Lecture 6: Presentations of deformation rings

Samit
November 6, 2009
Notes by Sam Lichtenstein

This lecture is about getting bounds for the dimension of deformation rings, by bounding the number of generators and relations. The reference for this lecture is Kisin's article in CDM, or stuff from his Hawaii notes.

1. Local setup and statement

Let $K/\mathbb{Q}_p$ be finite, $\mathcal{O} = \mathcal{O}_K, \pi$ a uniformizer, $k = \mathcal{O}/(\pi)$, Γ a profinite group satisfying the p-finiteness condition "Φ_p", and $\overline{\rho} : \Gamma \to \mathrm{GL}_n(k)$ a mod π representation. We consider deformations to complete local noetherian $\mathcal{O}$-algebras with residue field k. The framed deformation ring $R^{\square}_{\overline{\rho}}$ always exists, so we have a universal representation

$$\Gamma \overset{\rho^{\square}_{\mathrm{univ}}}{\to} \mathrm{GL}_n(R^{\square}_{\overline{\rho}}).$$

Assuming $\mathrm{End}_{\Gamma}\,\overline{\rho} = k$, we also know $R_{\overline{\rho}}$ exists, and we then get a universal deformation

$$\Gamma \overset{\rho^{\mathrm{univ}}}{\to} \mathrm{GL}_n(R_{\overline{\rho}}).$$

Recall that

$$D^{\square}_{\overline{\rho}}(k[\epsilon]) = \mathrm{Hom}_k(\mathfrak{m}_{R^{\square}}/(\mathfrak{m}^2_{R^{\square}}, \pi), k) \cong Z^1(\Gamma, \mathrm{ad}\,\overline{\rho})$$

and $D_{\overline{\rho}}(k[\epsilon]) = \mathrm{H}^1(\Gamma, \mathrm{ad}\,\overline{\rho})$ as k-vector spaces.

Theorem. *Let $r = \dim_k Z^1(\Gamma, \mathrm{ad}\,\overline{\rho})$. Then there exists an $\mathcal{O}$-algebra isomorphism*

$$\mathcal{O}[\![x_1, \ldots, x_r]\!]/(f_1, \ldots, f_s) \cong R^{\square}_{\overline{\rho}}$$

where $s = \dim_k \mathrm{H}^2(\Gamma, \mathrm{ad}\,\overline{\rho})$.

Corollary. (i) $\dim R^{\square}_{\overline{\rho}} \geq 1 + n^2 - \chi(\Gamma, \mathrm{ad}\,\overline{\rho}) = 1 + n^2 - h^0(\mathrm{ad}\,\overline{\rho}) + h^1(\mathrm{ad}\,\overline{\rho}) - h^2(\mathrm{ad}\,\overline{\rho})$.
(ii) $\dim R_{\overline{\rho}} \geq 2 - \chi(\Gamma, \mathrm{ad}\,\overline{\rho})$.

Proof of corollary. From $\mathcal{O}$ we get a contribution of 1. hence we get $\dim R^{\square}_{\overline{\rho}} \geq 1 + \dim Z^1 - h^2$. Now (i) follows formally noting that $\dim Z^0 = \dim C^0 = n^2$. (Use $h^1 = \dim Z^1 - \dim B^1$ and $\dim B^1 = \dim C^0 - \dim Z^0 = \dim C^0 - h^0$.) Then (ii) is immediate using the fact that $R^{\square}_{\overline{\rho}}$ is basically a PGL_n-bundle over $R_{\overline{\rho}}$. $\square$

2. Proof of Theorem 1

Using completeness [exercise] we can choose a surjection

$$\varphi : \mathcal{O}[\![x]\!] := \mathcal{O}[\![x_1, \ldots, x_r]\!] \twoheadrightarrow R^{\square}_{\overline{\rho}}.$$

(Send the x_i's to elements which reduce to a basis for the tangent space $Z^1(\Gamma, \mathrm{ad}\,\overline{\rho})$ of the framed deformation ring.) The problem is to show that the minimal number of generators of the kernel $\mathbb{J} = \ker\varphi \subset \mathcal{O}[\![x]\!]$ is at most s. Let $\mathbf{m} = \mathfrak{m}_{\mathcal{O}[\![x]\!]} \subset \mathcal{O}[\![x]\!]$ be the maximal ideal $(\pi, x_1, \ldots, x_r)$. It would suffice to construct a linear injection $(\mathbb{J}/\mathbf{m}\mathbb{J})^* \hookrightarrow \mathrm{H}^2(\Gamma, \mathrm{ad}\,\overline{\rho})$. There is a subtle technical problem in an attempt to construct such an injection. We explain the problem, and then the fix to get around it.

For each $\gamma \in \Gamma$ choose a set-theoretic lift $\widetilde{\rho}(\gamma) \in \mathrm{GL}_n(\mathcal{O}[\![x]\!]/\mathbf{m}\mathbb{J})$ of $\rho^{\square}(\gamma) \in \mathrm{GL}_n(\mathcal{O}[\![x]\!]/\mathbb{J}) = \mathrm{GL}_n(R^{\square})$. We need to make this choice so that $\widetilde{\rho}$ is a *continuous* function of γ. It is not clear if the map

$$\mathcal{O}[\![x]\!]/\mathbf{m}\mathbb{J} \twoheadrightarrow \mathcal{O}[\![x]\!]/\mathbb{J}$$

admits a continuous section as topological spaces, so it is not clear how to find a continuous $\widetilde{\rho}$. To handle this problem, we now prove:

Claim: For $r > 0$, let $\mathbb{J}_r = (\mathbb{J} + \mathbf{m}^r)/\mathbf{m}^r \in \mathcal{O}[\![x]\!]/\mathbf{m}^r$ and let $\mathbf{m}_r = \mathbf{m}/\mathbf{m}^r$. For $r \gg 0$, the natural map $\mathbb{J}/\mathbf{m}\mathbb{J} \to \mathbb{J}_r/\mathbf{m}_r\mathbb{J}_r$ is an isomorphism.

