Math 396. Stokes' Theorem on Riemannian manifolds
(or Div, Grad, Curl, and all that)

"While manifolds and differential forms and Stokes' theorems have meaning outside euclidean space, classical vector analysis does not." Munkres, *Analysis on Manifolds*, p. 356, last line. (This is false. Vector analysis makes sense on any oriented Riemannian manifold, not just $\mathbf{R}^n$ with its standard flat metric.)

1. Introduction

In standard books on multivariable calculus, as well as in physics, one sees Stokes' theorem (and its cousins, due to Green and Gauss) as a theorem involving vector fields, operators called div, grad, and curl, and certainly no fancy differential forms. To ensure that we have not made a big cheat by introducing elaborate machinery and naming some *other* result as "Stokes' theorem", we should show how the modern Stokes' theorem in fact implies the more classical versions (and in fact considerably extends their scope of validity, to Riemannian manifolds not necessarily given as isometrically embedded in $\mathbf{R}^2$ or $\mathbf{R}^3$). The general Stokes' Theorem concerns integration of compactly supported differential forms on arbitrary *oriented* C^∞ manifolds X, so it really is a theorem concerning the *topology* of smooth manifolds in the sense that it makes no reference to Riemannian metrics (which are needed to do any serious *geometry* with smooth manifolds). When we do endow X with a Riemannian metric, then we can translate differential forms into the language of vector fields (since an inner product on tangent spaces sets up a natural isomorphism between tangent spaces and their duals, which is where differential forms "live"). Moreover, with the help of the Hodge star operator and volume forms (all of which are defined in terms of the Riemannian metric), we will express the d-operator within the framework of C^∞ vector fields and C^∞ functions. This will enable us to recast the original (or perhaps more accurately, modern) theorem entirely in the language of vector fields, as was done classically before mathematicians really understood what was going on.

In the special case of $X = \mathbf{R}^3$ with its standard orientation and standard "flat" Riemannian metric, or for oriented smooth submanifolds of $\mathbf{R}^3$ (with their *induced* Riemannian metric), we will recover the three big theorems of classical vector calculus: Green's theorem (for compact 2-submanifolds with boundary in $\mathbf{R}^2$), Gauss' theorem (for compact 3-folds with boundary in $\mathbf{R}^3$), and Stokes' theorem (for oriented compact 2-manifolds with boundary in $\mathbf{R}^3$). In the 1-dimensional case we'll recover the so-called gradient theorem which computes certain line integrals and is really just a beefed-up version of the Fundamental Theorem of Calculus. It very much clarifies the logical structure of the proofs to actually work more generally. We will prove a "generalized divergence theorem" for vector fields on any compact oriented Riemannian manifold (with *no* restrictions on the dimension n), out of which Green's theorem and Gauss' theorem will drop out as special cases when $n = 2, 3$ respectively. In the incredibly special situation of dimension 3 we exploit the magical identity

$$1 + 1 = 3 - 1$$

to obtain a "general Stokes' theorem" for vector fields on oriented compact 2-manifolds with boundary inside of *any* ambient oriented 3-*dimensional* Riemannian manifold (not just $\mathbf{R}^3$, which is the classical case).

The moral of the story is that the difficulty with direct proofs of the classical theorems is entirely caused by the fact that these theorems involve an *enormous* amount of structures being piled on top of each other at the same time: the manifold structure, the orientations, the Riemannian metric, the Hodge star operator, the volume forms, etc. It *vastly* clarifies our understanding of the logic behind these theorems to have approached things in the way we have in this course: we see that for *any* oriented manifold with boundary, one has a Stokes' theorem in the context of compact supported differential forms. This has nothing to do with vector fields or Riemannian metric structure of any sort. If one *imposes* the extra data of a Riemannian metric, then one can ask for a translation of differential forms into the language of vector fields and functions (via the tangential inner products and volume form associated to the metric). And it is then almost just a matter of (careful) mathematical linguistics to translate the differential form version for any manifold into vector field language via the choice of metric. Roughly speaking, in each of the above classical situations we just have to prove a few vector identities in order to carry out the translation so as to obtain the classical results. In particular, we will see that classical vector calculus makes perfectly good sense in the general setting of (oriented) Riemannian manifolds (with boundary).

After reading this handout, you should take a look at your friends' multivariable calculus books and convince yourself that we really have proven the classical theorems, with the added benefit that our approach to integration avoids the mysticism which surrounds the pseudo-"definitions" of integration over surfaces and curves with "area elements" and "line elements" as in the big thick multivariable calculus books. This is not a point to be dismissed lightly: it is crucial that we have *not* just created an elaborate machine which spits out theorems that formally look like the classical results. You must convince yourself that the intuition lying behind the classical approach to (trying to) define the integrals on both sides of the classical theorems really is accurately captured by our *precise definitions* of how to integrate via partitions of unity (keeping in mind that all such sums are *finite* in the case of compact manifolds). More specially, when our definition of integration of differential forms is combined with the vector calculus translation made possible by the Riemannian metric tensor, then you must convince yourself that the resulting precise definitions of surface integrals, etc. as in our general vector calculus theorems really does give what one intuitively wants to be working with in those multivariable calculus books. If you think about the recipes in those books for actually *computing* their fancy integrals in terms of local coordinate systems, you'll see that it really is just our approach to integration in disguise (except that we don't have any of the mathematical imprecision which is inherent in the obscure "definitions" of those books: such definitions are incapable of providing an adequate foundation to actually *prove* things in a convincing manner, and that's why such books never present proofs for the classical theorems at a level of rigor that gets beyond a "plausibility argument").

2. Setup for the generalized divergence theorem

Let $(X, \mathrm{d}s^2)$ be a smooth Riemannian manifold with boundary and with constant positive dimension n. Choose an orientation μ on X. The boundary ∂X is naturally a smooth boundaryless manifold with constant dimension $n-1$ (compact when X is), and we give it the induced Riemannian metric. There is a uniquely determined smooth "outward unit normal field" $\widehat{N}$ along ∂X in X, in accordance with the recipe from class. We recall the basic mechanism here, since it will certainly be relevant to our *proofs*: for $p \in \partial X$ the 1-dimensional quotient $\mathrm{T}_p(X)/\mathrm{T}_p(\partial X)$ is naturally oriented, namely by the condition that the positive half-line is the one containing $\partial_{x_n}|_p$ for any local coordinate system $(U, \{x_1, \dots, x_n\})$ around p carrying an open neighborhood U of p onto an open in a closed half-space in $\mathbf{R}^n$ such that $x_n|_{U\cap\partial X} = c$ with $x_n \leq c$. Using the inner product, the normal line $\mathrm{T}_p(\partial X)^{\perp}$ in $\mathrm{T}_p(X)$ maps isomorphically onto the quotient $\mathrm{T}_p(X)/\mathrm{T}_p(\partial X)$ and so it acquires a natural orientation (namely, a normal vector is positive when it projects into the positive half-line of this quotient of $\mathrm{T}_p(X)$). If you think about a disk in $\mathbf{R}^2$ (or a nontrivial compact interval in $\mathbf{R}$ or solid ball in $\mathbf{R}^3$) with its standard orientation from the ambient Euclidean space, you'll see that this notion of "outward unit normal" is exactly in accordance with the classical notion via pictures.

