Math 396. Construction of vector fields

1. Motivation

Let $(X, \mathscr{O})$ be a C^p manifold with corners, $p > 0$. We have given a general definition of the notion of C^{p-1} vector field on an open set U in X, and in particular the notion of a *global* C^{p-1} vector field: the case $U = X$. But the only "obvious" element of $\mathrm{Vec}_X(X)$ is 0. How do we know it is any bigger? *Locally* where we have C^p coordinates it is obvious that there are lots of C^{p-1} vector fields. That is, for small open $U \subseteq X$ we can use C^p coordinates to construct many elements of $\mathrm{Vec}_X(U)$. But for "big" U not admitting a coordinate system, how can we build anything? In fact, a similar question comes up even for the C^p-structure: how do we know $\mathscr{O}(X)$ is any bigger than $\mathbf{R}$ (assuming X to be connected and non-empty, say)? Once again, for small U it is obvious that $\mathscr{O}(U)$ is very big. But in the absence of coordinates it is not immediately obvious how to "write down" C^p functions, and so in particular elements in $\mathscr{O}(X)$.

As an indication that something serious has to be done, we note that in the theory of complex-analytic manifolds (properly defined) it is *very* common that $\mathscr{O}(X) = \mathbf{C}$ and $\mathrm{Vec}_X(X) = \{0\}$ for many interesting compact complex manifolds X. Thus, if we are to get rather different results in the C^p case, clearly we have got to make use of something beyond definition-chasing with the general formalism of manifolds, and that special device has to be something not available in the complex-analytic case. The key device we are going to use are bump functions. As we learned in our study of real-analytic functions, the concept of a bump function is alien to the real-analytic world (due to uniqueness of analytic continuation).

In order to use bump functions effectively, we will need partitions of unity, and so for the first time (we're actually *constructing* global things, not just making definitions!) we have to impose the condition that our spaces be topologically reasonable: at least Hausdorff, and more specifically manifolds with corners (i.e., Hausdorff and second countable). For the remainder of this handout, we assume *X is Hausdorff and second-countable* (so all compact subsets of X are closed). In particular, as was shown in the handout on paracompactness, it is equivalent to say that X is Hausdorff and paracompact with countably many connected components.

Remark 1.1. A basic lesson of this handout is that the ring of global functions and the module of global vector fields on a C^p manifold are very "floppy" things.

2. Making many functions and vector fields

There is an important topological consequence of the Hausdorff and second countability assumptions on X that we shall use repeatedly without comment (and which was proved in the handout on paracompactness): *open covers of X have locally finite refinements.* As is explained in pages 50–52 of the course text, this guarantees the existence of *partitions of unity* subordinate to any open covering, and this is an absolutely fundamental device used in nearly all global constructions in differential geometry. Let us record the main result (Corollary 16 on page 52):

Theorem 2.1. *Let M be a C^p manifold with corners, $0 \leq p \leq \infty$. Let $\mathfrak{U} = \{U_\alpha\}$ be an open covering of M. There exists a C^p partition of unity subordinate to $\mathfrak{U}$: that is, a set $\{\phi_i\}$ of C^p functions $\phi_i : M \to [0,1]$ such that:*

1. *the supports $K_i = \mathrm{supp}(\phi_i)$ are compact and form a locally finite collection in M (i.e., each $m \in M$ admits an open neighborhood meeting only finitely many K_i's),*
2. *$\sum \phi_i(m) = 1$ for all $m \in M$ (by the first condition, this sum is locally finite – around each $m \in M$ there is an open on which all but finitely many ϕ_i's vanish – and so there is no subtle convergence issue for $\sum \phi_i$),*

(3) *each K_i is contained in some $U_{\alpha(i)}$.*

The course text asserts the theorem for smooth manifolds, but the proof only uses the para-compactness and Hausdorff properties of the topological space and the existence of a good local concept of "smooth function"; hence, the proof works *verbatim* in the C^p case on manifolds with corners for any $0 \leq p \leq \infty$ (check this, ignoring the second sentence in Theorem 13). Of course, for the local steps of the proof it is perfectly fine to use smooth bump functions in local charts (as they're certainly C^p for any p). In fact, since we've proved some general properties of paracompact spaces in an earlier handout, a few steps in the proof in the course text can be skipped.

