^p premanifold with corners on the topological quotient X/G such that the projection $\pi : X \to X/G$ is a local C^p isomorphism. Moreover, this has the following mapping property: if $f : X \to Y$ is a C^p map to another C^p premanifold with corners and f is G-invariant in the sense that $f(x) = f(x.g)$ for all $x \in X$ and $g \in G$ then the unique topological factorization $\overline{f} : X/G \to Y$ is C^p.*

The basic idea is that since $X \to X/G$ is a local homeomorphism, local coordinates on X/G should "come from" local coordinates on small opens in X. However, to verify that this gives a C^p-atlas on X/G we will need to somehow use that the action of G on X is a C^p action. The argument requires a bit of care, essentially because disjoint small opens in X may have non-disjoint images in X/G (and so the key is the observation that in such cases there is a unique element of g whose C^p action on X moves one of the opens so that it does meet the other, with overlap that maps homeomorphically onto the overlap of the image opens in X/G).

Proof. The C^p property for a continuous map between C^p premanifolds with corners is local on the source and target. Thus, once we find a C^p-structure on X/G making π a local C^p isomorphism, then the C^p property for $\overline{f}$ will follow from local considerations and the assumed C^p property of f. Our problem is therefore to find a C^p-structure on X/G making π a local C^p isomorphism, and to prove the uniqueness of such a C^p-structure on X/G. The uniqueness follows from the usual argument we have seen for uniqueness of objects with universal mapping properties. Namely, if we are given two C^p-structures $\mathscr{O}$ and $\mathscr{O}'$ on the topological space X/G that both make π a local C^p-isomorphism, and such that each structure has the mapping property for G-invariant C^p maps, then by the universal property for $\pi : X \to (X/G, \mathscr{O})$ applied to the C^p map $\pi : X \to (X/G, \mathscr{O}')$ we get the C^p property for the unique continuous factorization of π through itself, namely that the identity map from $(X/G, \mathscr{O})$ to $(X/G, \mathscr{O}')$ is a C^p map. Hence, $\mathscr{O}'(U) \subseteq \mathscr{O}(U)$ for each open $U \subseteq X/G$. Likewise, by swapping the roles of these two C^p-structures on X/G we get the reverse inclusion in the other direction, and so the two C^p structures would have to coincide.

Our problem is now to make a construction of a C^p-structure on X/G satisfying the desired properties in the theorem. Let $\{(\phi_i, U_i)\}$ be a covering of X by local C^p-charts such that the opens U_i map homeomorphically onto their open images $\overline{U}_i$ in X/G; here, $\phi_i : U_i \to V_i$ is a homeomorphism onto an open subset $\phi_i(U_i)$ in a sector Σ_i in a finite-dimensional $\mathbf{R}$-vector space V_i such that $\phi_j \circ \phi_i^{-1}$ is a C^p isomorphism between the open domains $\phi_i(U_i \cap U_j) \subseteq \Sigma_i$ and $\phi_j(U_i \cap U_j) \subseteq \Sigma_j$. We also take the U_i's so small that $(U_i.g) \cap U_i = \emptyset$ for all $g \in G$ with $g \neq 1$; such small opens exist since the G-action is free and properly discontinuous.

Let $\pi_i : U_i \to \overline{U}_i$ be the induced homeomorphism, and define $\overline{\phi}_i = \phi_i \circ \pi_i^{-1} : \overline{U}_i \to \Sigma_i$; this is a homeomorphism onto the open set $\phi_i(U_i)$ in Σ_i. We claim that the pairs $(\overline{\phi}_i, \overline{U}_i)$ form a C^p-atlas on X/G that the resulting C^p premanifold structure on X/G satisfies the desired properties (i.e., it makes $\pi : X \to X/G$ a local C^p-isomorphism and it is universal for G-invariant C^p maps from X to varying C^p premanifolds). We have to check that the homeomorphism

$$\overline{\phi}_j \circ \overline{\phi}_i^{-1} : \overline{\phi}_i(\overline{U}_i \cap \overline{U}_j) \simeq \overline{\phi}_j(\overline{U}_i \cap \overline{U}_j) \tag{1}$$