The boundary ∂X is now realized as an oriented submanifold of an oriented Riemannian manifold with boundary, via the following orientation. Let $\{x_1, \ldots, x_n\}$ be a local μ-positive coordinate system on an open $U \subseteq X$ such that x_n is maximized (and constant) along $\partial U = U \cap \partial X$. The resulting coordinate systems $\{x_1|_{\partial U}, \ldots, x_{n-1}|_{\partial U}\}$ form an oriented atlas on ∂X, and this defines the induced orientation $\partial\mu$ on ∂X. We write $\mathrm{d}V_X$ and $\mathrm{d}V_{\partial X}$ to denote the resulting volume forms on X and ∂X respectively. The relationship between $\partial\mu$ and $\widehat{N}$ at $p \in \partial X$ can be described as follows: (i) for $n > 1$, if $\{v_1, \ldots, v_{n-1}\}$ is an ordered basis of $T_p(\partial X)$ then its sign with respect to the orientation class $(\partial\mu)(p)$ is the $\mu(p)$-sign of the ordered basis $\{\widehat{N}(p), v_1, \ldots, v_{n-1}\}$ of $T_p(X)$, (ii) for $n = 1$, the sign $(\partial\mu)(p)$ is the $\mu(p)$-sign of the ordered basis $\{\widehat{N}(p)\}$ of the line $T_p(X)$.

In order to state the generalized divergence theorem, for any C^∞ vector field $\vec{F}$ on X we need to define a certain C^∞ function $\mathrm{div}(\vec{F})$, the *divergence* of $\vec{F}$ (it depends on the Riemannian metric). Using the inner product $\langle\cdot,\cdot\rangle_p$ on each $T_p(X)$ arising from $\mathrm{d}s^2(p)$, we get a natural isomorphism between $T_p(X)$ and its dual space $T_p^*(X)$ (assigning to each $v \in T_p(X)$ the linear functional $\langle v, \cdot\rangle_p$). Since the coefficients of the metric tensor are smooth, these fibral isomorphisms glue to a C^∞ vector bundle isomorphism $TX \simeq T^*X$. (More concretely, if we use Gram-Schmidt to make local orthonormal frames for TX then the isomorphism from tangent to contangent bundle carries this frame to its dual frame.) Hence, we may identify smooth vector fields with smooth 1-forms. More specifically, we make the:

Definition 2.1. Let $\vec{F}$ be a vector field on an open subset $U \subseteq X$. The 1-form $\omega_{\vec{F}}$ on U *dual to* (or *associated to*) $\vec{F}$ is the image of $\vec{F}$ under the isomorphism $\mathrm{Vec}_X(U) = (TX)(U) \simeq (T^*X)(U) = \Omega^1_X(U)$. Pointwise, for each $p \in U$ the linear functional $\omega_{\vec{F}}(p) \in T_p^*(X)$ on $T_p(X)$ is $\langle\vec{F}(p), \cdot\rangle_p$.

It is obvious from the definitions that the assignment $\vec{F} \mapsto \omega_{\vec{F}}$ is additive in $\vec{F}$, linear with respect to multiplication by C^∞-functions on U, and compatible with shrinking U.

Example 2.2. If $X = \mathbf{R}^n$ with the standard flat metric, then we have metric tensor coefficients $g_{ij} = \delta_{ij}$ relative to the standard global coordinates $\{x_1, \ldots, x_n\}$. Thus, we clearly get $\omega_{\partial_{x_i}} = \mathrm{d}x_i$. It follows that if $\vec{F} = \sum f_j \partial_{x_j}$ is a vector field on some open $U \subseteq \mathbf{R}^n$ (more classically written as $\vec{F}(p) = (f_1(p), \ldots, f_n(p))$ for each $p \in U$), then $\omega_{\vec{F}} = \sum f_j \mathrm{d}x_j$. This natural construction is determined (in a *coordinate-free manner*!) by the Riemannian manifold structure on $\mathbf{R}^n$. In particular, this is *not* an artificial *coordinate-dependent* definition that is pulled out of thin air solely for the purpose of making formulas in terms of 1-forms look like formulas in terms of vector fields (an approach that is regrettably used in several introductory texts).

Using the Riemannian structure on X, when given any C^∞ vector field $\vec{F}$ on an open set $U \subseteq X$ we get the dual C^∞ 1-form $\omega_{\vec{F}}$ on U. Via the Riemannian structure and orientation, we have a Hodge star operator that yields $\star(\omega_{\vec{F}}) \in \Omega^{n-1}_X(U)$. More generally, there are natural $C^\infty(U)$-linear isomorphisms

$$\star_r : \Omega^r_X(U) \to \Omega^{n-r}_X(U)$$

respecting shrinking on U and recovering the old Hodge star isomorphisms

$$\star_r(p) : \wedge^r(T_p^*(X)) \simeq \wedge^{n-r}(T_p^*(X))$$

from linear algebra for every integer r and every $p \in X$; recall that these fibral isomorphisms are given in terms of the inner product structure and orientation on $T_p(X)$. We usually write $\star$ rather than $\star_r$, since there seems little risk of confusion and such abuse of notation is standard (as r is always known from context).

We apply these considerations in the special case $r = 1$, so when given a C^∞ vector field $\vec{F}$ on $U \subseteq X$, we obtain an $(n-1)$-form $\star(\omega_{\vec{F}})$ in $\Omega^{n-1}_X(U)$. If we apply the d-operator to this, we get a differential form of top degree, and consequently over U can write this uniquely as a C^∞-function multiple of the nowhere-vanishing top degree *volume form* $\mathrm{d}V_X$ made out of the metric tensor and orientation. This coefficient function multiplier is to be called the *divergence* of $\vec{F}$:

Definition 2.3. For $\vec{F} \in C^\infty(U)$ with an open subset $U \subseteq X$, the function $\mathrm{div}(\vec{F}) \in C^\infty(U)$ is characterized by

$$\mathrm{d}(\star(\omega_{\vec{F}})) = \mathrm{div}(\vec{F})\,\mathrm{d}V_X|_U.$$

Example 2.4. Consider $X = \mathbf{R}^n$ and the standard orientation and standard Riemannian metric, with the standard linear coordinates $\{x_1, \dots, x_n\}$. If $\vec{F} = \sum F_j \partial_{x_j}$ is a C^∞ vector field on $U \subseteq X$, since the volume form determined by this metric and orientation is $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ (why?) we compute by unwinding the definitions (check!) that

$$\begin{aligned}
\mathrm{d}(\star(\omega_{\vec{F}})) &= \mathrm{d}(\star(\textstyle\sum F_j \mathrm{d}x_j)) \\
&= \mathrm{d}(\textstyle\sum F_j \star(\mathrm{d}x_j)) \\
&= \mathrm{d}(\textstyle\sum (-1)^{j-1} F_j \mathrm{d}x_1 \wedge \dots \widehat{\mathrm{d}x_j} \cdots \wedge \mathrm{d}x_n) \\
&= \textstyle\sum (-1)^{j-1} \mathrm{d}F_j \wedge \mathrm{d}x_1 \wedge \dots \widehat{\mathrm{d}x_j} \cdots \wedge \mathrm{d}x_n \\
&= (\textstyle\sum \partial F_j/\partial x_j) \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\
&= (\textstyle\sum \partial F_j/\partial x_j) \mathrm{d}V_X.
\end{aligned}$$