Proof. The map is surjective, and for injectivity we have to show that $\mathbb{J} \cap (\mathbf{m}\mathbb{J} + \mathbf{m}^r) = \mathbf{m}\mathbb{J}$ for large r. Certainly $\mathbf{m}\mathbb{J}$ lies in the intersection for all r, so since $\mathbb{J}/\mathbf{m}\mathbb{J}$ has finite length we see that the intersection stabilizes at some intermediate ideal for $r \gg 0$. This stabilizing ideal must then be the total intersection. But by Artin-Rees applied to $\mathbf{m}\mathbb{J}$ as a finite $\mathcal{O}[\![x]\!]$-module, the intersection of all $(\mathbf{m}\mathbb{J} + \mathbf{m}^r)$'s is $\mathbf{m}\mathbb{J}$. □

By the Claim, to prove the desired result about minimal number of generators of $\mathbb{J}$, we can replace $\mathcal{O}[\![x]\!]$ and $R^\square := R^\square_{\overline{\rho}}$ with their quotients by rth power of maximal ideal for some large r. The quotient of $R^\square$ by rth power of its maximal ideal is universal in the category of complete local noetherian $\mathcal{O}$-algebras whose maximal ideal has vanishing rth power (exercise!). So working within this full subcategory of local $\mathcal{O}$-algebras, we can still exploit universal mapping properties. But we gain the advantage that now our rings are of finite length as $\mathcal{O}/\pi^r$-modules, so in particular they're all discrete with their max-adic topology and hence the Galois representations which arise have open kernel. We can therefore find the required continuous section, working throughout with local rings whose maximal ideal has a fixed but large order of nilpotence.

So we now proceed in such a modified setting (so the definition of $\mathbb{J}$ changes accordingly, but the Claim shows that this does not affect $\mathbb{J}/\mathbf{m}\mathbb{J}$, which is to say the minimal number of generators of $\mathbb{J}$). In particular, in the new setting we will construct a k-linear injection of $\mathbb{J}/\mathfrak{m}\mathbb{J}$ into $\mathrm{H}^2(\Gamma, \mathrm{ad}\,\overline{\rho})$, thereby finishing the proof.

For $f \in (\mathbb{J}/\mathbf{m}\mathbb{J})^*$. let

$$\rho_f(\gamma, \delta) = f(\widetilde{\rho}(\gamma\delta)\widetilde{\rho}(\delta)^{-1}\widetilde{\rho}(\gamma)^{-1} - \mathbf{1}),$$

where we apply the map f "entry-wise" to the given matrix in $\mathrm{Mat}_{n\times n}(\mathbb{J}/\mathbf{m}\mathbb{J})$. That is, the map φ_f has the form

$$\Gamma^2 \to \mathrm{Mat}_{n\times n}(\mathbb{J}/\mathbf{m}\mathbb{J}) \xrightarrow{f} \mathrm{Mat}_{n\times n}(k).$$

Now we observe the following facts.

(1) $\varphi_f \in Z^2(\Gamma, \mathrm{ad}\,\overline{\rho})$.
(2) $[\varphi_f] \in \mathrm{H}^2(\Gamma, \mathrm{ad}\,\overline{\rho})$ is independent of the choice of lift $\widetilde{\rho}$.
(3) $f \mapsto [\varphi_f]$ is k-linear.
(4) $f \mapsto [\varphi_f]$ is injective, but more precisely we have $[\varphi_f] = 0 \Leftrightarrow$ we can choose $\widetilde{\rho}$ to be a homomorphism "mod $\mathbb{J}_f$" where $\mathbb{J}_f = \ker(\mathbb{J} \to \mathbb{J}/\mathbf{m}\mathbb{J} \xrightarrow{f} k) \Leftrightarrow f = 0 \Leftrightarrow \mathbb{J}_f = \mathbb{J}$.

Note that (4) provides the desired linear injection, and hence proves the theorem; (1)-(3) are necessary to make sense of (4).

Let us prove the facts above.

(1) This is a formal computation, which goes as follows. Note that we can identify $\mathrm{Mat}_{n\times n}(\mathbb{J}/\mathbf{m}\mathbb{J})$ under addition with $(\mathbf{1} + \mathrm{Mat}_{n\times n}(\mathbb{J}/\mathbf{m}\mathbb{J}))$ under multiplication, since $\mathbb{J} \subset \mathbf{m}$. Using this identification, we have

$$\mathrm{d}\,\varphi_f(\gamma, \delta, \epsilon) = \gamma\varphi_f(\delta, \epsilon) - \varphi_f(\gamma\delta, \epsilon) + \varphi_f(\gamma, \delta\epsilon) - \varphi_f(\gamma, \delta) \in \mathrm{Mat}_{n\times n}(k).$$

If we want to prove this is zero, it's enough to check "upstairs" in $\mathrm{Mat}_{n\times n}(\mathbb{J}/\mathbf{m}\mathbb{J})$, i.e. before applying f. Thus we really want to check that

$$(\widetilde{\rho}(\gamma)\widetilde{\rho}(\delta\epsilon)\widetilde{\rho}(\epsilon)^{-1}\widetilde{\rho}(\delta)^{-1}\widetilde{\rho}(\gamma)^{-1}) \times (\widetilde{\rho}(\gamma\delta)\widetilde{\rho}(\epsilon)\widetilde{\rho}(\gamma\delta\epsilon)^{-1}) \times (\widetilde{\rho}(\gamma\delta\epsilon)\widetilde{\rho}(\delta\epsilon)^{-1}\widetilde{\rho}(\gamma)^{-1}) \times (\widetilde{\rho}(\gamma)\widetilde{\rho}(\delta)\widetilde{\rho}(\gamma\delta)^{-1}) \stackrel{?}{=} \mathbf{1}\,.$$

