Math 396. Subbundles and quotient bundles

1. Motivation

We want to study the bundle analogues of subspaces and quotients of finite-dimensional vector spaces. Let us begin with some motivating examples.

Example 1.1. Let $i : H \hookrightarrow \mathbf{R}^n$ be a smooth embedded hypersurface. By the universal property of bundle pullback, the natural mapping of tangent bundles $\mathrm{d}i : TH \to T(\mathbf{R}^n)$ over $i : H \to \mathbf{R}^n$ uniquely factors through a C^∞ bundle mapping $f : TH \to i^*(T(\mathbf{R}^n))$ over H. On fibers over $h \in H$, this is the injection $\mathrm{T}_h(H) \to \mathrm{T}_{i(h)}(\mathbf{R}^n)$. If $t_1, \dots, t_n$ are the standard coordinate functions on $\mathbf{R}^n$ then $T(\mathbf{R}^n)$ is trivialized by the sections ∂_{t_j}, and so defines a bundle isomorphism $T(\mathbf{R}^n) \simeq \mathbf{R}^n \times \mathbf{R}^n$ over $\mathbf{R}^n$ (using projection to the first $\mathbf{R}^n$-factor on the target) that on fibers over $x \in \mathbf{R}^n$ is the linear isomorphism $\mathrm{T}_x(\mathbf{R}^n) \simeq \mathbf{R}^n$ via the ordered basis $\{\partial_{t_j}|_x\}$ of $\mathrm{T}_x(\mathbf{R}^n)$. Using this trivialization of $T(\mathbf{R}^n) \to \mathbf{R}^n$ we get an induced trivialization of the pullback $i^*(T(\mathbf{R}^n)) \to H$ via the pullback sections $i^*(\partial_{t_j})$.

Concretely, in the fiber $(i^*(T(\mathbf{R}^n)))_h \simeq T(\mathbf{R}^n)_{i(h)} \simeq \mathrm{T}_{i(h)}(\mathbf{R}^n)$ over $h \in H$ the section $i^*(\partial_{t_j})$ has value $\partial_{t_j}|_{i(h)}$. In this way, we have $i^*(T(\mathbf{R}^n)) \simeq H \times \mathbf{R}^n$ as C^∞ vector bundles over H, and so we get a bundle mapping

$$TH \xrightarrow{f} i^*(T(\mathbf{R}^n)) \simeq H \times \mathbf{R}^n$$

over H that is the composite injection

$$\mathrm{T}_h(H) \stackrel{\mathrm{d}i(h)}{\hookrightarrow} \mathrm{T}_{i(h)}(\mathbf{R}^n) \simeq \mathbf{R}^n$$

on fibers over $h \in H$. Visually, as h moves in H the tangent hyperplanes $\mathrm{T}_h(H)$ "move" in $\mathbf{R}^n$ in a manner that (in a sense to be made precise) depends "smoothly" on h. This is the local visualization motivating the idea of subbundles: nicely moving subspaces of fixed dimension in a fixed vector space.

Example 1.2. Give $\mathbf{R}^3$ its standard inner product, with associated standard norm. Let $X = \mathbf{R}^3 - \{0\}$, and for each point $x \in X$ let $S(x) \subseteq X$ be the sphere centered at the origin and passing through x. Consider the tangent plane $\mathrm{T}_x(S(x))$ inside of $\mathrm{T}_x(X) = \mathrm{T}_x(\mathbf{R}^3) \simeq \mathbf{R}^3$. As we vary x, how do these planes in $\mathbf{R}^3$ "move"? The simplest way to get our hands on the situation is to observe that the smooth surfaces $S(x)$ are level sets for the smooth function $f(u, v, w) = u^2 + v^2 + w^2$ on X that has no critical points, and so for each $x \in X$ the functional $\mathrm{d}f(x) : \mathrm{T}_x(\mathbf{R}^3) \to \mathrm{T}_{f(x)}(\mathbf{R}) \simeq \mathbf{R}$ (via the basis $\partial_t|_{f(x)}$ of $\mathrm{T}_{f(x)}(\mathbf{R})$) has as its kernel exactly the tangent plane to the level set $S(x)$ of f through x. Explicitly, the matrix for $\mathrm{d}f(x)$ (using basis $\partial_u|_x, \partial_v|_x, \partial_w|_x$ in $\mathrm{T}_x(\mathbf{R}^3)$) is $(u(x)\ v(x)\ w(x))$, and so

$$\mathrm{T}_x(S(x)) = \{a\partial_u|_x + b\partial_v|_x + c\partial_w|_x \in \mathrm{T}_x(X) \,|\, u(x)a + v(x)b + w(x)c = 0\}.$$

Using the trivializing frame $\{\partial_u, \partial_v, \partial_w\}$ of TX, for x in the open locus $W = \{w \neq 0\}$ in X we may solve the equation

$$u(x)a + v(x)b + w(x)c = 0$$

for c in terms of a and b, so by taking the pairs (a, b) of "free parameters" to be $(1, 0)$ and $(0, 1)$ we arrive at a "universal formula" for a basis of $\mathrm{T}_x(S(x)) \subseteq \mathrm{T}_x(X) = \mathbf{R}^3$ for $x \in W$, namely

$$s_1(x) = \partial_u|_x - (u(x)/w(x))\partial_w|_x, \;\; s_2(x) = \partial_v|_x - (v(x)/w(x))\partial_w|_x.$$

That is, over the open subset W the C^∞ sections

$$s_1 = \partial_u - (u/w)\partial_w, \;\; s_2 = \partial_v - (v/w)\partial_w$$

in $(TX)(W)$ have two properties: they are part of a trivializing frame for $TX|_W$ ($\{s_1, s_2, \partial_w|_W\}$) *and* fiberwise they give a basis for $\mathrm{T}_x(S(x))$ for all $x \in W$.

Though s_1 and s_2 do *not* extend to $(TX)(X)$ (since u/w and v/w "blow up" as we approach the boundary of W in X, whereas $\{\partial_u, \partial_v, \partial_w\}$ is a global frame for TX), over the other two opens $U = \{u \neq 0\}$ and $V = \{v \neq 0\}$ in X we may likewise find trivializing frames for $(TX)|_U$ and $(TX)|_V$ whose first two members give a fiberwise basis for $\mathrm{T}_x(S(x))$ at each point $x \in U$ and $x \in V$ respectively. Note that if $x_0 \in X$ lies in two or more of the opens U, V, and W, then the above procedures give different "families of bases" of the fiber spaces $\mathrm{T}_x(S(x))$ for x near x_0. The key point is this: there is no "smooth formula" for a trivializing frame of the family of planes $\mathrm{T}_x(S(x))$ inside of $\mathbf{R}^3$ as x varies across *all* of X. Indeed, even over the points x on a single sphere S in $\mathbf{R}^3$ centered at the origin these $\mathrm{T}_x(S)$'s are exactly the fibers of $TS \to S$ inside of the pullback of $T(\mathbf{R}^3)$ along $S \hookrightarrow \mathbf{R}^3$, and we know that the tangent bundle TS to a 2-sphere S has *no* non-vanishing continuous section over S (hairy ball theorem), let alone a continuous family of frames.