Thus, we conclude

$$\mathrm{div}(\vec{F}) = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}.$$

Of course, this computation works only in the standard coordinate system, since it is an oriented flat coordinate system (so the metric tensor acquires a particularly simple form in these coordinates). Since our definition of divergence of vector fields was intrinsic to the Riemannian structure and orientation, one can likewise compute divergence in *any* orientated coordinate system at all: once we compute the coefficient functions g_{ij} of the metric tensor relative to oriented coordinates, we can then compute a frame of orthonormal bases across all tangent spaces (by uniformly applying Gram-Schmidt) and then we can compute the Hodge star operator and volume form. Once this data is computed, the formula for divergence drops out by the same procedure used above. The crucial point here is that the *only* tricky aspect is to compute the g_{ij}'s. Once this is done, you should *never* switch to another coordinate system to carry out any other part of the calculation. It is exactly the power to work entirely within a single coordinate system (once the g_{ij}'s are known) that makes the coordinate-free language of metric tensors far superior to the more classical coordinate-dependent presentation of vector calculus (in terms of which one is always making painful switches back and forth between rectangular and other coordinate systems when trying to compute anything in a "non-rectangular" coordinate system).

You should compare our treatment of the divergence with that in various introductory texts, where it is often presented as an artifice of the rectangular coordinate system on $\mathbf{R}^n$ instead of more conceptually in a manner that makes sense for *any* smooth oriented Riemannian manifold (with boundary, or even with corners).

Remark 2.5. The divergence can be constructed *without* requiring orientations! (This is suggested by the fact that in the classical case in Example 2.4, the formula obtained is insensitive to reordering of the standard coordinate functions.) Suppose $(X, \mathrm{d}s^2)$ is a Riemannian manifold with corners, and let $U \subseteq X$ be an open subset. For $\vec{F} \in \mathrm{Vec}_X(U)$ we wish to uniquely define $\mathrm{div}(\vec{F}) \in C^\infty(U)$ such that on orientable open subsets U_0 of U it agrees with the divergence as defined over such subsets upon giving each a structure of oriented Riemannian manifold with corners. Since U is covered by orientable open subsets (e.g., coordinate balls), the uniqueness is immediate and for existence all we have to check is that the above construction in the oriented case is *independent* of the orientation.

It suffices to work over each of the open connected components of such U_0 separately, so we may suppose $U_0 = U = X$ is an orientable connected Riemannian manifold with corners and we want to prove that $\mathrm{div}(\vec{v})$ defined via an orientation on X is unaffected by negating the orientation (as this is the only other orientation, since X is now connected). Consider the equation $\mathrm{d}(\star(\omega_{\vec{F}})) = \mathrm{div}(\vec{F})\mathrm{d}V_X$ via the volume form and Hodge star over the oriented Riemannian manifold with corners X. If we negate the orientation then $\star$ is negated and $\mathrm{d}V_X$ is also negated! Hence, $\mathrm{div}(\vec{F})$ is unchanged. This completes the construction of the divergence in the absence of a global orientation.

3. Generalized divergence theorem and consequences

We can now state the general form of our result, which will then be specialized to recover classical theorems due to Green and Gauss.

First, here is the basic setup. Let $(X, \mathrm{d}s^2)$ be a smooth Riemannian manifold with boundary, and with constant positive dimension n. Fix an orientation μ on X. Give ∂X the induced Riemannian metric and induced orientation $\partial\mu$. Let $\widehat{N}$ be the C^∞ unit outward normal field along ∂X in X; this has *nothing* to do with the choice of the orientation μ, but recall that it is nicely related to $\partial\mu$ as we saw at the start of the previous section. Let $\mathrm{d}V_X$ and $\mathrm{d}V_{\partial X}$ denote the associated volume forms on X and ∂X arising from the Riemannian metric and orientations on each.

Let $\vec{F}$ be a compactly supported C^∞ vector field on X (an automatic property if X is itself compact, as in the classical cases). Consider the C^∞ functions $\mathrm{div}(\vec{F})$ on X and

$$\langle \vec{F}|_{\partial X}, \widehat{N}\rangle : p \mapsto \langle \vec{F}(p), \widehat{N}(p)\rangle_p$$

on ∂X. This latter function is C^∞ because the vector fields $\vec{F}|_{\partial X}$ and $\widehat{N}$ are C^∞ on the submanifold ∂X and the metric tensor coefficients defining the tangent space inner products are C^∞. Make sure you understand how to justify these C^∞ assertions.

The main theorem is:

Theorem 3.1. *With the above notation and hypotheses, and with all integrals understood to be taken with respect to the specified orientations, for any compactly supported $\vec{F}$ as above we have*

$$\int_X \mathrm{div}(\vec{F})\mathrm{d}V_X = \int_{\partial X} \langle \vec{F}|_{\partial X}, \widehat{N}\rangle\, \mathrm{d}V_{\partial X}.$$

Remark 3.2. We again stress that if X is compact, then the compactness assumption on the support of $\vec{F}$ is superfluous.

Remark 3.3. As a "safety check", let us verify *a priori* that the two sides of the divergence theorem exhibit the same sign-dependence on orientations. We may and do assume X is connected (by breaking up both sides of the proposed identity according to connected components of X); of course ∂X may be disconnected. Let μ and $\partial\mu$ respectively be the chosen orientation on X and the induced one on $\partial\mu$. Suppose we negate the orientation on X. Since $\partial(-\mu) = -\partial\mu$, the orientation on ∂X is negated as well. Since $\int_{X,-\mu} = -\int_{X,\mu}$ and $\int_{\partial X,-\partial\mu} = -\int_{\partial X,\partial\mu}$, it must be proved that the "integrands" exhibit the same sign-change behavior. The volume forms are each negated upon negating the orientations, so we need $\mathrm{div}(\vec{F})$ and $\langle \vec{F}|_{\partial X}, \widehat{N}\rangle$ have the same sign-dependence. By

Remark 2.5, divergence is orientation-independent. The definition of $\widehat{N}$ does not use orientations, so it is also orientation-independent.

Before proving Theorem 3.1, we record two important consequences: Green's theorem (for $n = 2$) and Gauss' theorem (for $n = 3$). We indulge in a bit of classical notation as well.