The trick is to insert the bracketed term (which is 1) below:

$$\widetilde{\rho}(\gamma)\widetilde{\rho}(\delta\epsilon)\widetilde{\rho}(\epsilon)^{-1}\widetilde{\rho}(\delta)^{-1}\widetilde{\rho}(\gamma)^{-1}\widetilde{\rho}(\gamma\delta)\overbrace{\widetilde{\rho}(\delta)^{-1}\widetilde{\rho}(\delta)}^{insert}\widetilde{\rho}(\epsilon)\widetilde{\rho}(\gamma\delta\epsilon)^{-1} \times \widetilde{\rho}(\gamma\delta\epsilon)\widetilde{\rho}(\delta\epsilon)^{-1}\widetilde{\rho}(\gamma)^{-1} \times (\widetilde{\rho}(\gamma)\widetilde{\rho}(\delta)\widetilde{\rho}(\gamma\delta)^{-1}) \stackrel{?}{=} \mathbf{1}\,.$$

Now observe that the bracketed terms below reduce to 0 in $\mathrm{Mat}_{n\times n}(k)$ and hence can be commuted with one another (!):

$$\widetilde{\rho}(\gamma)\overbrace{\widetilde{\rho}(\delta\epsilon)\widetilde{\rho}(\epsilon)^{-1}\widetilde{\rho}(\delta)^{-1}}^{I}\overbrace{\widetilde{\rho}(\gamma)^{-1}\widetilde{\rho}(\gamma\delta)\widetilde{\rho}(\delta)^{-1}}^{II}\widetilde{\rho}(\delta)\widetilde{\rho}(\epsilon)\widetilde{\rho}(\gamma\delta\epsilon)^{-1} \times \widetilde{\rho}(\gamma\delta\epsilon)\widetilde{\rho}(\delta\epsilon)^{-1}\widetilde{\rho}(\gamma)^{-1} \times (\widetilde{\rho}(\gamma)\widetilde{\rho}(\delta)\widetilde{\rho}(\gamma\delta)^{-1}) \stackrel{?}{=} \mathbf{1}\,.$$

After swapping I and II one sees that in fact everything cancels magically. (Is there is a "conceptual" proof of (1)?)

(2) This is similar to (1). First write $\widetilde{\rho}^{\text{new}}(\gamma) = a(\gamma)\widetilde{\rho}(\gamma)$ for some

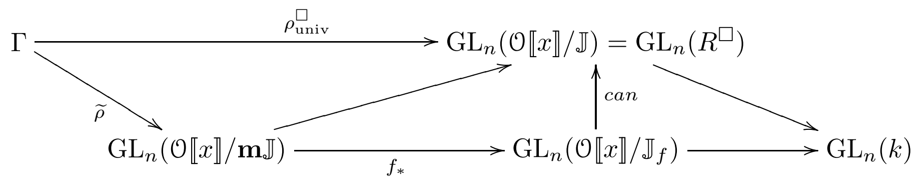
$$a : \Gamma \to \mathbf{1} + \mathrm{Mat}_{n\times n}(\mathbb{J}/\mathbf{m}\mathbb{J}).$$

The idea is to show formally that $a(\gamma)$ (which is of course a *continuous* 1-cocycle on Γ) changes φ_f by $\mathrm{d}\,a$. This is done with a similar "insert $\mathbf{1}$ cleverly and commute stuff" trick as in (1).

(3) OK.

(4) The last equivalence in (4) is clear. For the other two equivalences, the implications "$\Leftarrow$" are OK. The implication that $[\varphi_f] = 0$ implies we can choose $\widetilde{\rho}$ to be a homomorphism mod $\mathbb{J}_f$ follows from the previous calculation [omitted] that $\widetilde{\rho} \rightsquigarrow a \cdot \widetilde{\rho}$ changes φ by $\mathrm{d}\,a$. In particular, if φ is already a coboundary, then by changing the choice of lift we can make $\varphi = 0$, which is the same as saying our lift is a homomorphism mod $\mathbb{J}_f$. So the crux of the matter is the second "$\Rightarrow$".

Here's the situation. We have a diagram



We'd like to prove that $\mathcal{O}[\![x]\!]/\mathbb{J}_f \to \mathcal{O}[\![x]\!]/\mathbb{J}$ is an isomorphism. By the universality of $R^\square$ we get the map

$$\mathcal{O}[\![x]\!]/\mathbb{J} \overset{\exists!}{\to} \mathcal{O}[\![x]\!]/\mathbb{J}_f \overset{\text{can}}{\to} \mathcal{O}[\![x]\!]/\mathbb{J}$$

and again by universality the composition is the identity. Now it would be enough to check that $\mathbb{J} \subset \mathbb{J}_f$. Note that the image of x_i in $\mathcal{O}[\![x]\!]/\mathbb{J}$ maps to $x_i + a_i \in \mathcal{O}[\![x]\!]/\mathbb{J}_f$ where a_i is some element of $\mathbb{J}$. It will suffice to show that if $g(x_1, \dots, x_n) \in \mathbb{J}$ then g maps to g itself in $\mathcal{O}[\![x]\!]/\mathbb{J}_f$.

First we claim that $\mathbb{J} \subset (\mathbf{m}^2, \pi)$ [recall that $\mathbb{J} = \ker(\mathcal{O}[\![x]\!] \twoheadrightarrow R^\square)$]. Indeed, if $g \in \mathbb{J}$ then $g = g_0 + \sum g_i x_i + O(\mathbf{m}^2)$. Moreover $g_0 \in (\pi)$ and each g_i lies in (π) since the x_i's map to a *basis* of $\mathbf{m}/(\mathbf{m}^2, \pi)$. Thus $g \in (\mathbf{m}^2, \pi)$. Consequently, it's enough to show what we want for $g \in (\mathbf{m}^2, \pi)$. [This will be important later on!]