between open domains in Σ_i and Σ_j is a C^p isomorphism. (Of course, it suffices to merely prove that it is a C^p map in general, as then swapping the roles of i and j will ensure that the inverse is a C^p map too, so it is a C^p isomorphism.) This is a map between opens in $\phi_i(U_i)$ and $\phi_j(U_j)$ respectively, but the subtle part is that these open subsets of $\phi_i(U_i)$ and $\phi_j(U_j)$ are generally not contained in $\phi_i(U_i \cap U_j)$ and $\phi_j(U_i \cap U_j)$ respectively. Indeed, even if $\overline{U}_i \cap \overline{U}_j$ is non-empty, U_i and U_j may not even meet each other! The issue is that a point in $\overline{U}_i \cap \overline{U}_j$ corresponds to a pair of G-orbits $u_i.G$ and $u_j.G$ in X that meet (with $u_i \in U_i$ and $u_j \in U_j$), but all this implies is $u_i.g = u_j.g'$ for some $g, g' \in G$, or equivalently (by applying ${g'}^{-1}$ and renaming $g{g'}^{-1}$ as g) that $u_i.g = u_j$ for some $g \in G$. It could (and often does) happen that g cannot be taken to be the identity. (For example, taking $X = \mathbf{R}$ and $G = \mathbf{Z}$ acting by translations, the disjoint open subsets $I = (-1/2, 1/2)$ and $J = (1/2, 3/2)$ in $\mathbf{R}$ each map homeomorphically onto their images in the circle, and each is disjoint from its translates by non-zero integers, but the images of I and J in the circle have an enormous intersection.)

Choose $\overline{x} \in \overline{U}_i \cap \overline{U}_j$, and we wish to prove that (1) is a C^p-isomorphism between opens neighborhoods around $\overline{\phi}_i(\overline{x})$ and $\overline{\phi}_j(\overline{x})$. The point $\overline{x}$ is the image of unique points $u_i \in U_i$ and $u_j \in U_j$. Since $\pi_i : U_i \to \overline{U}_i$ and $\pi_j : U_j \to \overline{U}_j$ are bijective, the equality $\pi(u_i) = \overline{x} = \pi(u_j)$ in X/G says that the G-orbits of u_i and u_j meet, so (as we have just seen) there exists $g_0 \in G$ such that $u_i.g_0 = u_j$. Since U_i and U_j are open in X, and elements of G act on X by homeomorphisms of X, it follows that the homeomorphism $x \mapsto x.g_0$ from X onto itself carries the open set $U_i' = U_i \cap (U_j.g_0^{-1})$ around u_i onto the open set $U_j' = U_j \cap (U_i.g_0)$ around u_j. The images $\overline{U}_i' = \pi(U_i')$ and $\overline{U}_j' = \pi(U_j')$ are opens in $\overline{U}_i$ and $\overline{U}_j$ containing $\overline{x} = \pi(u_i) = \pi(u_j)$, with the maps $\pi_i' : U_i' \to \overline{U}_i'$ and $\pi_j' : U_j' \to \overline{U}_j'$ induced by π both homemorphisms. The subsets $\overline{\phi}_i(\overline{U}_i')$ and $\overline{\phi}_j(\overline{U}_j')$ are respectively open subsets in the open sets $\overline{\phi}_i(\overline{U}_i) = \phi_i(U_i) \subseteq V_i$ and $\overline{\phi}_j(\overline{U}_j) = \phi_j(U_j) \subseteq V_j$, and they respectively contain $\overline{\phi}_i(\overline{x})$ and $\overline{\phi}_j(\overline{x})$. Thus, it suffices to show that (1) is a C^p map between $\overline{\phi}_i(\overline{U}'_i \cap \overline{U}'_j)$ and $\overline{\phi}_j(\overline{U}'_i \cap \overline{U}'_j)$. (Certainly it induces a homeomorphism!) Now comes the crux:

Lemma 3.3. *The overlap $\overline{U}'_i \cap \overline{U}'_j$ is exactly the image under π of $U'_i.g_0 = U'_j$.*