To summarize, it is geometrically appealing to view the planes $\mathrm{T}_x(S(x)) \subseteq \mathrm{T}_x(X) \simeq \mathbf{R}^3$ as a "nicely varying family" parameterized by $x \in X$, and one feature of the niceness is that *locally* on X (namely, over each of the opens U, V, W that cover X) we *can* find a local frame for TX containing a subset whose specialization in the fiber $(TX)(x) = \mathrm{T}_x(X)$ at each point x in the domain of the frame is a basis of the subspace $\mathrm{T}_x(S(x))$. This is a sort of "local triviality" condition for the planes $\mathrm{T}_x(S(x))$ inside of the fibers $\mathrm{T}_x(X)$ of the vector bundle TX. We therefore wish to say that the collection of $\mathrm{T}_x(S(x))$'s is a "subbundle" of the vector bundle TX. The novelty is that this "subbundle" cannot be expected to exhibit the global triviality that is satisfied by TX but rather seems to only be trivializable over some opens that cover X.

The preceding two examples capture the essential idea for what a C^p subbundle of a C^p vector bundle $E \to X$ should be: a C^p-varying family of subspaces of the $E(x)$'s that is locally "trivialized" over X using subsets of local trivializing frames for $E \to X$. This is analogous to the fact from linear algebra that for any subspace of a finite-dimensional vector space we can find a basis of the subspace that extends to a basis of the entire vector space.

We conclude this preliminary discussion with a natural example of how the notion of subbundle (still to be defined!) arises in practice. Let $f : E' \to E$ be a C^p vector bundle morphism between C^p vector bundles over a C^p premanifold with corners X. For each $x \in X$, we get a subspace $\ker(f|_x) \subseteq E'(x)$ for the fiber map $f|_x : E'(x) \to E(x)$. For example, if $h : X = \mathbf{R}^3 - \{0\} \to \mathbf{R}$ is the map $h(u, v, w) = u^2 + v^2 + w^2$ then $\mathrm{d}h : TX \to T\mathbf{R}$ over h gives rise to a bundle mapping $f : E' = TX \to h^*(T\mathbf{R}) = E$ over X that on fibers is the mapping $\mathrm{d}h(x) : E'(x) = \mathrm{T}_x(X) \to \mathrm{T}_{h(x)}(\mathbf{R}) = E(x)$ whose kernel is the tangent plane $\mathrm{T}_x(S(x))$ to the standard sphere in $\mathbf{R}^3$ centered at the origin and passing through x. In other words, the study of the family of fibral kernels $\ker(f|_x) \subseteq E'(x)$ is a generalization of Example 1.2.

Let us prove a general theorem on kernels of varying families of matrices, as this will point the way to a good notion of "subbundle" that gives the right bundle generalization of the notion of subspace of a vector space.

Theorem 1.3. *Let $f : E' \to E$ be a C^p vector bundle map between C^p vector bundles over a C^p premanifold with corners X. The function $x \mapsto \dim(\ker f|_x)$ is locally constant if and only if there is a covering of X by opens U_i such that each $E'|_{U_i}$ admits a trivializing frame containing a subset whose specialization in the fiber over each point $x \in U_i$ is a basis of $\ker(f|_x)$.*

The local constancy condition on the dimensions of the kernels is crucial. In Example 1.2 if we had worked with $\mathbf{R}^3$ rather than $\mathbf{R}^3 - \{0\}$ (using the bundle mapping $T(\mathbf{R}^3) \to \widetilde{h}^*(T\mathbf{R})$ induced by the tangent mapping of $\widetilde{h} : (u, v, w) \mapsto u^2 + v^2 + w^2$ on $\mathbf{R}^3$) then at $x = 0$ the fiber map would be

0 and so the dimension of the kernel on that fiber would "jump" from 2 (the dimension at nearby fibers) to 3 (the dimension in the 0-fiber). In case of such dimension jumping at special points, one cannot expect the fibral kernels to arise as the fibers of a vector bundle (since the dimension of the fibers of a vector bundle is locally constant on the base).

Proof. The "if" direction is obvious (why?), so now we assume local constancy for the dimension of the fiberwise kernels and we seek to find the asserted local frames for E'. The problem is local on X, so we can assume E' and E are trivial and that the kernels of the maps $f|_x$ have a common dimension d. Let $\{s'_i\}$ and $\{s_j\}$ in $E'(X)$ and $E(X)$ be trivializing frames with $1 \leq i \leq n'$ and $1 \leq j \leq n$, so $r = n' - d$ is the common rank of the maps $f|_x$. Define $a_{ij} \in \mathscr{O}(X)$ by the condition $f(s'_j) = \sum_i a_{ij} s_i$. For any $x \in X$ some $r \times r$ submatrix in $(a_{ij}(x))$ is invertible, say for i and j running through r-tuples of indices I and J, and so by continuity for x' near x the "same" submatrix $(a_{ij}(x'))_{(i,j)\in I \times J}$ has non-vanishing determinant and so is invertible. Hence, we can cover X by opens U_α so that over U_α there are r-tuples of indices I_α and J_α such that the $r \times r$ submatrix $(a_{ij})_{(i,j)\in I_\alpha \times J_\alpha}$ is invertible over U_α. Since the theorem is a local problem over X, we may work separately over the U_α and so upon renaming each as X we may assume there is some $r \times r$ submatrix of (a_{ij}) that is invertible over X. By rearranging the indices, we can assume for convenience of notation that the *upper left* $r \times r$ submatrix (i.e., $1 \leq i, j \leq r$) that is invertible, though there could well be other invertible $r \times r$ submatrices (we simply have to *choose* one). In terms of the setup before we localized on X, this rearrangement of indices over each U_α will usually depend on U_α. (Compare with the considerations over U, V, W in Example 1.2, where we were really picking an invertible 1×1 submatrix in a 1×3 matrix over each of three opens that cover $X = \mathbf{R}^3 - \{0\}$.)