But if $g \in (\mathbf{m}^2, \pi)$ then under $\mathcal{O}[\![x]\!]/\mathbb{J} \to \mathcal{O}[\![x]\!]/\mathbb{J}_f$ we still have

$$g = g_0 + \sum g_i x_i + O(\mathbf{m}^2) \mapsto g_0 + \sum g_i(x_i + a_i) + O(\mathbf{m}^2),$$

and the observation is that when we subtract off g from this we get $\sum g_i a_i$ in the $O(\mathbf{m})$ term, which [by inspection] is in $\mathbf{m}\mathbb{J} \subset \mathbb{J}_f$. Similarly one sees that the higher order terms vanish mod $\mathbb{J}_f$.

This concludes the proof of (4), hence the claim, hence the theorem.

3. Completed tensor products

Example. Let R be a Noetherian ring, and consider $R[x] \otimes_R R[y] \cong R[x, y]$. However $R[\![x]\!] \otimes_R R[\![y]\!]$ is something weird, being just a part of $R[\![x, y]\!]$. It's easy to see that it does at least inject into $R[\![x, y]\!]$. The idea is that $M \otimes R^I \hookrightarrow M^I$ for any free R-module R^I (here I is an arbitrary index set) but this map fails to be an isomorphism.

To check the injectivity, note that it's OK for M finite free, which allows one to deduce it for M finitely presented, and then pass to a direct limit to conclude the general case. Applying this to $I = \mathbb{Z}$ and $M = R[\![x]\!]$ gives what we want in our case. But to see that our map $R[\![x]\!] \otimes R[\![y]\!] \hookrightarrow R[\![x, y]\!]$ is not surjective, observe that $\sum x^n y^n$ is not in the image!

Definition. Let $\mathcal{O}$ be a complete Noetherian local ring and R, S complete Noetherian local $\mathcal{O}$-algebras (meaning the structure maps are local morphisms). Assume at least one of the residue field extensions $\mathcal{O}/\mathfrak{m}_\mathcal{O} \subset R/\mathfrak{m}_R$ and $\mathcal{O}/\mathfrak{m}_\mathcal{O} \subset S/\mathfrak{m}_S$ is finite. Then set $\mathfrak{m} \lhd R \otimes_\mathcal{O} S$ to be the ideal generated by

$$\mathfrak{m}_R \otimes_\mathcal{O} S + R \otimes_\mathcal{O} \mathfrak{m}_S.$$

[Note: $(R\otimes_{\mathcal{O}} S)/\mathfrak{m}\cong \Bbbk_R\otimes_{\Bbbk_{\mathcal{O}}}\Bbbk_S$ is not necessarily a field, or even a local ring, but it is artinian.] Now define the completed tensor product $R\widehat{\otimes}_{\mathcal{O}}S$ to be the $\mathfrak{m}$-adic completion of $R\otimes_{\mathcal{O}} S$.

Universal property. $R\widehat{\otimes}_{\mathcal{O}}S$ is the coproduct in the category of complete semilocal Noetherian $\mathcal{O}$-algebras and continuous maps. It is thus the universal (i.e. initial) complete semilocal Noetherian $\mathcal{O}$-algebra equipped with continuous $\mathcal{O}$-algebra maps from R and S.

Example. We have $\mathcal{O}[\![x]\!]\widehat{\otimes}_{\mathcal{O}}\mathcal{O}'[\![y]\!]\cong\mathcal{O}'[\![x,y]\!]$ when $\mathcal{O}'$ is any complete Noetherian local $\mathcal{O}$-algebra. We also have

$$(\mathcal{O}[\![x_1,\dots,x_r]\!]/\mathbb{J})\widehat{\otimes}_{\mathcal{O}}(\mathcal{O}'[\![y_1,\dots,y_s]\!]/\mathbb{J}')\cong\mathcal{O}'[\![x_1,\dots,x_r,y_1,\dots,y_s]\!]/(\mathbb{J},\mathbb{J}')$$

in this setup.

4. Global setup and statement

Let F be a number field, and p a prime. Let S be a finite set of places of F containing $\{v|p\}$. Fix an algebraic closure $\overline{F}/F$ and let $F_S\subset\overline{F}$ be the maximal extension unramified outside S. Let $G_{F,S}=\mathrm{Gal}(F_S/F)$. Let $\Sigma\subset S$ be any subset of places [for now; later we'll impose conditions].

For $v\in\Sigma$, fix algebraic closures $\overline{F}_v/F_v$ and choose embeddings $\overline{F}\hookrightarrow\overline{F}_v$, or, what is the same thing, choices of decomposition group $\mathrm{Gal}(\overline{F}_v/F_v)=G_v\subset G_{F,S}$. Now let $K/\mathbb{Q}_p$ be a finite extension, and $\mathcal{O}$, π, and k be as above. Fix a character $\psi: G_{F,S}\to\mathcal{O}^\times$.

Let V_k be a finite dimensional continuous representation of $G_{F,S}$ over k such that $\det V_k=\psi \bmod \pi$.

Since we're fixing $\det=\psi$ in this subsection, we'll be dealing (from now on in this talk) with $\mathrm{ad}^0 V_k$ rather than $\mathrm{ad}\, V_k$. [More on this later.] A caution is in order: if $p|\dim V_k$ then $\mathrm{ad}^0 V_k$ is not a direct summand of $\mathrm{ad}\, V_k$. Usually the scalars in $\mathrm{ad}\, V_k$ give a splitting, but when $p|\dim V_k$ the scalars actually sit inside $\mathrm{ad}^0 V_k$. Hence we shall assume from now on that $p\nmid\dim V_k$.