Proof. Pick a point $\overline{x}'$ in this overlap, so it has the form $\pi(u'_i)$ for $u'_i \in U'_i = U_i \cap (U_j.g_0^{-1})$ and it also has the form $\pi(u'_j)$ for $u'_j \in U'_j = U_j \cap (U_i.g_0)$. Since the points $\pi(u'_i)$ and $\pi(u'_j)$ in X/G coincide, we have $u'_i.g_{\overline{x}'} = u'_j$ for some $g_{\overline{x}'} \in G$ (with $g_{\overline{x}'}$ *a priori* depending on $\overline{x}'$); note that $g_{\overline{x}'}$ is necessarily uniquely determined by $\overline{x}'$ because if $u'_i.g' = u'_j$ too then $u'_i.g' = u'_i.g_{\overline{x}'}$ and hence

$$u'_i.(g'g_{\overline{x}'}^{-1}) = (u'_i.g').g_{\overline{x}'}^{-1} = (u'_i.g_{\overline{x}'}).g_{\overline{x}'}^{-1} = u'_i.(g_{\overline{x}'}g_{\overline{x}'}^{-1}) = u'_i.1 = u'_i,$$

yet we took U_i (and hence U'_i) so small that the only element of G carrying a point of U'_i to a point of U'_i under the right action is the identity. This forces $g'g_{\overline{x}'}^{-1} = 1$, so $g' = g_{\overline{x}'}$, giving the asserted uniqueness of $g_{\overline{x}'}$.

We claim that $g_{\overline{x}'} = g_0$, which will settle the lemma. To see that $g_{\overline{x}'} = g_0$, note that $u'_i \in U_j.g_0^{-1}$, so $u'_i.g_0 \in U_j$ too, and hence

$$u'_j.(g_{\overline{x}'}^{-1}g_0) = (u'_i.g_{\overline{x}'}).(g_{\overline{x}'}^{-1}g_0) = u'_i.(g_{\overline{x}'}(g_{\overline{x}'}^{-1}g_0)) = u'_i.(g_{\overline{x}'}g_{\overline{x}'}^{-1}) = u'_i.g_0 \in U_j.$$

Thus, $g_{\overline{x}'}^{-1}g_0 \in G$ carries a point u'_j of U_j to a point of U_j, yet by hypothesis U_j is so small that it is disjoint from its translates by all non-identity elements of G. This forces $g_{\overline{x}'}^{-1}g_0 = 1$, so $g_{\overline{x}'} = g_0$. ■

By the lemma, each point in $\overline{\phi}_i(\overline{U}'_i \cap \overline{U}'_j)$ has the form $\overline{\phi}_i(\pi(u'))$ for a unique $u' \in U'_i.g_0 = U'_j$, so this point can also be written as $\overline{\phi}_i(\pi(u'.g_0^{-1})) = \phi_i(u'.g_0^{-1})$ since $u'.g_0^{-1} \in U'_i$. The image of this point under $\overline{\phi}_j \circ \overline{\phi}_i^{-1}$ is $\overline{\phi}_j(\pi(u')) = \phi_j(u')$ (as $u' \in U'_j$). Thus, the map $\overline{\phi}_j \circ \overline{\phi}_i^{-1}$ carries the point $\overline{\phi}_i(\pi(u')) = \phi_i(u'.g_0^{-1})$ to $\overline{\phi}_j(\pi(u')) = \phi_j(u')$, and so this is the restriction to $\phi_i(U'_i) \subseteq \phi_i(U_i)$ of $\phi_j \circ \rho(g_0) \circ \phi_i^{-1}$ with $\rho(g_0) : X \to X$ the right action $x \mapsto x.g_0$. (The point is that to extract u' from $u'.g_0^{-1}$ we apply the right action by g_0.) Hence, the problem of verifying that the $(\overline{\phi}_i, \overline{U}_i)$'s form a C^p-atlas on the topological space X/G has been reduced to the problem of checking that the map $\phi_j \circ \rho(g_0) \circ \phi_i^{-1}$ from $\phi_i(U'_i) = \phi_i(U_i \cap (U_j.g_0^{-1}))$ to $\phi_j(U'_j)$ is a C^p map. This is *exactly* the statement that the map $\rho(g_0) : X \to X$ is C^p as a map from $(U_i.g_0^{-1}) \cap U_j$ to $U_i \cap U_j.g_0$, since ϕ_i and ϕ_j give C^p-coordinates for X on U_i and U_j respectively. Since we assumed that $\rho(g)$ is globally C^p (on all of X) for all $g \in G$, we are therefore done! ■