Since $(a_{ij}(x))$ has rank *exactly* r for all $x \in X$ and the submatrix for $1 \leq i, j \leq r$ is invertible for *all* $x \in X$, the first r columns are linearly independent. Hence, the image of this matrix is spanned by the first j columns. In coordinate-free language, this says that for all $x \in X$ and all $j > r$ the vector $(f(s'_j))(x) = f|_x(s'_j(x)) \in E(x)$ is a unique linear combination of $(f(s'_1))(x), \dots (f(s'_r))(x)$. That is, for each $j > r$ if we consider the equation

$$(f(s'_j))(x) = \sum_{k=1}^{r} c_{kj} \cdot (f(s'_k))(x)$$

in $E(x)$ for r unknowns $c_{1j}, \dots, c_{rj}$ (with j fixed in the range $r < j \leq n' = r + d$), then this is a "universal" (typically overdetermined!) system of n' linear equations

$$a_{ij}(x) = \sum_{k=1}^{r} a_{ik}(x) c_{kj} \tag{1}$$

upon expanding the $f(s'_k)$'s and $f(s'_j)$ in the frame of the s_i's for $E(X)$ (here, $1 \leq i \leq n'$). But in our situation we know *a priori* that the system of equations (1) for $1 \leq i \leq n'$ and a fixed j satisfying $r < j \leq n' = r + d$ has a unique solution. Since $(a_{ij}(x))_{1 \leq i,j \leq r}$ is invertible, the subsystem of r equations for $1 \leq i \leq r$ has invertible coefficient matrix. We can therefore use this subsystem to uniquely solve for each c_{kj} via Cramer's formula, and we know *automatically* that such $c_{1j}, \dots, c_{rj}$ will have to satisfy all n' equations in (1) for $1 \leq i \leq n'$ (as we know in advance that there is a unique solution to the entire overdetermined system).

One consequence of the solution via Cramer's formula is that each $c_{kj} = c_{kj}(x)$ is a *rational* function of the $a_{ij}(x)$'s for $1 \leq i \leq r$ with denominator that is a determinant polynomial non-vanishing on X. In particular, each c_{kj} for $1 \leq k \leq r$ and $r + 1 \leq j \leq n'$ is a C^p function on X.

Hence, we get d sections

$$v_j = s'_{j+r} - \sum_{k=1}^{r} c_{k,j+r} s'_k \in E'(X) \tag{2}$$

with $1 \le j \le d$ and $c_{k,j+r} \in \mathscr{O}(X)$ such that $v_j(x) \in \ker(f|_x)$ for all $x \in X$. By inspection the d vectors $v_j(x) \in E'(x)$ are linearily independent (as $v_j(x)$ involves $s'_{j+r}(x)$ but $v_{j'}(x)$ does not for $j' \neq j$) and they lie in the d-dimensional $\ker(f|_x)$, so they must be a basis of this kernel. (This argument is a generalization of what we did in Example 1.2.)

Consider the n' sections $s'_1, \dots, s'_r, v_1, \dots, v_d \in E'(X)$. For each $x \in X$, the vectors $v_j(x) \in E'(x)$ are a basis of the kernel of $f|_x$ and the vectors $f|_x(s'_1(x)), \dots, f|_x(s'_r(x))$ are a basis of the image of $f|_x$, whence these n' vectors in $E'(x)$ are a basis. Hence, we have built a new frame for E' such that d of the members of the frame (namely, $v_1, \dots, v_d$ in (2)) give a fiberwise basis of the kernels of the maps $f|_x$. ■

The preceding considerations suggest a couple of different ways to define the notion of subbundle of a vector bundle. We begin with the most naive definition, and it will later be shown to admit equivalent reformulations in terms of local frames.

Definition 1.4. Let X be a C^p premanifold with corners. Let E be a C^p vector bundle over X. A C^p *subbundle* is a morphism $i : E' \to E$ of C^p vector bundles over X such that $i|_x : E'(x) \to E(x)$ is injective for all $x \in X$.

Applying pullback along a C^p map $f : X' \to X$ carries subbundles to subbundles: if $i : E' \to E$ over X is fiberwise injective, so is $f^*(i) : f^*E' \to f^*E$ over X'. (For the mapping $f^*(i)$, see Corollary 2.5 in the handout on pullback bundles.)

Example 1.5. Let $h : Y \hookrightarrow X$ be a C^p immersion between C^p premanifolds, with $p \ge 1$. The tangent mapping $\mathrm{d}h : TY \to TX$ over $h : Y \to X$ uniquely factors through a C^{p-1} vector bundle mapping $i : TY \to h^*(TX)$ as C^{p-1} vector bundles over Y. On fibers over $y \in Y$ this is the map $\mathrm{d}i(y) : \mathrm{T}_y(Y) \to \mathrm{T}_{h(y)}(X)$ that is injective since h is an immersion. Hence, i exhibits TY as a C^{p-1} subbundle of $h^*(TX)$ in the sense of the preceding definition (viewing the base Y as of class C^{p-1}). A special case of this is Example 1.1.

The reason we prefer to view subbundles as fiberwise injective bundle maps rather than as literal subsets of the target is because unlike in linear algebra we have to keep track of more than just the fibral linear structure, namely the differentiable structure, and so for such purposes it is best to keep the spaces E' and E "separate" from each other. An exception can be made in the case when there is no boundary or corner on X (and hence none on vector bundles over X) because in such cases we have the local immersion theorem available as a tool to tell us that certain subsets have a *unique* differentiable structure compatible with that on the ambient space. It will be essential in Stokes' theorem that we work with manifolds with boundary, and so we cannot restrict our attention to just the case of premanifolds. Hence, after we sort out special features of the case of subbundles of vector bundles over premanifolds we will return to the general setting of premanifolds with corners.

2. Properties of subbundles

We first wish to show that if X is a C^p premanifold (no corners or boundary), so there is a good notion of C^p embedding and all vector bundles over X also have no corners or boundary, then for a C^p subbundle $i : E' \to E$ the fiberwise images $i(E'(x)) \subseteq E(x)$ for $x \in X$ (i.e., the physical image $i(E') \subseteq E$) do determine the pair (E', i) up to unique C^p vector bundle isomorphism over X. This is analogous to "identifying" a linear injection with its image inside of the target vector space.

Lemma 2.1. *Let X be a C^p premanifold with corners. Let $i : E' \to E$ be a C^p subbundle of a C^p vector bundle E over X. The map i is a C^p closed immersion: it is topologically a closed embeddings, and as a map of premanifolds with corners it has injective tangent mappings. Moreover, if $f : E'_1 \to E$ is a bundle mapping with $f(E'_1) \subseteq i(E)$ then there is a unique C^p bundle mapping $\phi : E'_1 \simeq E'$ over X such that $i \circ \phi = f$, and if f is fiberwise injective with $f(E'_1) = i(E')$ then ϕ is a C^p bundle isomorphism. In particular, the subset $i(E') \subseteq E$ uniquely determines the pair (E', i) up to unique C^p isomorphism.*

The lemma suggests a natural question for C^p premanifolds with corners X: if $\Sigma \subseteq E$ is a subset such that $\Sigma \cap E(x)$ is a subspace of $E(x)$ for all $x \in X$, is Σ the union of the set of fibers of a C^p subbundle of E (necessarily unique up to unique C^p bundle isomorphism, by Lemma 2.1)? The necessary and sufficient conditions for an affirmative answer will be given in Theorem 2.5 in terms of local frames.