For each $v\in\Sigma$ fix a basis β_v of V_k. We're going to consider deformation functors (and the representing rings) with determinant conditions. Set $D_v^{\square,\psi}$ to be the functor of framed deformations of $V_k|_{G_v}$ with the basis β_v, with fixed determinant $\psi \bmod \pi$, and let $R_v^{\square,\psi}$ be the ring (pro-)representing it. This always exists. Likewise let $D_{F,S}^{\square,\psi}$ be the functor of deformations V_A of V_k with determinant $\psi \bmod \pi$, equipped with an A-basis $\widetilde{\beta}_v$ of V_A lifting β_v for each $v\in\Sigma$. Let $R_{F,S}^{\square,\psi}$ be the ring representing it. Again, this always exists.

We have analogous respective unframed counterparts R_v^ψ and $R_{F,S}^\psi$ under the usual condition that V_k has only scalar endomorphisms as a representation space for G_v and $G_{F,S}$ respectively.

Now *define* $R_\Sigma^{\square,\psi}=\widehat{\bigotimes}_{v\in\Sigma}R_v^{\square,\psi}$ [completed tensor product over $\mathcal{O}$]. Since each $R_v^{\square,\psi}$ has the same residue field, in this case the completed tensor product actually *is* local! Let $\mathfrak{m}_\psi^\square$ be its maximal ideal. Analogously define R_Σ^ψ and $\mathfrak{m}_\Sigma$. Denote the maximal ideal of the local ring $R_{F,S}^{\square,\psi}$ by $\mathfrak{m}_{F,S}^\square$ and likewise that of $R_{F,S}^\psi$ by $\mathfrak{m}_{F,S}$.

There is a natural R_Σ^ψ-algebra structure on $R_{F,S}^\psi$ via the universal property of $\widehat{\otimes}_{\mathcal{O}}$. Indeed, for each $v\in\Sigma$, by restricting the universal deformation of V_k valued in $R_{F,S}^\psi$ to $G_v\subset G_{F,S}$ the universal property of R_v^ψ induces a canonical local $\mathcal{O}$-algebra morphism $R_v^\psi\to R_{F,S}^\psi$. We then use the universal property of completed tensor products.

Theorem. *For $i\geq 1$ let h_Σ^i (resp. c_Σ^i) denote the k-dimension of the kernel (resp. cokernel) of the map*

$$\theta_i:\mathrm{H}^i(G_{F,S},\mathrm{ad}^0 V_k)\to\prod_{v\in\Sigma}\mathrm{H}^i(G_v,\mathrm{ad}^0 V_k).$$

Then we have an isomorphism of R_Σ^ψ-algebras

$$R_{F,S}^\psi\cong R_\Sigma^\psi[\![x_1,\dots,x_r]\!]/(f_1,\dots,f_{r+s})$$

where $r=h_\Sigma^1$ and $s=c_\Sigma^1+h_\Sigma^2-h_\Sigma^1$.

To get the desired presentation, as in the proof of Theorem 1, first consider a surjection

$$\mathbb{B}:=R_\Sigma^\psi[\![x_1,\dots,x_r]\!]\twoheadrightarrow R_{F,S}^\psi$$

where $r = \dim_k \operatorname{coker}(\mathfrak{m}_\Sigma/(\mathfrak{m}_\Sigma^2, \pi) \to \mathfrak{m}_{F,S}/(\mathfrak{m}_{F,S}^2, \pi))$; this surjectivity uses completeness. Dualizing, we have

$$r = \dim_k \ker(\operatorname{Hom}_k(\mathfrak{m}_{F,S}/(\mathfrak{m}_{F,S}^2, \pi), k) \to \operatorname{Hom}_k(\mathfrak{m}_\Sigma/(\mathfrak{m}_\Sigma^2, \pi), k)).$$

Using the computation from Mok's lecture, this is

$$\dim_k \ker \theta_1 = h^1_\Sigma.$$

The key point that makes these computations work is that the completed tensor product represents the product of the functors represented by the R_v^ψ, which is most easily checked by computing on artinian points (for which the completed tensor product collapses to an ordinary tensor product). That then brings us down to the elementary fact that the tangent space of the product of functors is the product of the tangent spaces.

Denote by $\mathbf{m}$ the maximal ideal of $\mathbb{B}$, and by $\mathbb{J}$ the kernel $\ker(\mathbb{B} \twoheadrightarrow R_{F,S}^\psi)$. Now comes a delicate technical point. Like in the proof of Theorem 1, we can set-theoretically lift $\rho : G_{F,S} \to \mathrm{GL}_n(R_{F,S}^\psi)$ to $\widetilde{\rho} : G_{F,S} \to \mathrm{GL}_n(\mathbb{B}/\mathbf{m}\mathbb{J})$, not necessarily a homomorphism, and there arises the problem of finding a continuous such $\widetilde{\rho}$. We seek a better method than the trick as earlier with finite residue fields because we wish to later apply the same technique to future situations involving characteristic-0 deformation theory, for which the residue field is a p-adic field and not a finite field. The reader who prefers to ignore this problem should skip the next section.

5. Continuity nonsense

To explain the difficulty and its solution, let us first formulate a general situation. Consider a surjective map $R' \twoheadrightarrow R$ between complete local noetherian rings with kernel $\mathbb{J}$ killed by $\mathfrak{m}_{R'}$, and assume that we are in one of two cases:

Case 1: residue field k is finite of characteristic p, so R and R' are given the usual max-adic topologies that are profinite. These topologies are the inverse limits of the discrete topologies on artinian quotients.

Case 2 (to come up later!): residue field k is a p-adic field and R and R' are $\mathbb{Q}_p$-algebras, whence uniquely k-algebras in a compatible way (by Hensel). Their artinian quotients are then finite-dimensional as k-vector spaces, and so are naturally topologized as such (making them topological k-algebras, with transition maps that are quotient maps, as for any k-linear surjections between k-vector spaces of finite dimension). Give R and R' the inverse limit of those topologies (which induce the natural k-linear topologies back on the finite-dimensional artinian quotients).