As an easy application of partitions of unity, let's construct lots of elements in $\mathscr{O}(X)$ for any C^p manifold with corners $(X, \mathscr{O})$. Let $\{U_\alpha\}$ be any open cover and $f_\alpha \in \mathscr{O}(U_\alpha)$; for example, the U_α's could be domains of C^p-charts (on which there is a plentiful supply of f_α's). Let $\{\phi_i\}$ be a C^p partition of unity subordinary to this cover, with the compact $K_i = \text{supp}(\phi_i)$ contained in $U_{\alpha(i)}$. Using Lemma 2 on page 33, and reviewing how $\{\phi_i\}$ is constructed from a locally finite refinement of $\{U_\alpha\}$, it is not difficult to see how to carry out the construction (with the help of some cutoff functions) so that many ϕ_i's are equal 1 on rather "large" subsets of coordinate balls. That is, we can arrange that many ϕ_i's are equal to 1 on a substantial part of K_i (so all other $\phi_{i'}$ vanish there).

Consider the product $\phi_i f_{\alpha(i)} \in \mathscr{O}(U_{\alpha(i)})$. This vanishes off of the compact K_i, so it vanishes on the *open* subset $U_{\alpha(i)} - K_i$ of $U_{\alpha(i)}$. Intuitively, $\phi_i f_{\alpha(i)}$ vanishes near the "edge" of $U_{\alpha(i)}$ in X. Hence, $\phi_i f_{\alpha(i)} \in \mathscr{O}(U_\alpha)$ and $0 \in \mathscr{O}(X - K_i)$ are C^p functions on open sets $U_{\alpha(i)}$ and $X - K_i$ that *cover* X, so they unique "glue" to a C^p function $F_i \in \mathscr{O}(X)$; this is called the "extension by zero" (it is only reasonable because $\phi_i f_{\alpha(i)}$ vanishes near the "edge" of $U_{\alpha(i)}$ in X).

By construction, since locally on X all but finitely many ϕ_i's vanish, it follows that locally on X all but finitely many F_i's vanish. Hence, the summation $F = \sum F_i$ is *locally* a finite sum and thus is a perfectly straightforward sum presenting no delicate convergence problems whatsoever. In particular, $F \in \mathscr{O}(X)$ (as this condition is local on X!). Note that for those i's such that $\phi_i = 1$ on a large subset $K_i' \subseteq K_i$, we have that $F_i|_{K_i'} = f_{\alpha(i)}|_{K_i'}$ and $F_j|_{K_i'} = 0$ for all $j \neq i$ (as $\phi_j|_{K_i'} = 0$ since all $\phi_r \geq 0$ with $\sum \phi_r = 1$ but $\phi_i|_{K_i'} = 1$). Hence, $F|_{K_i'} = f_{\alpha(i)}|_{K_i'}$. To summarize, we have constructed $F \in \mathscr{O}(X)$ such that on "large" subsets K_i' of the open $U_{\alpha(i)}$ the function F is equal to a prescribed function $f_{\alpha(i)}$. In this way, we see that there is an astoundingly large collection of elements of $\mathscr{O}(X)$ and that such elements may be built with prescribed restrictions on big closed subsets of many open coordinate domains with disjoint closures. This puts to rest any question of $\mathscr{O}(X)$ being "small".