Theorem 3.4. (Green) *Let $A \subset \mathbf{R}^2$ be a compact 2-submanifold with boundary $C = \partial A$ (we view A with the standard orientation as from $\mathbf{R}^2$). Give C the counterclockwise orientation (which really is the induced orientation!), and let $\widehat{N}$ denote the outward unit normal field along C relative to A (which really does coincide with the abstract outward unit normal construction!). Let $\mathrm{d}\ell$ denote the length form associated to the induced Riemannian metric on the oriented submanifold C of of $\mathbf{R}^2$.*

For any C^∞ vector field $\vec{F}$ in a neighborhood of A,

$$\int_A \operatorname{div}(\vec{F}) = \int_C \langle \vec{F}|_C, \widehat{N} \rangle \, \mathrm{d}\ell,$$

where the left side is the ordinary integral of a continuous integrand over the bounded rectifiable domain $A \subseteq \mathbf{R}^2$.

Note that the left side has nothing to do with the theory of orientations on manifolds, as it is a classical integral of a function over a subset of a Euclidean space. On the right side we use the counterclockwise orientation on C, and we also use the length form $\mathrm{d}\ell$ arising from this orientation of C (coupled with the induced Riemannian structure on C). If we were to use the clockwise orientation on C then the length form $\mathrm{d}\ell$ gets negated and so the overall integral on the right side of Green's theorem is unaffected. (The choice of $\widehat{N}$ has nothing to do with orientations of the plane, A, or C; it is a topological construction related to how C sits inside of A.) In practice C could be disconnected even if A is connected (e.g., take A to be an annulus), and so as an abstract manifold C may have more orientations than A. However, for Green's theorem it is *crucial* that we put on C an induced orientation from A (which is to say, exactly one among the clockwise or counterclockwise orientations, at least for each collection of components of C on a common component of A).

Here is the deduction of Green's theorem from the generalized divergence theorem.

Proof. Note that $\operatorname{div}(\vec{F})|_A = \operatorname{div}(\vec{F}|_A)$, with the latter divergence computed by viewing A as an oriented Riemannian manifold with boundary. Indeed, this follows from the fact that the volume form on $\mathbf{R}^2$ restricts to the volume form on A (with its induced Riemannian metric) since A is an embedded 2-manifold with boundary in $\mathbf{R}^2$. Thus, we can now apply generalized divergence theorem applied to $X = A$, upon noting that its Riemannian structure is such that the area form is $\mathrm{d}x \wedge \mathrm{d}y$, and hence the integral of a C^∞ function against the area form is a special case of the usual integral of a continuous function on A viewed as a (rectifiable) subset of $\mathbf{R}^2$. Note that A really is rectifiable because its topological boundary inside of $\mathbf{R}^2$ actually coincides with its "manifold boundary" C (why?), and since C is a compact 1-manifold it is covered the images of finitely many C^∞ maps from coordinate charts in $\mathbf{R}$, all of which have measure zero image inside of $\mathbf{R}^2$.

The one technical point which needs to be addressed is *why* the integral of a C^∞ function against the area form (an integral of a 2-form on a manifold $A \subseteq \mathbf{R}^2$) really does agree with its integral as an ordinary function on a reasonable subset of $\mathbf{R}^2$. The subtle aspect is that, strictly speaking, $\{x, y\}$ is not a valid coordinate system for computing integrals on A near points of C, as this coordinate system doesn't have one of the coordinates vanishing along C. In the handout on computing integrals, it was explained why the original 2-form integral over the oriented 2-manifold with boundary A really does agree with a corresponding (continuous) function integral over the subset A in $\mathbf{R}^2$. ∎

Next, we get Gauss' theorem for compact 3-folds in $\mathbf{R}^3$:

Theorem 3.5. (Gauss) *Let $R \subset \mathbf{R}^3$ be a compact 3-submanifold with boundary $S = \partial R$ (we view R with the standard orientation as from $\mathbf{R}^3$). Give S the standard outward normal orientation (which really is the induced orientation!), and let $\widehat{N}$ denote the outward unit normal field along S relative to R (which really does coincide with the abstract outward unit normal construction!). Let $\mathrm{d}A$ denote the area form associated to the induced Riemannian metric on the submanifold A of R (or of $\mathbf{R}^3$).*

For any C^∞ vector field $\vec{F}$ in a neighborhood of R,

$$\int_R \mathrm{div}(\vec{F}) = \int_S \langle \vec{F}|_S, \widehat{N}\rangle \, \mathrm{d}A,$$

where the left side is the ordinary integral of a continuous integrand over the bounded rectifiable domain $R \subseteq \mathbf{R}^3$.

As with Green's theorem, we can see *a priori* that the integrals on the two sides of Gauss' theorem are unaffected by passing to the opposite orientation on S.

Proof. This is an immediate consequence of the generalized divergence theorem applied to $X = R$, and again we have to refer to the handout on computing integrals in order to relate the function integral in the theorem with an integral of a differential form (the same technical issue arose in the proof of Green's theorem). ∎

With the classical special consequences having been discussed, let's now turn to the proof of Theorem 3.1. As we'll see, the hard part is the Stokes' theorem for differential forms, and the generalized divergence theorem itself will just be a special case when applied to a well-chosen differential form which is adapted to our choice of vector field.

Proof. (of Theorem 3.1). Let $\omega = \star(\omega_{\vec{F}}) \in \Omega^{n-1}_X(X)$, a compactly supported differential form (since $\vec{F}$ is compactly supported). By the very *definition* of divergence, we have

$$\mathrm{div}(\vec{F})\, \mathrm{d}V_X = \mathrm{d}\omega.$$

Thus, by Stokes' theorem (!) we get

$$\int_X \mathrm{div}(\vec{F})\, \mathrm{d}V_X = \int_X \mathrm{d}\omega = \int_{\partial X} \omega|_{\partial X}.$$

Thus, we just have to prove the identity

$$\star(\omega_{\vec{F}})|_{\partial X} = \langle \vec{F}|_{\partial X}, \widehat{N}\rangle \, \mathrm{d}V_{\partial X} \tag{3.1}$$

in $\Omega^{n-1}_{\partial X}(\partial X)$, with this understood to be an equality of numbers (with $\mathrm{d}V_{\partial X}$ a sign-valued function at each point of ∂X) in the case $n = 1$.

We check this identity at each point of ∂X, so choose $p \in \partial X$, and in some aspects we will need to treat the case $n = 1$ separately. At the outset, we allow any $n \geq 1$. Let $\{x_1, \dots, x_n\}$ be an oriented coordinate system on an open U around p with $x_n \leq c$, $\{x_n = c\}$ cutting out ∂U, and (if $n > 1$) $\{x_1, \dots, x_{n-1}\}$ restricting to an oriented coordinate system on ∂U. Let $\{e_1, \dots, e_n\}$ be the frame of orthonormal bases of the $T_q(X)$'s for $q \in U$ which we get by applying the Gram-Schmidt algorithm to the frame of *ordered* bases $\{\partial_{x_i}\}$. Since the $\partial/\partial x_j|_q$'s for $j < n$ give a basis of $T_q(\partial X)$ for $q \in \partial U$, it follows from the *construction* of the Gram-Schmidt algorithm (or rather, the property that it preserves the span of the first j vectors for every j) that the restriction

$$\{e_1|_{\partial U}, \dots, e_{n-1}|_{\partial U}\}$$

is a frame of orthonormal bases for the hyperplanes $T_q(\partial X) = T_q(\partial U)$.