Proof. Since i is a mapping over X, the property of it being a closed immersion is a local problem over X. Hence, by working locally we may suppose E' and E are both trivial, say with trivializing frames $\{s'_k\}$ and $\{s_j\}$. By hypothesis, $\{i|_x(s'_k(x))\}$ is a linearly independent set in $E(x)$ for all $x \in X$. That is,

$$i|_x(s'_k(x)) = \sum a_{jk}(x) s_j(x)$$

with $a_{jk} \in \mathscr{O}(X)$ and the $n \times n'$ matrix $(a_{jk}(x))$ having rank n' for all $x \in X$. Hence, for each x there exists an invertible $n' \times n'$ submmatrix, so by working locally on X and rearranging the indices (as in the proof of Theorem 1.3) we may arrange that the left $n' \times n'$ block $(a_{jk})_{1 \le j,k \le n'}$ is invertible over all of X.

Consider the ordered set of n vectors

$$\Sigma_x = \{i|_x(s'_1(x)), \dots, i|_x(s'_{n'}(x)), s_{n'+1}(x), \dots, s_n(x)\}$$

in the n-dimensional $E(x)$. The matrix of coefficients for this ordered set in terms of the frame of $s_j(x)$'s is an $n \times n$ matrix whose upper left $n' \times n'$ block is invertible, upper right $n' \times (n - n')$ block is 0, and lower right $(n - n') \times (n - n')$ block is the identity matrix. Thus, this matrix is invertible and so Σ_x is a basis of $E(x)$ for all $x \in X$. That is,

$$\Sigma = \{i(s'_1), \dots, i(s'_{n'}), s_{n'+1}, \dots, s_n\} \subseteq E(X)$$

is a trivializing frame for E. (Keep in mind that to get to this step we have already localized quite a bit on the original X, and also rearranged the original s_j's in the process.) In terms of the resulting bundle isomorphisms

$$E' \simeq X \times \mathbf{R}^{n'}, \ \ E \simeq X \times \mathbf{R}^n$$

associated to the frames $\{s'_k\}$ and Σ of E' and E, the composite C^p bundle map

$$X \times \mathbf{R}^{n'} \simeq E' \xrightarrow{i} E \simeq X \times \mathbf{R}^n$$

is the standard inclusion

$$(x, (a_1, \dots, a_{n'})) \mapsto (x, (a_1, \dots, a_{n'}, 0, \dots, 0)).$$

That is, it is the standard map $X \times \mathbf{R}^{n'} \to (X \times \mathbf{R}^{n'}) \times \mathbf{R}^{n-n'}$ defined by setting the last $n - n'$ coordinates in $\mathbf{R}^{n-n'}$ equal to 0. This completes the proof that i is a closed immersion.

Now suppose $f : E'_1 \to E$ is a bundle mapping with $f(E'_1) \subseteq i(E')$. We want to show that f uniquely factors through i via a bundle mapping $\phi : E'_1 \to E'$ and that ϕ is an isomorphism if f is fiberwise injective with $f(E'_1) = i(E')$. Once we build ϕ uniquely as a bundle mapping, then in the case that f is fiberwise injective with $f(E'_1) = i(E')$ it follows that $\phi : E'_1 \to E'$ is a bundle mapping that is a bijective on fibers, hence a linear isomorphism on fibers, and so ϕ is a C^p bundle isomorphism in such cases. The uniqueness of ϕ in general follows from the fiberwise injectivity of i, and certainly ϕ exists set-theoretically as a map $\phi : E'_1 \to E'$ over X that is linear on fibers. The only problem is to prove that this map is C^p, and this problem is local over X. As we have seen above, by working locally over X we can assume that there are trivializations $E' \simeq X \times \mathbf{R}^{n'}$ and $E \simeq X \times \mathbf{R}^n$ with $n' \leq n$ under which i is carried to the bundle mapping $X \times \mathbf{R}^{n'} \to X \times \mathbf{R}^n$ given by the standard inclusion of $\mathbf{R}^{n'}$ into the first n' coordinates of $\mathbf{R}^n$ on all fibers. Hence, the problem for proving ϕ is C^p becomes the problem of showing that a map $E'_1 \to X \times \mathbf{R}^n$ over X whose last $n - n'$ component maps are 0 has C^p factorization through the mapping $X \times \mathbf{R}^{n'} \to X \times \mathbf{R}^n$ given by the identity on X and the standard inclusion of $\mathbf{R}^{n'}$ into the first n' coordinates of $\mathbf{R}^n$. This follows from the universal properties of products and the fact that any Euclidean space $\mathbf{R}^N$ with its standard C^p structure is indeed a product of N copies of $\mathbf{R}$ as a C^p premanifold with corners. ■

The local picture in the lemma is quite attractive: a subbundle locally looks like a standard inclusion $X \times \mathbf{R}^{n'} \to X \times \mathbf{R}^n$ by appending 0's in the last $n - n'$ coordinates. Here is a coordinate-free formulation, essentially saying that subbundles admit local bundle complements (just as for any subspace V' in a finite-dimensional vector space V, we can find a subspace $V'' \subseteq V$ that is complementary to V' in the sense that the natural linear "addition" map $V' \oplus V'' \to V$ is an isomorphism).

Theorem 2.2. *A map of C^p vector bundles $f : E' \to E$ over X is a subbundle if and only if there is a covering of X by opens U_i such that $E'|_{U_i}$ is a direct summand of $E|_{U_i}$ as vector bundles over U_i: there exists a C^p vector bundle E''_i over U_i and a C^p vector bundle isomorphism*

$$\phi_i : E'|_{U_i} \oplus E''_i \simeq E|_{U_i}$$

over U_i such that $f|_{U_i}$ is the composite $\phi_i \circ j_i$ with $j_i : E'|_{U_i} \to E'|_{U_i} \oplus E''_i$ defined by $(\mathrm{id}, 0)$.