Corollary 3.4. *Let X be a C^p premanifold equipped with a free and properly discontinuous right action by G. For any locally closed C^p subpremanifold $X' \subseteq X$ that is G-stable, the induced map $X'/G \to X/G$ is an C^p embedding, and X'/G is closed* (*resp. open*) *in X/G if and only if X' is closed* (*resp. open*) *in X. Moreover, the mapping $X' \mapsto X'/G$ gives a bijection between the set of locally closed G-stable C^p subpremanifolds of X and the set of locally closed C^p subpremanifolds of X/G.*

Proof. Theorem 2.19 takes care of the topological apsects of the problem, and since the quotient maps $X \to X/G$ and $X' \to X'/G$ are local C^p isomorphisms it follows that the map $X'/G \to X/G$ is a homeomorphism onto a locally closed set in X/G and it is an immersion since $X' \to X$ is an immersion. Hence, this map of quotients as C^p premanifolds is an embedding. Conversely, if $\overline{Y}$ is an embedded C^p subpremanifold in X/G then we need to show that its G-stable locally closed preimage Y in X is a C^p subpremanifold. Since $X \to X/G$ is a local C^p isomorphism and the topological quotient $\overline{Y} = Y/G$ is a locally closed C^p subpremanifold in X/G, we can work *locally* on Y in X to see that it is a C^p subpremanifold (essentially by using the local immersion-theorem description of the embedded C^p subpremanifold $\overline{Y}$ in X/G). ■

We now revisit many of our examples.

Example 3.5. Let us consider a finite-dimensional $\mathbf{R}$-vector space V and a lattice L in V. We view V as a C^∞ manifold in the usual manner, and let $\pi : V \to V/L$ be the projection. For each $\lambda \in L$, the translation action $v \mapsto v + \lambda$ is certainly C^∞ on V, and hence the compact Hausdorff quotient V/L acquires a natural structure of C^∞ manifold. Concretely, to give local C^∞ coordinates near a point $\overline{v} \in V/L$ we choose a point $v \in V$ over it and we pick a small open $U \subseteq V$ that is disjoint from its L-translates by all nonzero $\lambda \in L$, so the projection $\pi_U : U \to \pi(U) = \overline{U}$ is a homeomorphism (as it is a continous bijective open map). Letting $\phi : U \to V$ be the inclusion (that gives U its C^∞ manifold structure as an open set in the C^∞ manifold V), we define $\overline{\phi} = \phi \circ \pi_U^{-1} : \overline{U} \to V$ to be the map that assigns to each $\overline{x} \in \overline{U}$ the unique point over it in U considered inside of V. The pair $(\overline{\phi}, \overline{U})$ gives a local C^∞-chart on V/L near $\overline{v}$.

To be more explicit, choose linear coordinates on V to identify V with $\mathbf{R}^n$. On the quotient $\mathbf{R}^n/L$, pick a point $\overline{x} \in \mathbf{R}^n/L$ and any $x \in \mathbf{R}^n$ over $\overline{x}$. Let $U \subseteq \mathbf{R}^n$ be a small open around x such that $U + \lambda$ is disjoint from U for all nonzero $\lambda \in L$. Let $\overline{U}$ be the image of U in $\mathbf{R}^n/L$. Local coordinates on $\overline{U}$ around $\overline{x}$ on $\mathbf{R}^n/L$ are given by restricting the standard coordinates on $\mathbf{R}^n$ to the open $U \subseteq \mathbf{R}^n$ around x and then precomposing these coordinates with the inverse of the homeomorphism $U \to \overline{U}$.