Now recall (by definition!) that the outward unit normal field $\widehat{N}$ along ∂X satisfies the property

$$\langle \widehat{N}, \partial/\partial x_n \rangle > 0$$

at each point of ∂U. This crucial positivity property ensures (even for $n = 1$) that for $q \in \partial U$ the final vector at the end of the Gram-Schmidt process applied to the ordered basis $\{\partial/\partial x_j|_q\}$ is in fact $\widehat{N}(q)$! Make sure you understand this, as it is the whole reason why the the unit normal field $\widehat{N}$ *as we defined it* really is relevant to the theorem at hand. By looking back at how the outward unit normal field $\widehat{N}$ is *defined* and at how the induced orientation on ∂X is *defined* in terms of $\widehat{N}$ and the orientation on X, it follows that for $q \in \partial U$, the ordered basis

$$\{\widehat{N}(q), e_1(q), \dots, e_{n-1}(q)\} \tag{3.2}$$

of $T_q(X)$ is an *oriented basis* relative to the given orientation on X (even if $n = 1$). Moreover, this is an orthonormal basis, so we can therefore compute the volume forms

$$\mathrm{d}V_X(q) = \widehat{N}(q)^* \wedge e_1(q)^* \wedge \cdots \wedge e_{n-1}(q)^*$$

and (for $n > 1$)

$$\mathrm{d}V_{\partial X}(q) = (e_1(q)^* \wedge \cdots \wedge e_{n-1}(q)^*)|_{T_q(\partial X)}$$

for $q \in \partial U$, using the associated dual bases. In the case $n = 1$, the "volume form" $\mathrm{d}V_{\partial X}(q)$ is the sign 1.

We may uniquely write our given vector field $\vec{F}|_U$ in the orthonormal frame (3.2) as

$$\vec{F}|_U = f_0 \widehat{N} + \sum_{j=1}^{n-1} f_j e_j \tag{3.3}$$

for suitable C^∞ functions f_j on U, with the summation understood to not be there in the case $n = 1$. Now assume temporarily that $n > 1$. By (3.3), the above volume form computation gives

$$\star(\omega_{\vec{F}})(q) = f_0(q) e_1(q)^* \wedge \cdots \wedge e_{n-1}(q)^* + \sum_{j=1}^{n-1} (-1)^j f_j(q) \widehat{N}(q)^* \wedge \dots \widehat{e_j(q)^*} \wedge \cdots \wedge e_{n-1}(q)^*$$

in $\wedge^{n-1}(T_q^*(X))$ for $q \in \partial U$. By mapping this into $\wedge^{n-1}(T_q^*(\partial X))$ all terms involving the vector $\widehat{N}(q)$ which is *perpendicular* to $T_q(\partial X)$ are killed (why?), so we get that $(\star(\omega_{\vec{F}}))|_{\partial U} \in \Omega^{n-1}_{\partial X}(\partial U)$ is given by

$$f_0(q) \cdot (e_1^* \wedge \cdots \wedge e_{n-1}^*)|_{T_q(\partial X)} = f_0(q) \mathrm{d}V_{\partial X}(q) \tag{3.4}$$

in fibers at any $q \in \partial U$. This gives a formula for the left side of (3.1) relative to our choice of local coordinates near $p \in \partial X$ for $n > 1$. The same formula holds for $n = 1$ in the sense that the pullback of the function (0-form) $\star(\omega_{\vec{F}})$ to ∂U is the function $f_0|_{\partial U}$. To verify this, note that it says exactly that for all $q \in \partial U$ the number $\star(\omega_{\vec{F}}(q))$ is equal to $f_0(q)$. We can prove this equality holds by noting that $\omega_{\vec{F}}(q) = f_0(q)\widehat{N}(q)$ with $\widehat{N}(q)$ a *positive* basis of the line $T_q(U)$ (so $\star(\widehat{N}(q)) = 1$).

It is obvious that for $q \in \partial U$ the inner product $\langle \vec{F}(q), \widehat{N}(q) \rangle_q$ is just the coefficient $f_0(q)$ in the *orthonormal* basis expansion (3.3). Hence, by the identity (3.4) that has been verified for all $n \geq 1$ we get

$$(\star(\omega_{\vec{F}}))|_{\partial X} = \langle \vec{F}|_{\partial X}, \widehat{N} \rangle \, \mathrm{d}V_{\partial X}$$

for all $n \geq 1$. More specifically, we have shown that both sides coincide on ∂U and hence at the initial arbitrary point $p \in \partial X$. ■

Although the proof of Theorem 3.1 may have seemed quite long and involved, if you study the proof you'll see that it is really just a glorified but ultimately straightfoward calculation: all of the hard work went into setting up the machinery to enable us to speak in the geometric language of Riemannian manifolds with corners (which really does recover the classical geometric intuition as in the classical context of vector calculus), and the real content of the proof truly is the differential forms version of Stokes' theorem (which has *nothing* to do with vector fields or Riemannian structures at all). Everything else in the proof was just a matter of correctly translating differential form constructions into appropriate vector field language in the context of Riemannian geometry.

4. The gradient theorem

We now prove a 1-dimensional specialization of Stokes' theorem which is called the gradient theorem (and is not endowed with anyone's name because it is really just a glorified version of the Fundamental Theorem of Calculus, as is Stokes' theorem in the 1-dimensional case). To give the setup, we first need to define the gradient of a function. This can be done quite generally, and doesn't even require an orientation:

Definition 4.1. Let X be a Riemannian manifold with corners. For an open $U \subseteq X$ and $f \in C^\infty(U)$, the C^∞ vector field $\mathrm{grad}(f)$ on U is defined by the condition

$$\omega_{\mathrm{grad}(f)} = \mathrm{d}f$$

in $\Omega^1_X(U)$.

In more concrete terms, for each $p \in X$ we have

$$\langle \mathrm{grad}(f)(p), \cdot \rangle_p = \mathrm{d}f(p)$$

as linear functionals on $T_p(X)$. Note that the gradient has nothing to do with orienations (e.g., the definition makes sense even if X is not orientable). As an example, suppose $X = \mathbf{R}^n$ with the standard Riemannian metric and $\{x_1, \dots, x_n\}$ are the standard linear coordinates, so

$$\mathrm{d}f = \sum (\partial_{x_j} f) \mathrm{d}x_j.$$

Since the ∂_{x_i}'s give a frame of orthonormal bases relative to the metric tensor at each point, upon writing $\mathrm{grad}(f) = \sum F_j \partial_{x_j}$ and pairing this against ∂_{x_i} for $1 \le i \le n$ we deduce $F_j = \partial f / \partial x_j$ for all j, so

$$\mathrm{grad}(f) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \, \partial_{x_j},$$

just as in classical vector calculus. Of course, if we had not used standard rectangular (and hence metric-flat) coordinates then the partials with respect to the coordinate functions would no longer generally be an orthonormal frame (i.e., the coefficient functions for the metric tensor would be more complicated) and hence the computation of the gradient would be more complicated (but the underlying method of computation goes the same way in *any* coordinate system).

