Math 396. Tangent spaces on products

1. Motivation

Let X_1 and X_2 be C^p premanifolds with corners, $1 \le p \le \infty$. Pick a point $\xi = (\xi_1, \xi_2) \in X_1 \times X_2$. We would like to make precise how the tangent space to $X_1 \times X_2$ at ξ is related to the tangent spaces of X_1 and X_2 at ξ_1 and ξ_2 respectively.

Example 1.1. Consider a cylinder in $\mathbf{R}^3$, say the zero locus C of $g(x,y,z) = x^2 + y^2 - 1$. This has a natural smooth manifold structure since $\mathrm{d}g(c) \neq 0$ for all $c \in C$. Also, as the picture suggests, it is abstractly a product $S^1 \times \mathbf{R}$ *as a C^∞ manifold* (and not merely as a set or topological space). This will be proved rigorously, using the C^∞ map $(\theta, t) \mapsto (\cos\theta, \sin\theta, t)$ from $S^1 \times \mathbf{R}$ to $\mathbf{R}^3$, in §1 in the upcoming handout on bijections and isomorphisms. If we stare at how the tangent plane to the cylinder at each point sits inside of the tangent space to $\mathbf{R}^3$ at the point, we can "see" the decomposition of each such plane into a direct sum of two (perpendicular) lines corresponding to tangent lines along the factor curves S^1 and $\mathbf{R}$ through the point on the cylinder. It is exactly this sort of decomposition seen physically for the cylinder that we seek to prove in general.

2. Slices

Let $\iota_j : X_j \to X_1 \times X_2$ be defined by $\iota_1(x_1) = (x_1, \xi_2)$ and $\iota_2(x_2) = (\xi_1, x_2)$; these are the inclusions along the horizontal and vertical "slices" through ξ.

Lemma 2.1. *The maps ι_j are C^p and homeomorphisms onto their images, with injective tangent maps.*

Proof. The topological aspect is clear, so we just have to show that the ι_j's are C^p with injective tangent maps. This is a local problem on the source and target, so since $X_1 \times X_2$ has local C^p-charts arising from products of local C^p-charts on the factors we may describe ι_j locally in C^p-coordinates as the linear inclusion $V_1 \to V_1 \oplus V_2$ via $v_1 \mapsto (v_1, 0)$ for vectors spaces V_1 and V_2 (really this map restricted to opens in sectors in such vector spaces). Thus, the C^p and injectivity properties are clear. ■

By the lemma, the maps $\mathrm{d}\iota_j(\xi_j) : \mathrm{T}_{\xi_j}(X_j) \to \mathrm{T}_\xi(X_1 \times X_2)$ are injections, and so we can view the tangent spaces along the factors as subspaces of the tangent space of the product. One approach for relating $\mathrm{T}_\xi(X_1 \times X_2)$ and the $\mathrm{T}_{\xi_j}(X_j)$'s is to proceed in an *ad hoc* manner, as follows. If $\{x_1^{(1)}, \dots, x_{n_1}^{(1)}\}$ and $\{x_1^{(2)}, \dots, x_{n_2}^{(2)}\}$ are local C^p coordinates on X_1 and X_2 near ξ_1 and ξ_2, then $\{\partial_{x_j^{(1)}}|_{\xi_1}\}$ and $\{\partial_{x_i^{(2)}}|_{\xi_2}\}$ are respective bases of $\mathrm{T}_{\xi_1}(X_1)$ and $\mathrm{T}_{\xi_2}(X_2)$ whose respective images under $\mathrm{d}\iota_1(\xi_1)$ and $\mathrm{d}\iota_2(\xi_2)$ give the basis of $\mathrm{T}_\xi(X_1 \times X_2)$ arising from the $x_j^{(1)}$'s and $x_i^{(2)}$'s considered as providing local C^p coordinates on $X_1 \times X_2$ near ξ. (Strictly speaking, it is the functions $x_j^{(1)} \circ \pi_1$ and $x_i^{(2)} \circ \pi_2$ that are such coordinates, where $\pi_j : X_1 \times X_2 \to X_j$ is the C^p projection, and $\mathrm{d}\iota_j(\xi_j)(\partial_{x_i^{(j)}}|_{\xi_j}) = \partial_{x_i^{(j)} \circ \pi_j}|_\xi$.) This viewpoint is very useful in practice, but we prefer to explain the decomposition of the tangent space of a product into the direct sum of the tangent spaces along the factors by a procedure that is not coordinate-dependent and hence is intrinsic (but will recover the procedure just described when local C^p coordinates are given).

3. Decomposition via projections

Consider the linear map

$$\mathrm{T}_{\xi_1}(X_1) \oplus \mathrm{T}_{\xi_2}(X_2) \to \mathrm{T}_\xi(X_1 \times X_2). \tag{1}$$

defined by $(v_1, v_2) \mapsto \mathrm{d}\iota_1(\xi_1)(v_1) + \mathrm{d}\iota_2(\xi_2)(v_2)$. This gives a coordinate-free version of the decomposition of tangent planes along a cylinder in $\mathbf{R}^3$, and if we choose local coordinates around ξ_1 and ξ_2 in X_1 and X_2 then this recovers the inverse of the *ad hoc* procedure with partials along coordinate directions as suggested above. In general, we have:

Theorem 3.1. *The linear map* (1) *is an isomorphism, with inverse given by*

$$v \mapsto (\mathrm{d}\pi_1(\xi)(v), \mathrm{d}\pi_2(\xi)(v)),$$

where $\pi_j : X_1 \times X_2 \to X_j$ *is the* C^p *projection.*

Proof. The map $\pi_j \circ \iota_j$ is the identity on X_j, so by the Chain Rule $\mathrm{d}\pi_j(\xi) \circ \mathrm{d}\iota_j(\xi_j)$ is the identity on $\mathrm{T}_{\xi_j}(X_j)$. The maps $\pi_2 \circ \iota_1$ and $\pi_1 \circ \iota_2$ factor through 1-point spaces, and hence induce the zero map on tangent spaces. Thus, by the Chain Rule $\mathrm{d}\pi_2(\xi) \circ \mathrm{d}\iota_1(\xi_1) = 0$ and $\mathrm{d}\pi_1(\xi) \circ \mathrm{d}\iota_2(\xi_2) = 0$. (Lest this look like a trick, it can be seen very concretely in terms of coordinates: the point is that the projection π_2 does not depend on the coordinates in the X_1 direction, so the Jacobian matrix for $\pi_2 \circ \iota_1$ must be zero; the same goes for $\pi_1 \circ \iota_2$.) This shows that composing (1) with the map in the statement of the theorem gives the identity, and so for dimension reasons it follows that we have inverse linear maps. ■