Proof. Working locally as in the proof of Lemma 2.1, we have seen that if f is a subbundle then upon shrinking X we can choose local frames so that there are isomorphisms $E' \simeq X \times \mathbf{R}^{n'}$ and $E \simeq X \times \mathbf{R}^n$ such that f is identified with the standard inclusion $X \times \mathbf{R}^{n'} \to X \times \mathbf{R}^n$ by appending 0's in the last $n - n'$ coordinates. Define $E'' = X \times \mathbf{R}^{n-n'}$ and define the isomorphism $\phi : E' \oplus E'' \simeq E$ by

$$E' \oplus E'' \simeq (X \times \mathbf{R}^{n'}) \oplus (X \times \mathbf{R}^{n-n'}) \simeq X \times (\mathbf{R}^n \oplus \mathbf{R}^{n-n'}) = X \times \mathbf{R}^n \simeq E.$$

This has the desired relationship with f, and so settles one direction of the theorem. Conversely, if the local direct summand condition holds then f is trivially injective on fibers and so it is a subbundle. ■

The following corollary shows that C^p subbundles behave very much like linear subspaces of vector spaces in terms of factorization of maps:

Corollary 2.3. *If $f : E' \to E$ is a C^p subbundle and $h : V \to E$ is a C^p bundle morphism such that on fibers $V(x) \to E(x)$ factors through the subspace $E'(x) \subseteq E(x)$ for all $x \in X$ then h uniquely factors through f via a C^p bundle morphism $V \to E'$ over X.*

Proof. The set-theoretic factorization exists and is unique (and respects projection to X and linear structure on the fibers), so the only problem is to prove that it is a C^p map. This is local over X, so by Theorem 2.2 we can shrink X to get to the case when there is a C^p bundle isomorphism $\phi : E' \oplus E'' \simeq E$ satisfying $\phi|_x(s', 0) = f|_x(s')$ for all $x \in X$ and $s' \in E'(x)$, and we can study $\phi^{-1} \circ h$ rather than h. That is, we may assume $E = E' \oplus E''$ with f the standard inclusion $E' \to E' \oplus E''$ via the 0-map to E'' and the identity map to E'. The assumption on h says that the composite mapping of bundles $V \to E' \oplus E'' \to E''$ is the zero map on fibers and hence is 0. Thus, if we let $h' : V \to E'$ be the composite of h and the "projection" $E' \oplus E'' \to E'$ then $h|_x = (h'|_x, 0)$ as maps from $V(x)$ to $E'(x) \oplus E''(x)$ for all $x \in X$. That is, $h = (h', 0) = j \circ h'$ is a factorization of h through the standard inclusion $j : E' \to E' \oplus E''$. ■

Corollary 2.4. *If $f : E' \to E$ is a subbundle then every local frame for E' locally extends to a local frame for E. That is, for any open set $U \subseteq X$ and trivialization $\{s'_j\}$ of $E'|_U$ there is a covering of U by opens U_α such that the restrictions $f(s'_j)|_{U_\alpha} \in E(U_\alpha)$ are a subset of a frame for $E|_{U_\alpha}$.*

Proof. The problem is local on X, so we may assume $U = X$ and (by Theorem 2.2) that there is a C^p isomorphism of bundles $E' \oplus E'' \simeq E$ such that the composite

$$E' \overset{(\mathrm{id},0)}{\to} E' \oplus E'' \simeq E$$

is f. There is a covering of X by opens U_α such that $E''|_{U_\alpha}$ admits a frame $\{s''_k\}$. Hence, upon renaming U_α as X, the sections $(s'_j, 0)$ and $(0, s''_k)$ of $E'(X) \oplus E''(X) = (E' \oplus E'')(X)$ for all j and k give a trivialization of $E' \oplus E''$ because they give a basis on all fibers. Their image in $E(X)$ therefore gives a trivialization of E. But the direct summand isomorphism was rigged to carry a section $(s', 0)$ to $f(s')$, and so the $f(s'_j)$'s are a subset of a trivializing frame for E. ■

We now answer the question: what ways of selecting subspaces of fibers of a vector bundle gives a subbundle? By Lemma 2.1, this set-theoretic question is most reasonable (in the sense of having an answer that is unique up to unique isomorphism) when X is a premanifold with corners. The answer is a mixture of manifold conditions and linear algebra in the fibers:

Theorem 2.5. *Assume X is a C^p premanifold with corners and let $E \to X$ be a C^p vector bundle. Let $Z \hookrightarrow E$ be a C^p closed immersion from a C^p premanifold with corners Z such that $Z \cap E(x) \subseteq E(x)$ is a linear subspace whose dimension is locally constant as a function of $x \in X$. Also assume that locally near each $z \in Z$, the map $Z \to E$ splits as a slice-inclusion by a factor of a product decomposition. Finally, assume that for every $z \in Z$ there is a local C^p section to π around $\pi(z) \in X$ carrying $\pi(z)$ back to z.*

This Z admits a unique structure of a C^p vector bundle over X such that the given mapping $Z \to E$ makes it a subbundle of E over X. If X has no corners then the assumption on the existence of local C^p-sections to $Z \to X$ through all points of Z is automatically satisfied.

In view of the preceding results, the conditions on Z are certainly necessary in order that Z be a C^p subbundle of E over X. Also, by the immersion theorem, if X has no corners then the local hypothesis on the structure of the map $Z \to E$ is equivalent to $Z \to E$ being an immersion.

Proof. The given conditions provide a linear structure on the fibers of $\pi : Z \to X$ such that the C^p map $Z \to E$ is a linear injection on fibers, so the only problem is to show that with its fibral linear structure the map $Z \to X$ satisfies the local C^p triviality condition in the definition of C^p vector bundle. This problem is local on X, so we may assume E has constant rank n and (by the local constancy hypothesis) that $Z \cap E(x)$ has a common dimension d for all $x \in X$. We next check that if X has no corners then $Z \to X$ has local C^p-sections through all points of Z. The zero-section $X \to E$ lands inside of Z set-theoretically (since $0 \in E(x)$ is contained in $Z \cap E(x)$ for all $x \in X$), so by the immersion theorem (no corners!) the zero section $X \to E$ uniquely factors through $Z \to E$ as a C^p *map* $0 : X \to Z$. Hence, there is a composite of C^p maps

$$X \overset{0}{\to} Z \overset{\pi}{\to} X$$

that is the identity, so by the Chain Rule the tangent maps for $\pi : Z \to X$ are surjective. By the submersion theorem (no corners!), it follows that for every $z \in Z$ there is a local C^p section to π around $\pi(z) \in X$ carrying $\pi(z)$ back to z. Hence, if we choose any $x_0 \in X$ and pick a basis $v_1, \ldots, v_d$ of $Z \cap E(x_0) \subseteq E(x_0)$ then by shrinking X around x_0 we can find C^p sections $s_1, \ldots, s_d$ of $Z \to X$ such that $s_j(x_0) = v_j$ for $1 \leq j \leq d$.

Choose $x_0 \in X$. We aim to prove the local triviality condition for $\pi : Z \to X$ over an open neighborhood of $x_0 \in X$. Shrinking X around x_0 so that E has a trivialization, consider the resulting coordinates for the d linearly independent vectors $s_j(x)$ with respect to the x-fibers of the chosen frame of E. These form the columns of an $n \times d$ matrix whose value at x_0 has d independent columns (as the vectors $s_j(x_0) = v_j \in E(x_0)$ are linearly independent by construction of the v_j's). Thus, by equality of row rank and column rank for matrices there are d independent rows at x_0 and hence there is a $d \times d$ submatrix that is invertible at x_0 and therefore invertible at x near x_0. By shrinking X around x_0, we may thereby arrange that the d vectors $s_j(x)$ in the d-dimensional subspace $Z \cap E(x) \subseteq E(x)$ are linearly independent and hence a basis of $Z \cap E(x)$ for *all* $x \in X$.