In both cases, let $\rho : G \to \mathrm{GL}_n(R)$ be a continuous representation. We seek to make an obstruction class in a "continuous" $\mathrm{H}^2(G, \operatorname{ad}\rho)$ (over k) for measuring whether or not ρ can be lifted to a *continuous* representation into $\mathrm{GL}_n(R')$. The problem is to determine if ρ has a continuous set-theoretic lifting (moreover with with a fixed determinant if we wish to study deformations with a fixed determinant, assuming that p doesn't divide n).

We saw earlier how to handle Case 1 when R is *artinian*, by a trick. That trick rested on ρ at artinian level factoring through a *finite* quotient of G. Such an argument has no chance of applying when k is a p-adic field in interesting cases, and we're sure going to need that later when studying generic fibers of deformation rings and proving smoothness by proving vanishing of a p-adic H^2. So we need an improvement of the method from artinian Case 1 which addresses the following two points:

(i) what to do when k is p-adic,

(ii) how to incorporate additional things like working with a fixed determinant.

Actually, (ii) will be very simple once we see how to deal with (i), as we will see below. This is important because in practice we want to deal with more general constraints than just "fixed determinant" and so we want a general method which works for any "reasonable property", not just something ad hoc for the property of fixed determinant.

To deal with (i) (and along the way, (ii)), we will use a variant on fix from artinian Case 1. That argument allows us to reduce to deal with the case when R and R' are artinian, but we need to show in that *artinian setting* we can make a continuous set-theoretic lifting without the crutch of "factoring through finite quotient of G" (which is available for finite k but not p-adic k).

First conjugate so the reduction $\rho_0 : G \to \mathrm{GL}_n(k)$ lands in $\mathrm{GL}_n(\mathcal{O}_k)$. Then by using the method from Brian's talk on p-adic points of deformation rings, we can find a finite flat local $\mathcal{O}_k$-algebra $\mathcal{O}_k$-lattice A

inside of R with residue field equal to that of $\mathcal{O}_k$ and containing the compact $\rho(G)$, and then we can find a similar such A' in R' mapping onto A. We'd like to lift

$$\rho : G \to \mathrm{GL}_n(A)$$

to $\mathrm{GL}_n(A')$ set-theoretically in a continuous way. Note that $\mathrm{GL}_n(A') \to \mathrm{GL}_n(A)$ is surjective.

The point is that $\mathrm{GL}_n(A)$ and $\mathrm{GL}_n(A')$ are respectively open in $\mathrm{GL}_n(R)$ and $\mathrm{GL}_n(R')$ with subspace topologies that arise from the ones on A inside R and A' inside R' which are their natural topologies as finite free $\mathcal{O}_k$-modules. This makes them *profinite*, much as $\mathrm{GL}_n(R)$ and $\mathrm{GL}_n(R')$ were in the case of finite k. So we have reduced ourselves to the following situation, in which we will use an argument suggested by Lurie that also gives another approach for handling the case of finite k as well.

Let $H' \to H$ be a continuous surjective map of profinite groups, and $\rho : G \to H$ a continuous homomorphism. We claim that there is a continuous set-theoretic lifting $G \to H'$ of ρ that also respects properties like "fixed det" in the case of intended applications. To see this, let $F \twoheadrightarrow G$ be a surjection from a "free profinite group". The composite map

$$F \twoheadrightarrow G \to H$$

can be lifted continuously to $F \to H'$ even as a homomorphism by individually lifting from H to H' the images of each member of the "generating set" for the free profinite F. Those individual lifts can be rigged to have a desired det, or whatever other "reasonable homomorphic property" can be checked pointwise through a surjection, and so such a property is inherited by the map $F \to H'$. But what about $G \to H'$? If we can find a continuous set-theoretic section of $F \twoheadrightarrow G$ then composing that section with $F \to H'$ will give the required $G \to H'$. So our continuity problems will be settled once we prove the following fact.

Claim: If $f : G' \to G$ is a continuous homomorphism between profinite groups then it has a continuous section (as topological spaces).

Proof. For closed normal subgroups $N' \lhd G'$ and $N := f(N') =$ closed normal in G, consider continuous sections $s : G/N \to G'/N'$ to the induced quotient map $G'/N' \to G/N$ arising from f. For example, such an s exists if $N' = G'$ (so $N = G$). If (N', s) and (M', t) are two such pairs with N' containing M', say $(M', t) \geq (N', s)$ if

$$t : G/M \to G'/M' \qquad \textit{and} \qquad s : G/N \to G'/N'$$

are compatible via the projections $G/M \twoheadrightarrow G/N$ and $G'/M' \twoheadrightarrow G'/N'$.

I claim that the criterion for Zorn's Lemma is satisfied. Let $\{(N'_i, s_i)\}$ be a chain of such pairs, and let $N' = \bigcap N'_i$. Then the natural map

$$G'/N' \to \varprojlim G'/N'_i$$

is surjective (since an inverse limit of surjections $G'/N' \to G'/N'_i$ between compact Hausdorff spaces), yet also injective and thus a homeomorphism. Likewise, for $N := \bigcap N_i$ the map $G/N \to \varprojlim G/N_i$ a homeomorphism, and I claim that $N = f(N')$. Indeed, if x is in N then $f^{-1}(x)$ meets each N'_i in a non-empty closed set, and these satisfy the finite intersection property since $\{N'_i\}$ is a chain ordered by inclusion, so $f^{-1}(x)$ contains a point in the intersection N' of all N'_i. That says x is in $f(N')$ as desired. (The inclusion of $f(N')$ inside of N is clear.)

It follows that the compatible continuous sections $s_i : G_i/N_i \to G'_i/N'_i$ induced upon passing to the projective limit define a continuous section

$$s : G/N \to G'/N',$$

so (N', s') is an upper bound on the chain $\{(N'_i, s_i)\}$.

