Math 396. The topologists' sine curve

We want to present the classic example of a space which is connected but not path-connected. Define

$$S = \{(x, y) \in \mathbf{R}^2 \mid y = \sin(1/x)\} \cup (\{0\} \times [-1, 1]) \subseteq \mathbf{R}^2,$$

so S is the union of the graph of $y = \sin(1/x)$ over $x > 0$, along with the interval $[-1, 1]$ in the y-axis. Geometrically, the graph of $y = \sin(1/x)$ is a wiggly path that oscillates more and more frequently (between the lines $y = \pm 1$) as we get near the y-axis (more precisely, over the tiny interval $1/(2\pi(n + 1)) \leq x \leq 1/(2\pi n)$ the function $\sin(1/x)$ goes through an entire wave).

We'll write S_+ and S_0 for these two parts of S (i.e., S_+ is the graph of $y = \sin(1/x)$ over $x > 0$ and $S_0 = \{0\} \times [-1, 1]$). It is clear that S_+ is path-connected (and hence connected), as is the graph of any *continuous* function (we use $t \mapsto (t, \sin(1/t))$ to define a path from $[a, b]$ to join up $(a, \sin(1/a))$ and $(b, \sin(1/b))$ for any $0 < a \leq b$, and then reparameterize the source variable to make our domain $[0, 1]$). We will show that S is connected but is *not* path-connected. Intuitively, a path from S_+ that tries to get onto the y-axis part of S cannot get there in finite time, due to the crazy wiggling of S_+. Of course, we have to convert this idea into precise mathematics.

1. Connectedness of S

We begin with a lemma which shows how to recover S from S_+. This will enable us to show that S is connected.

Lemma 1.1. *The closure of S_+ in $\mathbf{R}^2$ is equal to S.*

The point of the lemma is that we'll show the closure of a connected subset of a topological space is always connected, so the connectedness of S_+ and this lemma then implies the connectedness of S. The fact that S turns out to not be path-connected then shows that forming closure can destroy the property of path connectedness for subsets of a topological space (even a metric space).

Proof. To show that S lies in the closure of S_+, we have to express each $p \in S$ as a limit of a sequence of points in S_+. If $p \in S_+$ we use the constant sequence $\{p, p, \dots\}$. If $p = (0, y)$ with $|y| \leq 1$, we argue as follows. Certainly $y = \sin(\theta)$ for some $\theta \in [-\pi, \pi]$, whence $y = \sin(\theta + 2n\pi)$ for all positive integers n. Thus, for $x_n = 1/(\theta + 2n\pi) > 0$ we have $\sin(1/x_n) = y$ for all n. Since $x_n \to 0$ as $n \to \infty$, we have $(x_n, \sin(1/x_n)) = (x_n, y) \to (0, y)$. Geometrically, this is the infinite sequence of points where the horizontal line through y cuts the graph of $\sin(1/x)$.

Now that we have shown that the set S containing S_+ lies inside the closure of S_+, to show that it is the closure of S_+ we just have to show that S is closed (as the closure of S_+ in $\mathbf{R}^2$ is the unique minimal closed subset of $\mathbf{R}^2$ which contains S_+). Let $\{(x_n, y_n)\}$ be a sequence in S with limit $(x, y) \in \mathbf{R}^2$. We must prove $(x, y) \in S$. Since $x = \lim x_n$ and $y = \lim y_n$, we know that $x \geq 0$ and $|y| = \lim |y_n| \leq 1$. If $x = 0$, then clearly $(x, y) = (0, y) \in S$ since $|y| \leq 1$. If $x > 0$, then upon dropping the first few terms of the sequence we can assume $x_n > 0$ for all n. Then $(x_n, y_n) \in S$ must lie on S_+, so $y_n = \sin(1/x_n)$. Since the function $t \mapsto \sin(1/t)$ on $(0, \infty)$ is continuous, from the condition $x_n \to x$ we conclude

$$y = \lim y_n = \lim \sin(1/x_n) = \sin(1/x).$$

Thus, $(x, y) \in S_+ \subseteq S$ once again. ∎

Thanks to the lemma, the connectedness of S is an immediate consequence of the following general fact (applied to the topological space $\mathbf{R}^2$ and the connected subset S_+):

Theorem 1.2. *Let X be a topological space and Y a connected subset. Then the closure $\overline{Y}$ of Y in X is connected.*

Proof. Without loss of generality, $Y \neq \emptyset$. Suppose that $\{U, V\}$ is a separation of $\overline{Y}$. That is, U and V are disjoint opens of $\overline{Y}$ with union equal to $\overline{Y}$. We want one of them to be empty. The intersections $U' = U \cap Y$ and $V' = V \cap Y$ give a separation of Y (why?), so by connectedness of Y we have that one of U' or V' is empty and the other is equal to Y. Without loss of generality, we may suppose $U' = Y$ and $V' = \emptyset$.

Since U is closed in $\overline{Y}$, it has the form $U = \overline{Y} \cap Z$ for some closed subset Z in X. But $Y = U' \subseteq U \subseteq Z$, so by closedness of Z it follows that $\overline{Y} \subseteq Z$. Then

$$U = \overline{Y} \cap Z = \overline{Y},$$

and by disjointness V must then be empty. Hence, $\overline{Y}$ indeed has no non-trivial separations, so it is connected. ■

2. S is not path-connected

Now that we have proven S to be connected, we prove it is not path-connected. More specifically, we will show that there is no continuous function $f : [0, 1] \to S$ with $f(0) \in S_+$ and $f(1) \in S_0 = \{0\} \times [-1, 1]$. Assuming such an f exists, we will deduce a contradiction. Thanks to path-connectedness of S_0, we can extend our path to suppose $f(1) = (0, 1)$. Choose $\varepsilon = 1/2 > 0$. By continuity, for some small $\delta > 0$ we have $\|f(t) - (0, 1)\| < 1/2$ whenever $1 - \delta \leq t \leq 1$. If you draw the picture, you'll see that the graph of $\sin(1/x)$ keeps popping out of the disc around $(0, 1)$ of radius $1/2$, and that will contradict the existence of a *continuous* path f.

To be precise, consider the image $f([1-\delta, 1])$, which must be *connected* since f is *continuous* and $[1 - \delta, 1]$ is connected. Let $f(1 - \delta) = (x_0, y_0)$. Consider the composite of $f : [1 - \delta, 1] \to \mathbf{R}^2$ and projection to the x-axis. Both such maps are continuous, hence so is their composite, so the image of the composite map is a connected subset of $\mathbf{R}$ which contains 0 (the x-coordinate of $f(1)$) and x_0 (the x-coordinate of $f(1 - \delta)$). But since connected subsets of $\mathbf{R}$ must be intervals, it follows that the set of x-coordinates of points in $f([1 - \delta, 1])$ includes the entire interval $[0, x_0]$. Thus, for all $x_1 \in (0, x_0]$ there exists $t \in [1 - \delta, 1]$ such that $f(t) = (x_1, \sin(1/x_1))$.

In particular, if $x_1 = 1/(2n\pi - \pi/2)$ for large n then $0 < x_1 < x_0$ yet $\sin(1/x_1) = \sin(-\pi/2) = -1$. Thus, the point $(1/(2n\pi - \pi/2), -1)$ has the form $f(t)$ for some $t \in [1 - \delta, 1]$, and hence this point lies within a distance of $1/2$ from the point $(0, 1)$. But that's a contradiction, since the distance from $(1/(2n\pi - \pi/2), -1)$ to $(0, 1)$ clearly at least 2 (as is the distance between any point on the line $y = 1$ and any other point on the line $y = -1$).