Consider the bundle morphism $\phi : X \times \mathbf{R}^d \to E$ over X defined by the s_j's. This map is fiberwise injective with image Z because it carries each x-fiber $\mathbf{R}^d$ to a linear subspace of $E(x)$ spanned by the *basis* $s_j(x)$ of $Z \cap E(x)$. In particular, $X \times \mathbf{R}^d \to E$ is a subbundle, whence it is a topological embedding and injective on tangent spaces (by Lemma 2.1). But $Z \to E$ is another C^p map with the same properties and the same image subset inside of E. It follows that the map $\phi : X \times \mathbf{R}^d \to E$ and the given C^p inclusion $Z \to E$ each factor uniquely through each other as C^p *maps* (the C^p property holds because both maps locally can be described as slice-inclusions of a factor of a product space), and the resulting C^p maps $X \times \mathbf{R}^d \to Z$ and $Z \to X \times \mathbf{R}^d$ are inverse to each other (and linear on fibers). Thus, we have a C^p isomorphism $X \times \mathbf{R}^d \simeq Z$ over X respecting the linear structure on fibers; this is the desired local trivialization. ■

The last result we prove concerning subbundles is a criterion for the fiberwise kernels of a bundle mapping to "be" a subbundle of the source bundle. This was hinted at in Theorem 1.3.

Theorem 2.6. *Let $f : E' \to E$ be a C^p bundle morphism over X. The following conditions are equivalent:*

1. *the function $x \mapsto \dim(\ker f|_x)$ is locally constant on x,*
2. *there is a subbundle $E'_0 \to E'$ such that the image of $E'_0(x)$ in $E(x)$ is $\ker f|_x$ for each $x \in X$.*

When these conditions hold, the $\mathscr{O}(U)$-linear map $E'_0(U) \to E'(U)$ is an isomorphism onto $\ker \underline{f}_U = \ker(E'(U) \to E(U)) \subseteq E'(U)$ *for every open set $U \subseteq X$.*

When the conditions of the theorem are satisfied, then the $\mathscr{O}$-module $U \mapsto \ker \underline{f}_U$ is locally free of finite rank because it is $\mathscr{O}$-linearly isomorphic to $\underline{E'_0}$. We then call the subbundle E'_0 the *kernel* of f and denote it $\ker f$ (so $(\ker f)(x) = \ker(f|_x)$ for all $x \in X$). If the dimensions of the kernels $\ker(f|_x)$ are *not* locally constant in x then we do *not* speak of the kernel of f because there is no good way to make sense of it as a vector bundle. (It is always possible to make sense of a good notion of kernel by working with $\mathscr{O}$-modules not necessarily arising from vector bundles, but that goes beyond the level of the course and so we won't discuss it any further here.)

Proof. The implication (2) $\Rightarrow$ (1) is trivial: $E'_0(x) \to E'(x)$ is an injection, so $\dim(\ker f|_x) = \dim E'_0(x)$ is locally constant in x. To prove that (1) implies (2), let $\mathscr{K}(U) = \ker \underline{f}_U \subseteq E'(U)$ for open $U \subseteq X$. We shall begin by proving that $\mathscr{K}$ is locally free of finite rank. This is a local question, so we may work locally over X to arrange that (by Theorem 1.3) E' is trivial with a frame $\{s'_1, \ldots, s'_{n'}\}$ whose first d elements give a fiberwise basis of $\ker(f|_x)$ for all $x \in X$. For open $U \subseteq X$, suppose $s' \in E'(U)$ maps to 0 in $E(U)$ (i.e., $s' \in \mathscr{K}(U)$). We may uniquely write $s' = \sum a_i s'_i|_U$ with $a_i \in \mathscr{O}(U)$. Since $s'(x) = \sum a_i(x) s'_i(x)$ lies in the span $\ker(f|_x)$ of the $s'_i(x)$'s for $i \leq d$ for all

$x \in U$, we have $a_i(x) = 0$ for all $x \in U$ and $i > d$. That is, $a_i = 0$ for all $i > d$. This shows that $\mathscr{K}(U) \subseteq E'(U)$ is the *free* $\mathscr{O}(U)$-submodule spanned by the $s'_i|_U$'s for $i \le d$. We have proved that the sections $s'_1, \dots, s'_d$ in $\mathscr{K}(X)$ have restriction in $\mathscr{K}(U)$ that is an $\mathscr{O}(U)$-module basis of $\mathscr{K}(U)$ for all opens $U \subseteq X$. In other words, $\mathscr{K}$ is locally free of finite rank.

Since $\mathscr{K}$ is locally free of finite rank, it arises from a unique C^p vector bundle E'_0 on X, and the mapping $\underline{E'_0} \simeq \mathscr{K} \to \underline{E'}$ of $\mathscr{O}$-modules arises from a unique map of C^p vector bundles $E'_0 \to E'$ over X. Using the isomorphism $\underline{E'_0}(x) \simeq E'_0(x)$ between fibers of $\mathscr{O}$-modules and vector bundles, the preceding local calculation in frames shows that E'_0 admits local frames ($\{s'_1, \dots, s'_d\}$ in $\mathscr{K}(X) = E'_0(X)$ in the preceding notation, at least after working *locally on* X) whose values in each fiber $E'_0(x)$ map to a basis of $\ker(f|_x) \subseteq E'(x)$ for each x in small opens that cover X. Hence, the fiber map $E'_0(x) \to E'(x)$ is injective with image $\ker f|_x$ for all $x \in X$. ■

Example 2.7. Beware that if the $\mathscr{O}$-module $U \mapsto \ker \underline{f}_U$ is locally free of finite rank, it does *not* necessarily happen that the conditions of the preceding theorem are satisfied. For example, let $X = \mathbf{R}$ with standard coordinate t and let $E' = E = X \times \mathbf{R}$ be trivial line bundles (with associated $\mathscr{O}$-module $\mathscr{O}$). Let $f : E' \to E$ be the map $(x, c) \mapsto (x, cx)$ that corresponds to the $\mathscr{O}$-module map $\underline{f} : \mathscr{O} \to \mathscr{O}$ given by multiplication by $t|_U$. That is, $\underline{f}|_U : \mathscr{O}(U) \to \mathscr{O}(U)$ is exactly $h \mapsto th$. This is certainly injective for all U, even if U contains the origin (since a continuous function h vanishing on a punctured neighborhood of the origin must also vanish at the origin by continuity). Hence, $\ker \underline{f}_U = 0$ for all open U; this is certainly locally free of finite (vanishing!) rank. However, $\ker(f|_x)$ has a dimension-jump at the origin: its dimension is 1 at $x = 0$ and is 0 for $x \neq 0$.