Example 3.6. Let us specialize the preceding example to the case of $\mathbf{R}/\mathbf{Z}$, as it reveals a delicate issue. We get a C^∞-structure on this compact space, and for any point $\overline{\xi} \in \mathbf{R}/\mathbf{Z}$ represented by $\xi \in \mathbf{R}$, a local coordinate near $\overline{\xi}$ is given by the standard coordinate on $\mathbf{R}$ restricted to $(\xi - 1/2, \xi + 1/2)$ (or really the composition of this restriction with the inverse of the map projecting this interval into $\mathbf{R}/\mathbf{Z}$). This example is a little subtle, because we have also identified $\mathbf{R}/\mathbf{Z}$ topologically with the circle S^1 in $\mathbf{R}^2$, on which there are natural C^∞-charts as well (generalizing to all standard spheres). It is crucial to know that we get the same C^∞ manifold structure on this topological space via both methods! Roughly speaking, this says that locally on the circle the inverse trig functions depends in a C^∞ manner on the standard coordinates of the plane and vice-versa. We let the reader fill in the details of an argument along such lines; we will later see a much more effective way to deal with such issues. As but one application of this agreement of C^∞-structures: we may identify C^∞ functions on S^1 with C^∞ functions on $\mathbf{R}$ that are translation-invariant by $\mathbf{Z}$ (or $2\pi\mathbf{Z}$).

The preceding problem of comparing two different C^∞-structures arises in many settings. Here is another one where we can solve it "by hand", and we give the explicit calculations. We have exhibited $\mathbf{P}^n(\mathbf{R})$ as a quotient of the standard n-sphere $S^n \subseteq \mathbf{R}^{n+1}$ by the antipodal map. The standard n-sphere has a natural C^∞ manifold structure arising from how it sits in $\mathbf{R}^{n+1}$ (as was sketched in class and worked out in the homework). Since the antipodal map on S^n is induced by negation $(x_1, \dots, x_{n+1}) \mapsto (-x_1, \dots, -x_{n+1})$ in rectangular coordinates on $\mathbf{R}^{n+1}$, it is easy to verify (do it!) that the antipodal map on S^n is a C^∞ map with respect to the C^∞ structure on S^n. Hence, the quotient procedure puts a C^∞ structure on the topological quotient $\mathbf{P}^n(\mathbf{R})$ of S^n modulo the antipodal map. But we have already made a C^∞-structure on $\mathbf{P}^n(\mathbf{R})$ by covering it with $n+1$ Euclidean spaces (as a special case of the general C^∞-atlases we made on Grassmannians). So once again there arises a question: are these two C^∞-structures on the topological space $\mathbf{P}^n(\mathbf{R})$ the same?

To verify the equality of these C^∞-structures, let us first give $\mathbf{P}^n(\mathbf{R})$ its C^∞-structure through the atlas of $n+1$ charts $U_0, \dots, U_n$ modelled on Euclidean spaces. Recall that U_i is the set of hyperplanes in $\mathbf{R}^{n+1}$ whose defining equation involves x_i, and with coordinate system $\phi_i : U_i \simeq \mathbf{R}^n$ given by sending any $H \in U_i$ to the coefficients $(a_j)_{j\neq i}$ in the unique linear equation $\sum a_j x_j = 0$ for H with $a_i = 1$. We let $\mathbf{P}^n(\mathbf{R})'$ denote projective n-space with the C^∞-structure induced as a quotient of S^n. The problem of comparison of C^∞ structures reduces to checking that the continuous antipodal-quotient map $h : S^n \to \mathbf{P}^n(\mathbf{R})$ is C^∞ and in fact a local C^∞ isomorphism. Indeed, assuming this is checked, then the universal property of the quotient manifold structure implies that the unique factorization of h through a continuous map $\overline{h} : \mathbf{P}^n(\mathbf{R})' \to \mathbf{P}^n(\mathbf{R})$ has $\overline{h}$ a C^∞ map, yet $\overline{h}$ is even a local C^∞ isomorphism because the quotient map $\pi : S^n \to \mathbf{P}^n(\mathbf{R})'$ is a local C^∞ isomorphism and we are assuming that $h = \overline{h} \circ \pi$ is a local C^∞ isomorphism. But with $\overline{h}$ a local C^∞-isomorphism with respect to the two C^∞-structures on projective n-space, the observation that $\overline{h}$ is the identity map on the underlying topological space of projective n-space then settles the problem (as then a sufficiently refined atlas for each C^∞-structure on $\mathbf{P}^n(\mathbf{R})$ is also an atlas for the other).