Now fix a Riemannian manifold X without boundary and with constant positive dimension n (such as $\mathbf{R}^2$ or $\mathbf{R}^3$ in the classical case). Let C be an oriented embedded submanifold with boundary inside of X with constant dimension 1, and give C the induced Riemannian metric, so the boundary ∂C is assigned a collection of signs $\varepsilon(p) \in \{\pm 1\}$ for each $p \in \partial C$. Let $\mathrm{d}\ell$ denote the length form on C. We let $\vec{T}$ denote the tangent field on C dual to $\mathrm{d}\ell$, as in Definition 2.1, so it is a C^∞ vector field. Moreover, the vector field $\vec{T}$ is a field of unit vectors that lie in the half-lines of the tangent lines as determined by the orientation (i.e., $\vec{T}(p) \in T_p(C)$ is the unique unit vector in the "positive" half of the tangent line at each $p \in C$). Since $\vec{T}$ points in the direction of the orientation, we see (by the definition of "induced orientation" from a 1-manifold onto its 0-dimensional boundary) that for $p \in \partial C$, the sign $\varepsilon(p)$ is equal to -1 if and only if $\vec{T}(p)$ is an outward unit normal vector at p to the oriented Riemannian manifold (with boundary) C.

Here is the gradient theorem in its abstract form (to be followed by its classical formulations):

Theorem 4.2. *Let X, C, $\mathrm{d}\ell$, and $\vec{T}$ be as above. Then for any $f \in C^\infty(X)$ for which $f|_C \in C^\infty(C)$ is compactly supported, the C^∞ inner product function $\langle \mathrm{grad}(f)|_C, \vec{T}\rangle$ is compactly supported on C and*

$$\int_C \langle \mathrm{grad}(f)|_C, \vec{T}\rangle \mathrm{d}\ell = \sum_{p \in \partial C} \varepsilon(p) f(p).$$

Note that the sum on the right side is finite, as f is compactly supported. The right side is negated if we negate the orientation on C (and hence on ∂C). The left side is as well, for if we negate the orientation on C then $\int_C$, the 1-form $\mathrm{d}\ell$, and the tangent field $\vec{T}$ all get negated.

Before we prove the gradient theorem, we note that this version immediately implies the classical special case in which $X = \mathbf{R}^n$ (usually with $n = 2$ or $n = 3$) with its usual Riemannian metric structure and C is an oriented compact 1-manifold with boundary inside of $\mathbf{R}^n$, given the induced metric. In such cases the compactness conditions on support are automatically satisfied and the theorem really just says how to compute a "line integral" of a gradient along a compact curve in Euclidean space. Of course, if our curve C is connected and we have a parametrization $\sigma : [0,1] \to C$ which is a C^∞ isomorphism, then the left side of the gradient theorem just becomes an ordinary 1-variable integral in terms of which we replace $\mathrm{d}\ell$ with $\sqrt{g}\mathrm{d}t$ where $\sigma^*(\mathrm{d}s^2) = g\mathrm{d}t \otimes \mathrm{d}t$ is the pullback of the metric tensor to the parameter line. In the extremely special case with C a line segment along a coordinate axis in $\mathbf{R}^n$, then the gradient theorem really is just the fundamental theorem of calculus along that coordinate line.

Here is the proof of the gradient theorem, as usual obtained by stripping away the language to express it as a special case of the general Stokes' theorem.

Proof. By the very definition of the induced orientation, the right side of the gradient theorem is exactly the 0-dimensional integral $\int_{\partial C} f|_{\partial C}$, and by Stokes' theorem is this equal to $\int_C \mathrm{d}(f|_C)$. It therefore suffices to prove the general identity

$$\mathrm{d}(f|_C) = \langle \mathrm{grad}(f)|_C, \vec{T}\rangle \mathrm{d}\ell$$

for $f \in C^\infty(X)$ (which will moreover imply that the coefficient function on the right side has compact support when $f|_C$ is compactly supported). By the very *definition* of gradient we have

$$\langle \mathrm{grad}(f)(p), \vec{T}(p)\rangle_p = ((\mathrm{d}f)(p))(\vec{T}(p)) = ((\mathrm{d}f)(p)|_{T_p(C)})(\vec{T}(p))$$

for $p \in C$, and $(\mathrm{d}f)(p)|_{T_p(C)} = \mathrm{d}(f|_C)(p)$ for $p \in C$ since d commutes with pullback (such as along $C \hookrightarrow X$), so we seek to prove

$$\mathrm{d}(f|_C)(p) = (\mathrm{d}(f|_C)(p))(\vec{T}(p)) \cdot \mathrm{d}\ell(p)$$

in $T_p^*(C)$ for $p \in C$.

In other words, for a 1-dimensional oriented inner product space V over $\mathbf{R}$ (such as $T_p(C)$) and any $\phi \in V^\vee$ (such as $\mathrm{d}(f|_C)(p)$ on $T_p(C)$), we want to prove

$$\phi = \phi(t) \cdot \alpha$$

where $t \in V$ is the unique unit vector in the positive half-line (as determined by the orientation) and α is the length form. To check the identity it suffices to evaluate both sides on the basis $\{t\}$

of V, and so we just need to prove $\alpha(t) = 1$. But this follows from the computation of the volume form α relative to the oriented orthonormal basis $\{t\}$ of V! ■

5. The classical Stokes' theorem

We now finally turn our attention to the special case of 2-manifolds with boundary inside of 3-manifolds (all with Riemannian structure and orientations). In this case a new operator, the *curl*, emerges. This gives rise to another special instance of Stokes' theorem, adapted to the peculiar features of the curl on 3-manifolds. In fact, this special case is what is called Stokes' theorem in classical vector calculus (with the ambient 3-manifold taken to be $\mathbf{R}^3$ with its standard flat metric).

In order to state the classical Stokes' theorem (which for us will be a theorem about 3-dimensional Riemannian manifolds, though we'll also record the special case when things happen inside of $\mathbf{R}^3$), we first need to discuss the curl operator on vector fields. It is here that dimension 3 plays a special role. Let X be an oriented Riemannian manifold with boundary, and assume it has constant dimension 3. As has been discussed above, we get Hodge star operators between r-forms and $(3-r)$-forms on X (and likewise with 3 replaced by $\dim X$ quite generally). The quirk with $\dim X = 3$ is that for any C^∞ vector field $\vec{F}$ on an open set $U \subseteq X$ the dual 1-form $\omega_{\vec{F}} \in \Omega^1_X(U)$ satisfies

$$\mathrm{d}\omega_{\vec{F}} \in \Omega^2_X(U) = \Omega^{3-1}_X(U)$$

thanks to the fundamental identity

$$1 + 1 = 3 - 1.$$

Thus, we can apply the Hodge star operator to get

$$\star(\mathrm{d}\omega_{\vec{F}}) \in \Omega^1_X(U),$$

another 1-form. This C^∞ 1-form must then arise from a unique C^∞ vector field on U, and it is this resulting vector field which we call the *curl* of $\vec{F}$:

Definition 5.1. Let $\vec{F}$ be a C^∞ vector field on an open subset $U \subseteq X$. We define $\mathrm{curl}(\vec{F})$ to be the unique C^∞ vector field on U which satisfies

$$\omega_{\mathrm{curl}(\vec{F})} = \star(\mathrm{d}(\omega_{\vec{F}})).$$

Just as with the divergence and gradient, this definition is conceptual and is determined by the Riemannian metric tensor and orientation (though it only makes sense in the 3-dimensional case). If we negate the orientation on X then the Hodge star is negated and hence the curl is negated.