Example 2.8. There is a very important class of C^p bundle maps $f : E' \to E$ for which the criteria in Theorem 2.6 are satisfied: those f such that $f|_x : E'(x) \to E(x)$ is surjective for all $x \in X$. Indeed, in such cases $\dim \ker(f|_x) = \dim E'(x) - \dim E(x)$ is a difference of locally constant functions and hence is locally constant.

3. Bundle quotients

In the preceding section we made an exhaustive study of the bundle analogue of a linear injection. We now take up the analogue of a linear surjection and a quotient by a subbundle.

Definition 3.1. A C^p *bundle surjection* is a C^p bundle map $f : E' \to E$ such that $f|_x : E'(x) \to E(x)$ is surjective for all $x \in X$.

Recall that in linear algebra, a linear map between finite-dimensional vector space is injective (resp. surjective) if and only if its dual map is surjective (resp. injective). The same holds for vector bundles: a C^p bundle mapping $f : E' \to E$ with dual map $f^\vee : E^\vee \to E'^\vee$ (inducing $(f|_x)^\vee : E(x)^\vee \to E'(x)^\vee$ on x-fibers for all $x \in X$) has the property that f is a bundle surjection (resp. subbundle) if and only if $f^\vee$ is a subbundle (resp. bundle surjection). Indeed, these conditions are fibral by definition, so the old result from linear algebra gives what we need.

Remark 3.2. It is natural to ask if a bundle surjection $f : E' \to E$ induces a surjection $E'(U) \to E(U)$ for all opens $U \subseteq X$. A moment's reflection shows that this is *not* obvious, but in fact the answer is affirmative for manifolds with corners. The proof requires partitions of unity, as we shall see in Theorem 3.6. (Beware, however, that such surjectivity on U-sections is generally *false* when working in the complex-analytic case.)

The first order of business is to prove that bundle surjections satisfy some of the nice mapping properties of linear surjections in linear algebra. Let $f : E' \to E$ be a C^p bundle surjection. By Example 2.8, there is a well-defined subbundle $E'_0 = \ker f$ inside of E', and since $f|_x$ is surjective it induces a linear isomorphism $E'(x)/E'_0(x) \simeq E(x)$ for all $x \in X$. In view of the universal mapping property of quotients in linear algebra, the follows result is appealing:

Theorem 3.3. *For a C^p vector bundle surjection $f : E' \to E$, if $h : E' \to E''$ is a C^p bundle mapping such that h kills the subbundle* $\ker f$ *(or equivalently, $h|_x : E'(x) \to E''(x)$ kills* $\ker(f|_x) \subseteq E'(x)$ *for all $x \in X$) then h uniquely factors through f via a C^p bundle mapping $\overline{h} : E \to E''$.*

Proof. Set-theoretically (working on fibers over X) we get the map $\overline{h}$ over X and it is linear on fibers. The problem is to prove that it is C^p, so we can work locally on X. By Theorem 2.2, we may therefore suppose (by working locally over X) that the subbundle $E'_0 = \ker f$ is a direct summand of E'. That is, we may assume there is a C^p bundle isomorphism $g : E'_0 \oplus E'_1 \simeq E'$ identifying the subbundle inclusion $E'_0 \to E'$ with the standard inclusion of E'_0 into $E'_0 \oplus E'_1$. It is harmless to replace h with $h \circ g$, so we may suppose $E' = E'_0 \oplus E'_1$ with $f : E'_0 \to E'$ the standard inclusion. Using the standard C^p inclusion $E'_1 \to E'_0 \oplus E'_1 = E'$, the composite map $\phi : E'_1 \to E' \stackrel{f}{\to} E$ induces an isomorphism on fibers (since $E'_0(x) = \ker(f|_x)$, and $f|_x$ is surjective for all $x \in X$), whence it is a C^p bundle isomorphism.

Define $\overline{h}$ to be the composite C^p bundle map

$$E \stackrel{\phi^{-1}}{\simeq} E'_1 \to E'_0 \oplus E'_1 = E' \stackrel{h}{\to} E''.$$

Passing to fibers over X, this is seen to work: $\overline{h} \circ f = h$. ■

The method of proof of Theorem 3.3 shows that bundle surjections have a nice characterization in terms of local frames, generalizing the fact from linear algebra that a basis of a quotient V/W lifts to part of a basis of V:

Theorem 3.4. *A C^p bundle map $f : E' \to E$ is a bundle surjection if and only if X is covered by opens on which E admits a local frame that lifts to part of a local frame of E'. In such cases, for any direct sum decomposition $E'|_U \simeq (\ker f)|_U \oplus E''$ as C^p vector bundles over an open $U \subseteq X$ (extending the inclusion of $(\ker f)|_U$ into $E'|_U$) the composite*

$$E'' \stackrel{(0,\mathrm{id})}{\to} (\ker f)|_U \oplus E'' \simeq E'|_U \to E|_U$$

is an isomorphism of C^p vector bundles, and so any local frame of E locally lifts to part of a local frame of E'.

When f in Theorem 3.4 is a bundle surjection, we get the subbundle $\ker f$ in E'. Thus, in such cases Theorem 2.2 ensures that the base space X admits a covering by opens U over which there exist direct summand decompositions as considered in the statement of the theorem.

Proof. If a local frame for $E|_U$ lifts to part of one for $E'|_U$ then since local frames give a basis on fibers over their domain it follows that $f|_x$ must be surjective for all $x \in U$. Thus, if such U's exist that cover X then $f|_x$ is surjective for all $x \in X$ (i.e., f is a bundle surjection). Conversely, assume f is a bundle surjection, so we do have a C^p subbundle $\ker f$ in E'. By Theorem 2.2, there exists an open covering $\{U_i\}$ of X and C^p vector bundles E''_i on U_i fitting in to direct sum decompositions $E'|_{U_i} \simeq (\ker f)|_{U_i} \oplus E''_i$ as C^p vector bundles (extending the inclusion of $(\ker f)|_{U_i}$ into $E'|_{U_i}$). The claim that the composite C^p bundle map $E''_i \to E|_{U_i}$ is an isomorphism may be checked on fibers over U_i, where it is obvious because $(\ker f)(x) = \ker(f|_x)$ inside of $E'(x)$ for all $x \in X$ since $E''(x) \subseteq E'(x)$ is a complement to the kernel of the linear *surjection* $f|_x : E'(x) \to E(x)$. ■