How about $\text{Vec}_X(X)$ for $p \geq 1$? In fact, the *exact* same method works, since $\text{Vec}_X(U)$ is a module over $\mathscr{O}'(U)$ compatible with restriction on the open set $U \subseteq X$, where $\mathscr{O}'$ is the "underlying C^{p-1}-structure". We pick opens $\{U_\alpha\}$ covering X with $\vec{v}_\alpha \in \text{Vec}_X(U_\alpha)$. We let $\{\phi_i\}$ be a C^{p-1} partition of unity subordinary to the covering by the U_α's (or a C^p partition of unity, if you prefer), and we consider the C^{p-1} vector field $\vec{V}_i \in \text{Vec}_X(X)$ given by "extension of zero" of the C^{p-1} vector field $\phi_i \vec{v}_{\alpha(i)} \in \text{Vec}_X(U_{\alpha(i)})$ that vanishes away from the compact K_i (why?). As in the case of functions, we form the locally finite sum $\vec{V} = \sum \vec{V}_i \in \text{Vec}_X(X)$ and we note that this gives rise to a vast collection of elements of $\text{Vec}_X(X)$ (with $\vec{V}|_{K_i'} = \vec{v}_{\alpha(i)}|_{K_i'}$ if $\phi_i|_{K_i'} = 1$). Thus, $\text{Vec}_X(X)$ is very big.

3. Global definition

As a further application of the technique of bump functions, for smooth manifolds with corners we link up the definition of global vector fields from the course text and the definition used in class; our definition is promotes local thinking and works in the real-analytic and complex-analytic cases (whereas the definition of the course text and virtually all introductory books on differential geometry is the wrong concept in such cases).

Let $(M, \mathscr{O})$ be a smooth manifold with corners. For any open set $U \subseteq M$, $\mathscr{O}(U)$ is the $\mathbf{R}$-algebra of smooth functions on U. Let $\mathrm{Der}_{\mathbf{R}}(\mathscr{O}(U))$ be the set of $\mathbf{R}$-linear derivations

$$D : \mathscr{O}(U) \to \mathscr{O}(U)$$

of the $\mathbf{R}$-algebra $\mathscr{O}(U)$. That is, D is an $\mathbf{R}$-linear map satisfying the *Leibnitz rule*: $D(fg) = fD(g) + gD(f)$ in $\mathscr{O}(U)$. The fundamental distinction between this and the notion of $\mathbf{R}$-linear derivation of $\mathscr{O}|_U$ as discussed in class is that the latter concept was a collection $D = \{D_{U'}\}_{U' \subseteq U}$ of $\mathbf{R}$-linear derivations $D_{U'}$ of $\mathscr{O}(U')$ for *every* open $U' \subseteq U$ with a compatibility condition on the $D_{U'}$'s via restriction maps for inclusions among open subsets of U. In contrast, the notion we have just introduced is the data of a single $\mathbf{R}$-linear derivation D on $\mathscr{O}(U)$ and we do not give ourselves any additional data of $D_{U'}$'s on $\mathscr{O}(U')$'s for *all* open $U' \subseteq U$.

Note that $\mathrm{Der}_{\mathbf{R}}(\mathscr{O}(U))$ is an $\mathscr{O}(U)$-module in an obvious way: for $D_1, D_2 \in \mathrm{Der}_{\mathbf{R}}(\mathscr{O}(U))$ we define $(D_1 + D_2)(f) = D_1(f) + D_2(f)$ (this *is* an $\mathbf{R}$-linear derivation of $\mathscr{O}(U)$; check!), and for $h \in \mathscr{O}(U)$ and $D \in \mathrm{Der}_{\mathbf{R}}(\mathscr{O}(U))$ we define $hD : f \mapsto h \cdot D(f)$ (again, check that this really is an $\mathbf{R}$-linear derivation of $\mathscr{O}(U)$). Of course, one should check that these definitions really do satisfy the axioms for $\mathrm{Der}_{\mathbf{R}}(\mathscr{O}(U))$ to be an $\mathscr{O}(U)$-module, but this is straightforward definition-chasing and so is left to the reader.

If we are given $\vec{v} \in \mathrm{Vec}_M(U)$, then we have seen how to make

$$D_{\vec{v}} = \{D_{\vec{v},U'}\}_{U' \subseteq U} \in \mathrm{Der}_M(\mathscr{O}|_U),$$

so in particular we get $D_{\vec{v}} \stackrel{\text{def}}{=} D_{\vec{v},U} \in \mathrm{Der}_{\mathbf{R}}(\mathscr{O}(U))$. Explicitly, for all $f \in \mathscr{O}(U)$ the smooth function $D_{\vec{v}}(f) \in \mathscr{O}(U)$ has value $\vec{v}(u)(f_u) \in \mathbf{R}$ at $u \in U$ (with f_u the germ of f in $\mathscr{O}_u$); the function $D_{\vec{v}}(f)$ is usually just denoted $\vec{v}(f) \in \mathscr{O}(U)$, as in our course text and virtually all works in differential geometry. The mapping