Now we apply Zorn's Lemma to get a maximal element (N', s). This is a continuous section $s : G/N \to G'/N'$ where $N = f(N')$. I claim $N' = \{1\}$, so we will be done. If not, then since $N' \cap U'$ for open normal subgroups U' in G' define a base of opens in N' around 1 (as N' gets its profinite topology as subspace topology from G'), there must exist such U' so that $N' \cap U'$ is a proper subgroup of N'. Replacing G' with $G'/(N' \cap U')$ and G with quotient by image of $N' \cap U'$ in G brings us to the case where N is finite and *non-trivial* yet (N', s) retains the maximality property (no continuous section using a proper [closed] subgroup of N' normal in G'). We seek a contradiction.

Since N' and N are finite, the quotient maps $q' : G' \twoheadrightarrow G'/N'$ and $q : G \twoheadrightarrow G/N$ are covering spaces with finite constant degree > 0. By total disconnectedness, these covering spaces admit sections. Composing s

with a section to q' gives a continuous section $G/N \to G'$ to

$$G' \xrightarrow{f} G \xrightarrow{q} G/N.$$

Composing such a section with q gives a continuous map $t : G \to G'$ so that $f(t(g)) = g \bmod N$, so by profiniteness of G and finiteness of N we get an open normal subgroup U in G such that for each representative g_i of G/U there exists $n_i \in N$ such that $f(t(g_i u)) = n_i g_i u$ for all $u \in U$. But $n_i = f(n_i')$, so replacing t on $g_i U$ with $(n_i')^{-1} t$ for each i gives a new t so that $f(t(g_i u)) = g_i u$ for all $u \in U$ and all i, which is to say $ft = \mathbf{1}_G$. This exhibits a continuous section t to f, contradicting that N was arranged to be nontrivial and maximal with respect to the preceding Zorn's Lemma construction. Hence, in fact N above is $\{1\}$ so we are done. □

6. Proof of Theorem 4

Returning to the situation of interest, we now have a continuous $\widetilde{\rho}$ that can even be arranged to satisfy $\det \widetilde{\rho} \equiv \psi \bmod \mathbf{m}\mathbb{J}$. Still following the argument from the proof of Theorem 1, define for $f \in \mathrm{Hom}_k(\mathbb{J}/\mathbf{m}\mathbb{J}, k)$ the continuous 2-cocycle φ_f as before, and observe that this time the determinant condition entails that $[\varphi_f] \in \mathrm{H}^2(G_{F,S}, \mathrm{ad}^0 V_k)$. The proof of the well-definedness of $[\varphi_f]$ is as before. Also we still have the equivalence that $[\varphi_f] = 0$ if and only if $\widetilde{\rho}$ can be chosen to be a homomorphism $\bmod \ker f$.

Now for the restriction of ρ to each G_v, we know we can find a continuous lift, namely coming from the universal representation ρ_v at v:

$$G_v \xrightarrow{\rho_v} \mathrm{GL}_n(R_v^\psi) \to \mathrm{GL}_n(R_\Sigma^\psi) \to \mathrm{GL}_n(\mathbb{B})$$

where the other maps are the obvious ones. Hence the class $[\varphi_f]|_{G_v} \in \mathrm{H}^2(G_v, \mathrm{ad}^0 V_k)$ is always trivial. In other words, we have a k-linear map $\mathrm{Hom}_k(\mathbb{J}/\mathbf{m}\mathbb{J}, k) \xrightarrow{\Phi} \ker \theta_2$ satisfying $f \mapsto [\varphi_f]$; the target has dimension h^2_Σ by definition. Therefore [easy exercise] it suffices to show that $\dim_k \ker \Phi \le c^1_\Sigma$. (All we need is the inequality, because we can always throw in extra trivial "relations" $f_i = 0$ into the denominator of $R^\psi_{F,S}$.)

Let $I = \ker(\mathfrak{m}_\Sigma/(\mathfrak{m}_\Sigma^2, \pi) \to \mathfrak{m}_{F,S}/(\mathfrak{m}_{F,S}^2, \pi))$. Then $\mathrm{Hom}_k(I, k) \cong \mathrm{coker}(\theta_1)$. So it is enough to construct a linear injection $\ker \Phi \hookrightarrow \mathrm{Hom}_k(I, k)$.

Step 1: Observe that $I = \ker(\mathbf{m}/(\mathbf{m}^2, \pi) \to \mathfrak{m}_{F,S}/(\mathfrak{m}_{F,S}^2, \pi))$ because we chose the x_i's to map onto a basis of $\mathrm{coker}(\mathfrak{m}_\Sigma/(\mathfrak{m}_\Sigma^2, \pi) \to \mathfrak{m}_{F,S}/(\mathfrak{m}_{F,S}^2, \pi))$. (In other words, none of the extra stuff in $\mathbf{m}$ dies when we map to $\mathfrak{m}_{F,S}$.)

Step 2: We next claim that $\mathbb{J}/\mathbf{m}\mathbb{J}$ surjects onto I. To prove this, first note that the map $\mathbb{J}/\mathbf{m}\mathbb{J} \to \mathbf{m}/(\mathbf{m}^2, \pi)$ comes from tensoring

$$0 \to \mathbb{J} \to \mathbf{m} \to \mathfrak{m}_{F,S} \to 0$$

over $\mathbb{B}$ with $\mathbb{B}/\mathbf{m}$ and then reducing mod π. We need to show that this map is surjective onto I. Fix $x \in I \subset \mathbf{m}/(\mathbf{m}^2, \pi)$. We know

$$\mathbb{J}/\mathbf{m}\mathbb{J} \twoheadrightarrow \ker(\mathbf{m}/\mathbf{m}^2 \to \mathfrak{m}_{F,S}/\mathfrak{m}_{F,S}^2).$$