In the definition of the curl operator, we have *not* mentioned any particular (oriented) coordinate system! Of course, given a specific *oriented* coordinate system $\{x_1, \dots, x_n\}$ around a point, once we compute the metric tensor coefficients in this coordinate system we can then compute the Hodge star and hence can compute the curl operator relative to the basis $\{\partial/\partial x_i|_p\}$ of the $T_p(X)$'s for p in the domain of the coordinate system.

Example 5.2. If $X = \mathbf{R}^3$ with its standard orientation and standard flat Riemannian metric, then in the standard oriented linear coordinates $\{x, y, z\}$ the vector curl applied to

$$\vec{F} = F_1 \partial/\partial x + F_2 \partial/\partial y + F_3 \partial/\partial z$$

is given by exactly the classical formula from vector calculus books. If you've gone through the example of the divergence which we computed above on $\mathbf{R}^n$, you should have not much difficulty carrying out the curl calculation, so we leave it as an exercise. But once again, the crucial point is that the calculation goes the same way in *any* (oriented) coordinate system once the metric tensor coefficients are known (though obviously the specific shape of the formula will ultimately depend on the metric tensor coefficient functions). In particular, to compute the vector curl in spherical coordinates on $\mathbf{R}^3$ you should *not* first compute in rectangular and then do change of coordinates. This is a complete nightmare. It is far more intelligent to just compute the metric tensor coefficients in spherical coordinates, and then carry out the rest entirely in the spherical coordinate system.

The power of the modern language of differential geometry is that it provides a coordinate-free way to discuss concepts such as vector fields, curl, etc. in a manner which works the same way in *all* coordinate systems. The *method* of computation is always the same. This represents a vast improvement on the classical vector calculus language in $\mathbf{R}^3$ which lacks an *intrinsic* (i.e., coordinate-free) geometric formulation and hence is welded to the rectangular coordinate system (thereby making it seem to be a huge mess to compute things in other coordinate systems or to freely move between different coordinate systems without getting overwhelmed with messy formulas).

With the vector curl under our belts, we can now give the statement of the "classical Stokes' theorem" in suitably general form. As with the generalized divergence theorem (i.e., Theorem 3.1), we first have to introduce some terminology. Let $(X, \mathrm{d}s^2)$ be a Riemannian manifold *without* boundary and with constant dimension 3, and give X an orientation. Let $Z \hookrightarrow X$ be a smooth closed submanifold with boundary and with constant dimension 2, and give Z the induced Riemannian metric. Also assume that Z is orientable, and choose an orientation on Z (note that Z is not the boundary of anything, so there is *no* induced orientation of X on Z, just as the orientation of $\mathbf{R}^3$ does not "induce" an orientation on all surfaces in $\mathbf{R}^3$: some surfaces such as the Möbius strip don't even admit an orientation!) Let $\mathrm{d}A$ denote the resulting "area form" on the oriented Riemannian manifold (with boundary) Z. We let $\widehat{N}$ denote the outward unit normal field along Z in X, arising from the orientations on Z and X and from the Riemannian metrics (in accordance with the universal recipe via outward unit normal fields to oriented hypersurfaces in an oriented Riemannian manifold).

Remark 5.3. In contrast with the outward normal field along the boundary of a Riemannian manifold with boundary (as in the generalized divergence theorem and its corollaries due to Green and Gauss), the normal field $\widehat{N}$ just defined along Z in X *does* depend on some orientations, namely for both Z and X. (If we negate exactly one of these then $\widehat{N}$ is negated, but if we negate both then $\widehat{N}$ is unchanged.)

Consider the boundaryless 1-manifold ∂Z, which we give the induced orientation from Z and the induced metric (from Z or X, it comes to the same). Let $\mathrm{d}\ell$ denote the resulting length form on the oriented 1-dimensional Riemannian manifold ∂Z. We define the *tangent field* $\vec{T}$ along ∂Z to be the smooth vector field dual to the length form $\mathrm{d}\ell$ as in Definition 2.1. Thus, $\vec{T}(p) \in T_p(\partial Z)$ satisfoes

$$(\mathrm{d}\ell(p))(\vec{T}(p)) = 1$$

for all $p \in \partial Z$. Due to the relationship between the length form $\mathrm{d}\ell$ and both the orientation and Riemannian structure on ∂Z, this tangent field consists entirely of *unit* vectors (so it is classically called the *unit tangent field in the direction of the orientation on* ∂Z, the justification of this terminilogy being left as an exercise to work out). The vector field $\vec{T}$ and the 1-form $\mathrm{d}\ell$ each change by a sign if we negate the orientation on Z, and neither has anything to do with the orientation on X.

With our ambient 3-manifold X, its 2-dimensional submanifold (with boundary) Z, the unit outward normal field $\widehat{N}$ along Z and tangent field $\vec{T}$ along ∂Z, and area form $\mathrm{d}A$ on Z and length form $\mathrm{d}\ell$ on ∂Z all understood (as determined by the various Riemannian metrics and orientations which we have specified), we're now ready to state the original theorem of Stokes.

Theorem 5.4. (Stokes) *With notation as introduced above, let $\vec{F}$ be a C^∞ vector field on an open neighborhood U of Z in X with $\vec{F}|_Z$ compactly supported (so $\vec{F}|_{\partial Z}$ is compactly supported). Then the C^∞ function $\langle \mathrm{curl}(\vec{F})|_Z, \widehat{N}\rangle$ on Z is compactly supported and*

$$\int_Z \langle \mathrm{curl}(\vec{F})|_Z, \widehat{N}\rangle \, \mathrm{d}A = \int_{\partial Z} \langle \vec{F}|_{\partial Z}, \vec{T}\rangle \, \mathrm{d}\ell.$$

Remark 5.5. When Z is compact (so ∂Z is also), then the compact support hypotheses are automatically satisfied. The classical case of this theorem is the special case $X = \mathbf{R}^3$ (with standard orientation and Riemannian metric) and Z a compact 2-manifold with boundary inside of $\mathbf{R}^3$. In this case, the above theorem is *literally* the theorem called "Stokes' theorem" in vector calculus books. Unlike the case of the theorems of Green and Gauss, for which one side of the theorem was just an ordinary function integral (so we had to do a little bit of work to identify an integral of a differential form with an integral of a function), in the case of the theorem of Stokes there is no such interpretation of either side of the equation. However, you should think carefully about the naive intuition behind surface elements and line elements in classical vector calculus books to convince yourself that the two sides of Theorem 5.4 really do recover the means by which one *computes* the surface and line integrals in classical vector calculus. This matter of comparison with classical concepts of integration of "surface elements" and "line elements" is explained in detail in the handout on how to compute integrals.