$$\mathrm{Vec}_M(U) \to \mathrm{Der}_{\mathbf{R}}(\mathscr{O}(U))$$

defined by $\vec{v} \mapsto D_{\vec{v}}$ is trivially checked to be an $\mathscr{O}(U)$-linear map. That is, for $\vec{v}_1, \vec{v}_2 \in \mathrm{Vec}_M(U)$ and $h_1, h_2 \in \mathscr{O}(U)$,

$$D_{h_1\vec{v}_1 + h_2\vec{v}_2} = h_1 D_{\vec{v}_1} + h_2 D_{\vec{v}_2};$$

this amounts to comparing what happens when each side is applied to an arbitrary $f \in \mathscr{O}(U)$, and it is a simple calculation left to the reader.

The entire preceding discussion could have been carried out without knowing anything about whether or not $\mathrm{Vec}_M(U)$ or $\mathscr{O}(U)$ are "big". It is interesting to note that in the setting of complex-analytic manifolds, there are many examples where $\mathrm{Vec}_M(M)$ nonzero (and even quite large) but $\mathscr{O}(M) = \mathbf{C}$ (so $\mathrm{Der}_{\mathbf{C}}(\mathscr{O}(M)) = 0$). This does *not* contradict the fact that in the complex-analytic case one has $\mathrm{Vec}_M(U) \simeq \mathrm{Der}_M(\mathscr{O}|_U)$, as an element $D = \{D_{U'}\}_{U' \subseteq U}$ may well be nonzero even if $D_U = 0$! The purpose of mentioning the complex-analytic case here (even though we have not discussed it in any rigorous manner) is to emphasize that one cannot expect to determine by "general nonsense" definition-chasing alone whether or not the $\mathscr{O}(U)$-linear map we have built from $\mathrm{Vec}_M(U)$ to $\mathrm{Der}_{\mathbf{R}}(\mathscr{O}(U))$ is an isomorphism. We have got to use something specific to the setting of C^p manifolds if we are to verify an isomorphism result in this direction; we will use bump functions Here is the result:

Theorem 3.1. *Let M be a smooth manifold with corners. For any open set $U \subseteq M$, the natural $\mathscr{O}(U)$-linear map*

$$\mathrm{Vec}_M(U) \to \mathrm{Der}_{\mathbf{R}}(\mathscr{O}(U))$$

is an isomorphism.

Incredibly, the right side of the isomorphism in this theorem is usually taken as the *definition* of the left side! The conclusion of the theorem is in fact a very convenient device in the study of smooth manifolds. The proof only uses local bump functions, not partitions of unity, so the proof works if M is merely Hausdorff and not assumed to be second countable (or paracompact).

Proof. We may and do rename U as M (thereby cutting down on the amount of notation). We have to prove two things: if $\vec{v} \in \mathrm{Vec}_M(M)$ satisfies $D_{\vec{v}} = 0$ then $\vec{v} = 0$ (this gives injectivity), and if $D : \mathscr{O}(M) \to \mathscr{O}(M)$ is an $\mathbf{R}$-linear derivation then $D = D_{\vec{v}}$ for some $\vec{v} \in \mathrm{Vec}_M(M)$ (this gives surjectivity).

For injectivity, suppose that for all $f \in \mathscr{O}(M)$ we have $D_{\vec{v}}(f) = 0$; that is, the point derivation $\vec{v}(m) : \mathscr{O}_m \to \mathbf{R}$ kills the u-germ of f for all $f \in \mathscr{O}(M)$. We wish to conclude $\vec{v}(m) = 0$ for all $m \in M$. By Lemma 3.2 in the handout on "globalization via bump functions", the natural map of $\mathbf{R}$-algebras $\mathscr{O}(M) \to \mathscr{O}_m$ sending each $f \in \mathscr{O}(M)$ to its m-germ $[(M, f)]_m$ is *surjective.* Thus, it follows trivially that $\vec{v}(m)$ vanishes for all $m \in M$, which is to say $\vec{v} = 0$. This gives injectivity.