Theorem 3.3 gives a precise sense in which for a bundle surjection $f : E' \to E$, the data of E and the mapping f from E' to E are uniquely determined by E' and the subbundle $\ker f$ in E': those C^p bundle maps $E' \to E''$ that factor through f are exactly the ones that vanish on $\ker f$, in which case they *uniquely* factor through f. In other words, $f : E' \to E$ is the “simplest” C^p bundle map that kills the subbundle $\ker f$ in the sense that all others uniquely factors through it. This suggests we try to reverse the procedure: given a C^p subbundle E'_0 in a C^p vector bundle E', does there exist a C^p bundle surjection $f : E' \to E$ whose kernel subbundle $\ker f$ is equal to E'_0? (Analogue in linear algebra: for a subspace V'_0 in a finite-dimensional vector space V', does there exist a linear surjection $V' \to V$ with kernel V'_0? Answer: $V = V'/V'_0$ with the natural projection to it from V'.) If so, we would be justified in calling the pair $(E, f : E' \to E)$ the *quotient* of E' modulo the subbundle E'_0, and denoting it E'/E'_0 (equipped with the bundle surjection $E' \to E'/E'_0$). The discussion preceding Theorem 3.3 shows that in such a situation, the natural map $E'(x)/E'_0(x) \to (E'/E'_0)(x)$ must be an isomorphism. Does such a quotient by a subbundle always exist?

Theorem 3.5. *Let E'_0 be a C^p subbundle of a C^p vector bundle E'. The quotient E'/E'_0 exists.*

Proof. The idea is to use bundle duality to convert problems for quotients into problems for subbundles, and to use double duality to return to the original setup. Let $i : E'_0 \to E'$ be the subbundle inclusion, so the dual map $i^\vee : E'^\vee \to E'^\vee_0$ is a bundle surjection. Hence, it has a *subbundle* kernel $\ker(i^\vee)$. Let $j : \ker(i^\vee) \to E'^\vee$ be the natural inclusion map. Consider the dual vector bundle $E'' = (\ker i^\vee)^\vee$ equipped with the mapping

$$h : E' \simeq E'^{\vee\vee} \stackrel{j^\vee}{\to} E''.$$

By passing to fibers (which commutes with the kernel and duality operations we have used) and using the double duality theorem from linear algebra, it is seen that for all $x \in X$ the map $h|_x : E'(x) \to E''(x)$ is *surjective* with kernel $E'_0(x)$. Since h is a bundle surjection, we get a subbundle $\ker h$ inside of E' with x-fiber $E'_0(x)$ for all $x \in X$. By Corollary 2.3 the subbundle inclusions of $\ker h$ and E'_0 into E' each factor uniquely through each other (as they do so on all fibers over X), and the resulting C^p bundle maps $\ker h \to E'_0$ and $E'_0 \to \ker h$ are inverse isomorphisms because they are so on all fibers over X. In other words, the given subbundle E'_0 in E' is $\ker h$. The pair (E'', h) therefore satisfies the requirements to be a quotient of E' modulo E'_0. ∎

The next theorem address the question of surjectivity on U-sections for a bundle surjection:

Theorem 3.6. *Assuming that X is a C^p manifold with corners, if $f : E' \to E$ is a bundle surjection of C^p vector bundles then $E'(U) \to E(U)$ is surjective for all opens U in X.*

Note that we do impose the mild topological conditions that X be Hausdorff and second countable (though paracompactness would suffice in place of second countability), since we use partitions of unity in the proof.

Proof. We may rename U as X, so we pick $s \in E(X)$ and we want to lift it to $s' \in E'(X)$. By Theorem 3.4, we may cover X by opens U_i such that $E'|_{U_i}$ contains a subbundle E''_i mapping isomorphically to $E|_{U_i}$. In particular, there exists $s'_i \in E''_i(U_i) \subseteq E'(U_i)$ that lifts $s|_{U_i} \in E(U_i)$.

Let $\{\phi_\alpha\}$ be a C^p partition of unity subordinate to the covering $\{U_i\}$ of X, so the compact support K_α of ϕ_α in X is contained in some $U_{i(\alpha)}$. The product $\phi_\alpha s'_{i(\alpha)} \in E(U_{i(\alpha)})$ is supported in the compact subset K_α and so its extension by zero to all of X is an element $s'_\alpha \in E(X)$ vanishing in fibers outside of K_α. Since the collection of K_α's is locally finite, the sum $s' = \sum s'_\alpha$ is locally finite and hence makes sense as an element in $E(X)$. Its image in $E'(X)$ is equal to the locally finite sum of sections $\phi_\alpha s \in E'(X)$ because ϕ_α vanishes outside of $U_{i(\alpha)}$ and inside of $U_{i(\alpha)}$ the product $\phi_\alpha s$ equals $\phi_\alpha s|_{U_{i(\alpha)}}$. Hence, the locally finite sum $\sum_\alpha \phi_\alpha s$ in $E'(X)$ is $(\sum \phi_\alpha)s = 1 \cdot s = s$. ∎

Example 3.7. We conclude with an example that ties together subbundle kernels and quotients, and provides the only reasonable generality in which there are good vector-bundle notions of "image" and "cokernel" for a map of bundles that is not assumed to be a bundle surjection. Briefly, we have to require that the fiber-ranks are constant (a reasonable condition if we are to try to realize the fibral images as the fibers of a subbundle of the target bundle).

Let $f : E' \to E$ be a C^p map of C^p vector bundles over X, and assume that $x \mapsto \operatorname{rank}(f|_x)$ is locally constant on X. Under this assumption,

$$\dim \ker(f|_x) = \dim E'(x) - \operatorname{rank}(f|_x)$$

is a difference of locally constant functions and thus is locally constant. (Conversely, if $\dim \ker(f|_x)$ is locally constant in x then so is $\operatorname{rank}(f|_x)$.) It follows from Theorem 2.6 that there is a kernel subbundle $\ker f$ in E' whose x-fiber is $\ker(f|_x) \subseteq E'(x)$ for all $x \in X$. Since the map $f : E' \to E$ kills the subbundle $\ker f \subseteq E'$, it uniquely factors through a C^p bundle mapping $E'/(\ker f) \to E$ that is injective on fibers (as on x-fibers it is the map $E'(x)/\ker(f|_x) \to E(x)$ induced by the fiber map $f|_x : E'(x) \to E(x)$) and hence is a subbundle. We call the quotient of E by the subbundle $E'/\ker f$ the *cokernel* of f (it does induce the cokernel quotient of $E(x)$ on fibers over $x \in X$). The interested reader can check that the dual map $f^\vee$ has kernel of locally constant rank (since on fibers it is the dual space to the fiber of the bundle $\operatorname{coker} f$ that has locally constant rank), and the dual mapping $(\operatorname{coker} f)^\vee \to E^\vee$ is a subbundle that is killed by $f^\vee$ with the resulting bundle map $(\operatorname{coker} f)^\vee \to \ker(f^\vee)$ an isomorphism (as may be checked on fibers). In other words, for a bundle map with locally constant rank the dual of its cokernel bundle "is" the kernel of its dual map, exactly as in linear algebra.