Thus, our version really does establish the theorem one wants to have in vector calculus, with the bonus that it has been transported to the generality of an arbitrary oriented 3-dimensional Riemannian manifold so as to free oneself from the obscure specificity of the standard oriented flat Riemannian manifold $\mathbf{R}^3$ which is really of no relevance at all in the proof of the theorem.

Remark 5.6. The right side of Stokes' theorem as stated above has nothing to do with the orientation on X. If we negate the orientation on Z then $\int_{\partial Z}$, $\vec{T}$, and $\mathrm{d}\ell$ are all negated and so the right side changes by a sign. Hence, we should check *a priori* that the left side of the proposed identity is independent of the orienation on X and changes by a sign if we negate the orientation on Z. It suffices to study what happens when we fix one of the two orientations and negate the other. Keeping the orientation on Z fixed but negating the one on X causes $\mathrm{curl}(\vec{F})$ to change by a sign but (by Remark 5.3) also causes $\widehat{N}$ to change by a sign! Hence, there is no overall effect in such cases (just like for the right side). Keeping the orientation on X fixed but negating the orientation on Z, we see that $\int_Z$ and $\widehat{N}$ change by a sign (see Remark 5.3 for the latter) and also the area form $\mathrm{d}A$ on the oriented Riemannian manifold Z changes by a sign. Thus, the overall effect on the left side is multiplication by -1 in such cases (just like for the right side).

Proof. Let $\eta = \omega_{\vec{F}} \in \Omega^1_X(U)$ be dual to $\vec{F}$ via the Riemannian structure on U. Thus, $\eta|_Z \in \Omega^1_Z(Z)$ is compactly supported (as it vanishes at points where $\vec{F}|_Z$ vanishes). By definition of vector curl, we also have

$$\omega_{\mathrm{curl}(\vec{F})} = \star(\mathrm{d}\eta).$$

Since $\star \circ \star = (-1)^{r(n-r)}$ on r-forms on an n-manifold, for $n = 3$ and $r = 1$ we get $\star \circ \star = 1$ on 1-forms on U, so

$$\mathrm{d}\eta = \star(\omega_{\mathrm{curl}(\vec{F})}). \tag{5.1}$$

By the same sort of calculation which proved (3.1), but applied to the vector field $\mathrm{curl}(\vec{F})$ and the 2-manifold Z inside of X (of dimension $n = 3$), we thereby obtain

$$\langle \mathrm{curl}(\vec{F})|_Z, \widehat{N}\rangle \, \mathrm{d}A = (\mathrm{d}\eta)|_Z$$

inside of $\Omega^2_Z(Z)$. But $(\mathrm{d}\eta)|_Z = \mathrm{d}(\eta|_Z)$ since d commutes with pullback (in this case, pullback along the closed embedding of Z into X), and this is compactly supported on Z since $\eta|_Z$ is. In particular, we see that $\langle \mathrm{curl}(\vec{F})|_Z, \widehat{N}\rangle$ is indeed compactly supported on Z.

By Stokes' theorem applied to the compactly supported 1-form $\eta|_Z$ on the 2-manifold (with boundary) Z, we get:

$$\int_Z \langle \mathrm{curl}(\vec{F})|_Z, \widehat{N}\rangle \, \mathrm{d}A = \int_Z (\mathrm{d}\eta)|_Z = \int_Z \mathrm{d}(\eta|_Z) = \int_{\partial Z} \eta|_{\partial Z}.$$

Thus, our problem comes down to proving the identity

$$\eta|_{\partial Z} = \langle \vec{F}|_{\partial Z}, \vec{T}\rangle \, \mathrm{d}\ell \tag{5.2}$$

in $\Omega^1_{\partial Z}(\partial Z)$, where we again recall that $\vec{T}$ is the tangent field along ∂Z which is dual to the nowhere-vanishing length form $\mathrm{d}\ell$.

Since $\eta = \omega_{\vec{F}}$ in $\Omega^1_X(U)$, where we recall that the construction $\vec{F} \rightsquigarrow \omega_{\vec{F}}$ arises from the Riemannian structure on U, it follows from that $\eta|_Z = \omega_{\vec{F}|_Z}$ in $\Omega^1_Z(Z)$ via the Riemannian structure on Z. This assertion (i.e., that $\omega_{\vec{F}|_Z} = \omega_{\vec{F}}|_Z$) is pointwise nothing more than the statement that if $i : W \hookrightarrow V$ is an inclusion of finite-dimensional inner product spaces over $\mathbf{R}$ and $w \in W$ is some vector, with i respecting the inner products (think of the inclusion of $T_p(Z)$ into $T_p(X)$ for $p \in Z$), then the restriction of $\langle i(w), \cdot\rangle_V$ to W is just $\langle w, \cdot\rangle_W$. Quite generally, this same method shows that the $\vec{F} \rightsquigarrow \omega_{\vec{F}}$ construction commutes with restriction to any submanifold (not just a hypersurface).

Thus, if we rename $\vec{F}|_Z$ as $\vec{G}$, then we want to prove the general identity

$$\langle \vec{G}|_{\partial Z}, \vec{T}\rangle \mathrm{d}\ell = \omega_{\vec{G}}|_{\partial Z}$$

for any C^∞ vector field $\vec{G}$ on Z. But by the exact same argument as we just went through in the preceding paragraph, the right side is $\omega_{\vec{G}|_{\partial Z}}$. Thus, if we rename $\vec{G}|_{\partial Z}$ as $\vec{H}$, we want to prove that for any C^∞ vector field $\vec{H}$ along ∂Z, $\langle \vec{H}, \vec{T}\rangle \mathrm{d}\ell = \omega_{\vec{H}}$ in $\Omega^1_{\partial Z}(\partial Z)$. But quite generally for any 1-dimensional oriented Riemannian manifold C with length form $\mathrm{d}\ell$ and dual tangent field $\vec{T}$, we claim that

$$\langle \vec{H}, \vec{T}\rangle \mathrm{d}\ell = \omega_{\vec{H}}$$

for any C^∞ vector field $\vec{H}$ on C. Evaluating both sides at a point $p \in C$, this comes down to the following general assertion concerning a 1-dimensional oriented $\mathbf{R}$-vector space V endowed with an inner product (such as the tangent line $V = T_p(C)$): if $v \in V$ is any vector, $\phi \in V^\vee$ is the length form, and $t \in V$ is the vector dual to ϕ, then

$$\langle v, t\rangle \phi = \langle v, \cdot\rangle$$

in $V^\vee$. It suffices to check this equality when evaluating both sides on the basis $\{t\}$, and since $\phi(t) = 1$ by definition of t this case is clear. ■