We now choose an $\mathbf{R}$-linear derivative $D : \mathscr{O}(M) \to \mathscr{O}(M)$ and we seek to construct $\vec{v} \in \mathrm{Vec}_M(M)$ such that $D_{\vec{v}} = D$. Let us first pick $m \in M$, and define $\vec{v}(m) \in \mathrm{T}_m(M)$. Since M is *smooth,* to give a tangent vector at m is to give an $\mathbf{R}$-linear mapping $\mathscr{O}_m \to \mathbf{R}$ satisfying the "Leibnitz Rule at m". Observe that the $\mathbf{R}$-linear mapping

$$D|_m : \mathscr{O}(M) \to \mathbf{R}$$

defined by $D|_m(f) = (Df)(m)$ is $\mathbf{R}$-linear in f and satisfies the Leibnitz at m (simply evaluate at m for the identity $D(fg) = fD(g) + gD(f)$). As was shown in §3 in the handout on "globalization via bump functions", such a mapping $D|_m$ uniquely has the form $\vec{v}(m) \circ \pi_m$ where $\pi_m : \mathscr{O}(M) \to \mathscr{O}_m$ is the natural $\mathbf{R}$-algebra surjection sending any f to its germ at m and $\vec{v}(m) : \mathscr{O}_m \to \mathbf{R}$ is a point-derivation at m (i.e., an element in $\mathrm{T}_m(M)$, as M is smooth).

Consider the set-theoretic vector field $\vec{v}_D$ on M whose value at each $m \in M$ is $\vec{v}(m)$ as just defined; that is, $(Df)(m) = \vec{v}_D(m)(f_m)$ with $f_m \in \mathscr{O}_m$ the germ of f at m. We shall prove that $\vec{v}_D$ is a smooth vector field on M, and that it recovers D. For any $f \in \mathscr{O}(M)$ we have

$$(\vec{v}_D f)(m) \stackrel{\text{def}}{=} \vec{v}_D(m)(f_m) = (Df)(m),$$

so the set-theoretic function $\vec{v}_D(f) : M \to \mathbf{R}$ agrees with $Df \in \mathscr{O}(M)$. Hence, once $\vec{v}_D$ is proved to be a smooth vector field it must satisfy the requirements to solve our problem.

Let $(\{x_1, \ldots, x_n\}, U)$ be a smooth coordinate chart on M, so $\vec{v}_D|_U = \sum a_j \partial_{x_j}$ as set-theoretic vector fields on U. It must be proved that $a_j : U \to \mathbf{R}$ is a smooth function. This problem is local on U, so pick $u_0 \in U$ and let $\phi \in \mathscr{O}(M)$ be a global smooth function that is supported in some compact subset $K \subseteq U$ and equal to 1 near u_0. (Again, see Lemma 2 on page 33 for the construction of such bump functions.) In particular, each $\phi x_j \in \mathscr{O}(U)$ is supported in the compact $K \subseteq U$ and thus extends by zero to a smooth function $\widetilde{x}_j$ on M. We claim that each a_i near u_0 is equal to $D(\widetilde{x}_i) \in \mathscr{O}(M)$, whence we get the desired smoothness of each a_i near the arbitrary point $u_0 \in U$ (thereby settling the proof). By the definition of $\vec{v}_D$, for $u \in \mathrm{int}_M(K)$ and $x_{i,u} \in \mathscr{O}_u$ the germ associated to x_i, this germ is also induced by $\widetilde{x}_i \in \mathscr{O}(M)$ (as $\phi = 1$ near u) and so

$$a_i(u) = \sum a_j(u)(\partial_{x_j}|_u)(x_{i,u}) = \vec{v}_D(u)(x_{i,u}) = D(\widetilde{x}_i)(u).$$

This shows a_i and $D(\widetilde{x}_i)$ agree on the open set $\mathrm{int}_M(K)$ around u_0 in U, as desired. ∎