Having reduced ourselves to showing that $h : S^n \to \mathbf{P}^n(\mathbf{R})$ is a local C^∞-isomorphism, we now bring out the C^∞-atlases to describe $\overline{h}$ in local C^∞ coordinates. One atlas on S^n is given by the $2(n+1)$ charts arising from the hemispheres $S^n_{i,-} = S^n \cap \{x_i < 0\}$ and $S^n_{i,+} = S^n \cap \{x_i > 0\}$ which project onto the ith coordinate hyperplane in $\mathbf{R}^{n+1}$. That is, for $0 \le i \le n$ and $\varepsilon = \pm 1$, on $S^n_{i,\varepsilon}$ we have the C^∞ parameterization via the open unit ball in $\mathbf{R}^n$:

$$\{(a_j)_{j\neq i} \in \mathbf{R}^n \mid \sum_{j\neq i} a_j^2 < 1\} \simeq S^n_{i,\varepsilon}$$

defined by

$$(a_j)_{j\neq i} \mapsto (a_0, \dots, a_{i-1}, \varepsilon \cdot \sqrt{1 - \sum_{j\neq i} a_j^2}, a_{i+1}, \dots, a_n).$$

A point $(a_0, \dots, a_n) \in S^n$ lies in $S^n_{i,+}$ or in $S^n_{i,-}$ precisely when the hyperplane $\sum a_j x_j = 0$ has $a_j \neq 0$, which is to say that it lies in $U_i \subseteq \mathbf{P}^n(\mathbf{R})$. Hence, under the topological quotient map $\pi : S^n \to \mathbf{P}^n(\mathbf{R})$ the preimage of U_i is $S^n_{i,+} \cup S^n_{i,-}$. The local C^∞-isomorphism property for h therefore is equivalent to saying that the maps $S^n_{i,\varepsilon} \to U_i$ are local C^∞-isomorphisms when expressed in terms of the standard coordinates on these domains. This map is

$$(a_j)_{j\neq i} \mapsto (\varepsilon a_j / s(\underline{a}))_{j\neq i}$$

with $s(\underline{a}) = \sqrt{1 - \sum_{j\neq i} a_j^2}$. This is obviously C^∞. In this latter formula the sum of the squares of the coordinates is $(\sum_{j\neq i} a_j^2)/(1 - \sum_{j\neq i} a_j^2)$, and $y \mapsto y/(1-y)$ is a C^∞-isomorphism of $(0,1)$ onto $(0,\infty)$ (with inverse $t \mapsto t/(1+t)$), so the C^∞ map $S^n_{i,\varepsilon} \to U_i = \mathbf{R}^n$ has open image $\mathbf{R}^n - \{0\}$ and is a C^∞ isomorphism onto this image: a C^∞ inverse map is

$$(t_j)_{j\neq i} \mapsto (\varepsilon t_j / S(\underline{t}))_{j\neq i}$$

with $S(\underline{t}) = \sqrt{1 + \sum_{j\neq i} t_j^2}$. This completes the verification that the two C^∞-structures on $\mathbf{P}^n(\mathbf{R})$ coincide.

Example 3.7. On both the Klein bottle and the Möbius strip (with finite or infinite height) we get structures of C^∞ manifolds as respective quotients of $S^1 \times S^1$ and $(-a, a) \times S^1$ or $\mathbf{R} \times S^1$. What this means in concrete terms is that to give local coordinates at a point of either of these surfaces, we simply use the "pushforward" of local coordinates on $S^1 \times S^1$ and $(-a, a) \times S^1$ (or $\mathbf{R} \times S^1$) on small opens. The standard procedure on these products is to use coordinates from the factors, and for S^1 we typically use "angular" coordinates arising from the presentation of S^1 as $\mathbf{R}/\mathbf{Z}$ via trigonometric functions.

Example 3.8. Our analysis of the quotient of $\mathbf{R}^2 - \{0\} = \mathbf{C}^\times$ by the rotation action of the additivie group G of integers mod n works *verbatim* in the C^∞ setting (with "homeomorphism" replaced by "C^∞ isomorphism", and so on). Hence, we have a C^∞ isomorphism of $(\mathbf{R}^2 - \{0\})/G$ with $\mathbf{R}^2 - \{0\}$ carrying the canonical projection over to the nth-power map on $\mathbf{R}^2 - \{0\} = \mathbf{C}^\times$. In view of the general mapping property for C^∞ quotients by free and properly discontinuous C^∞ actions of discrete groups, the consequence is this: if we are given a C^∞ map $f : \mathbf{C}^\times \to Y$ to a C^∞ premanifold Y and $f(\zeta z) = f(z)$ for all $z \in \mathbf{C}^\times$ and nth roots of unity ζ in $\mathbf{C}^\times$ (this is the G-invariance condition in disguise), then the unique set-theoretic map $\overline{f} : \mathbf{C}^\times \to Y$ defined by $\overline{f}(z) = f(z')$ for *any* nth root z' of z is in fact a C^∞ mapping.

Example 3.9. If we replace $\mathbf{C}^\times$ with the circle $S^1 = \mathbf{R}/2\pi\mathbf{Z}$, the same exact method identifies the C^∞ quotient of S^1 by the $2\pi/n$-rotation with S^1 as a C^∞ manifold; the quotient map is thereby identified with the nth-power map $S^1 \to S^1$. In the special case $n = 2$, this says that the C^∞ quotient C of S^1 by the 180-degree rotation $w \mapsto -w$ is identified with S^1, with the quotient map given by the squaring map $S^1 \to S^1$.

Now consider the Möbius strip M_∞ with infinite height. This is the C^∞ quotient of $\mathbf{R} \times S^1$ by $(t, \theta) \mapsto (-t, \theta + \pi)$. Letting G denote the group of order 2, G acts on $\mathbf{R} \times S^1$ with quotient M_∞ and it acts on S^1 (via 180-degree rotation) with quotient C that is also a circle, and these G-actions are compatible with the projection $p : \mathbf{R} \times S^1 \to S^1$. Thus, there is an induced C^∞ mapping $\overline{p} : M_\infty \to C$ on quotients by the G-actions. For each $c \in C$, the fiber $\overline{p}^{-1}(c)$ is identified with $\mathbf{R}$ (considered as $p^{-1}(\theta)$ for either of the two antipodal points $\theta \in S^1$ over $c \in C$).

For a small open $U \subseteq S^1$ with arc length less than a half-circle, U maps isomorphically (in the C^∞ sense) onto its image $\overline{U}$ in C and likewise $\mathbf{R} \times U$ maps isomorphically onto its image in M_∞. Thus, over the small open $\overline{U} \subseteq C$ the map $\overline{\pi}^{-1}(\overline{U}) \to \overline{U}$ is identified with the map $\mathbf{R} \times U \to U$. We shall see that the *global* geometry of the mapping $\overline{\pi} : M_\infty \to C$ onto the circle is more subtle: it cannot be identified with the projection $\mathbf{R} \times C \to C$ in a manner that respects the *a priori* identification of each fiber $\overline{\pi}^{-1}(c)$ with a copy of the real line, even though we have just seen that locally over C such "splittings" can be found. It is such global "twistedness" of M_∞ that is the hallmark of the distinction between local and global geometry.

Example 3.10. We conclude by revisiting the significance of the case of split actions. Now we let X be a C^p premanifold with corners, and we consider the C^p quotient mapping $X \to X/G$. As we have seen in the topological considerations, we may cover X/G by (arbitrarily small) open subsets $\overline{U} \subseteq X/G$ for which the open G-stable preimage U of $\overline{U}$ is topologically identified with the split situation via the action mapping $U_0 \times G \to U$ for a suitable open subset $U_0 \subseteq U$ such that the translates $U_0.g$ for varying $g \in G$ are pairwise disjoint. This homeomorphism $U_0 \times G \to U$ is even a C^p *isomorphism* when the left side is given the product structure (with G considered to be a discrete 0-dimensional C^p premanifold).

More precisely, the left side is a disjoint union of copies of U_0 indexed by $g \in G$, with G acting through right multiplication on the indices, and with this C^p structure on $U_0 \times G$ that recovers the given C^p structure from U_0 on the open set $U_0 \times \{g\}$, the evident G-action on $U_0 \times G$ is a C^p action and the mapping $U_0 \times G \to U$ is also a local C^p isomorphism (and hence a C^p isomorphism). The reason is that locally on source and target this restricts to the map $U_0 \to U_0.g$ given by the restricting the C^p automorphism $x \mapsto x.g$ of X to the open subset U_0. The importance is this: *locally* over X/G the quotient situation $X \to X/G$ is identified with a split example in a manner that respects the C^p structures. This is extremely useful for reducing some questions of local nature to the split case when everything can be sorted out "by hand".