We can lift x to $\widetilde{x} \in \mathbf{m}/\mathbf{m}^2$. Since x maps to zero in $\mathfrak{m}_{F,S}/(\mathfrak{m}_{F,S}^2, \pi)$, $\widetilde{x}$ maps to $\pi r \bmod \mathfrak{m}_{F,S}^2$ for some $r \in R^\psi_{F,S}$. But now we can just choose some $\widetilde{r} \in \mathbb{B}$ mapping to $r \in R^\psi_{F,S}$ (i.e. mod $\mathbb{J}$). Now replace $\widetilde{x}$ with $\widetilde{x} - (\pi \widetilde{r} \bmod \mathfrak{m}^2)$ so that $\widetilde{x}$ has vanishing image in $\mathfrak{m}_{F,S}/\mathfrak{m}_{F,S}^2$. That says $\widetilde{x}$ is in the image of $\mathbb{J}/\mathfrak{m}\mathbb{J}$ in $\mathfrak{m}/\mathfrak{m}^2$, so x is hit by $\mathbb{J}/\mathfrak{m}\mathbb{J}$ as desired.

Step 3: By Step 2 we get $\mathrm{Hom}_k(I, k) \hookrightarrow \mathrm{Hom}_k(\mathbb{J}/\mathbf{m}\mathbb{J}, k) \supset \ker \Phi$. So we need to show that $\ker \Phi \subset \mathrm{Hom}_k(I, k)$. In other words, if $[\varphi_f] = 0$ then we claim that $f : \mathbb{J}/\mathbf{m}\mathbb{J} \to k$ should factor through I, or equivalently vanish on $K = \ker(\mathbb{J}/\mathbf{m}\mathbb{J} \twoheadrightarrow I)$. Or equivalently, we need to show that $K = \mathbb{J} \cap (\mathbf{m}^2, \pi) \subset \mathbb{J}_f = \ker f$. But in fact this is really what we showed at the end of the proof of Theorem 1 when we showed property (4) of Φ.

7. The framed case

Let

$$\eta : \mathfrak{m}_\Sigma^\square/((\mathfrak{m}_\Sigma^\square)^2, \pi) \to \mathfrak{m}_{F,S}^\square/((\mathfrak{m}_{F,S}^\square)^2, \pi).$$

Then

$$R_{F,S}^{\square,\psi} \cong R_\Sigma^{\square,\psi}[\![x_1, \ldots, x_{r^\square}]\!]/(f_1, \ldots, f_{r^\square + s^\square}),$$

where $r^{\square} = \dim_k \operatorname{coker} \eta$ and $r^{\square} + s^{\square} = h^2_{\Sigma} + \dim_k \ker \eta$.

The proof is the same as in the unframed case, just with extra squares floating around all over the place. But now our H's have turned into Z's (that is, elements of the tangent space which were cohomology classes are now cocycles) so it's better to phrase the result as above.

8. Formulas for r's and s's

Theorem. *Suppose that $\{v|p\} \subset \Sigma$, that $\{v|\infty\} \subset S$, and that $S - \Sigma$ contains at least one finite prime. Then (with notation as above)*

$$s = \sum_{v|\infty, v \notin \Sigma} \dim_k (\operatorname{ad}^0 V_k)^{G_v}.$$

Remark. We also have $r^{\square} \geq \#\Sigma - 1, r^{\square} \stackrel{?}{=} r + \#\Sigma - 1, s^{\square} = s - \#\Sigma + 1$.

Proof. Let $Y = \operatorname{ad}^0 V_k$ and $X = Y^{\vee}(1)$. (In the notation of Rebecca's talk, $X = Y'$; it is written as a "twisted Pontrjagin dual" here because instead of being Hom into $\mathbb{Q}/\mathbb{Z}$ (trivial G-module) the target is given the action of the cyclotomic character.) Recall the end of the Poitou-Tate exact sequence (from Rebecca's talk)

$$\mathrm{H}^2(G_{F,S}, Y) \to \prod_{v \in S} \mathrm{H}^2(G_v, Y) \to \mathrm{H}^0(G_{F,S}, X)^{\vee} \to 0.$$

Split the product into two pieces:

$$\prod_{v \in S} \mathrm{H}^2(G_v, Y) = \prod_{v \in \Sigma} \mathrm{H}^2(G_v, Y) \times \prod_{v \in S - \Sigma} \mathrm{H}^2(G_v, Y).$$

The claim is that as long as the second factor is nonzero (which it is by hypothesis), it surjects onto $\mathrm{H}^0(G_{F,S}, X)^{\vee}$. Indeed, trivially $\mathrm{H}^0(G_{F,S}, X) \hookrightarrow \mathrm{H}^0(G_v, X)$ since restricting to the decomposition group gives more invariants. Dually, we have $\mathrm{H}^0(G_v, X)^{\vee} \twoheadrightarrow \mathrm{H}^0(G_{F,S}, X)^{\vee}$. But by the Tate pairing, $\mathrm{H}^0(G_v, X)^{\vee} \cong \mathrm{H}^2(G_v, Y)$. On each factor, the last map in the Tate-Poitou sequence is none other than the composition $\mathrm{H}^2(G_v, Y) \cong \mathrm{H}^0(G_v, X)^{\vee} \twoheadrightarrow \mathrm{H}^0(G_{F,S}, X)^{\vee}$. Thus the claim is true.

Now we do a little diagram chase. We have

$$\mathrm{H}^2(G_{F,S}, Y) \to \prod_{v \in \Sigma} \mathrm{H}^2(G_v, Y) \times \prod_{v \in S - \Sigma} \mathrm{H}^2(G_v, Y) \to \mathrm{H}^2(G_{F,S}, X)^{\vee} \to 0.$$

The claim is that $\mathrm{H}^2(G_{F,S}, Y) \twoheadrightarrow \prod_{v \in \Sigma} \mathrm{H}^2(G_v, Y)$. Indeed, given $(a_v)_{\Sigma} \in \prod_{v \in \Sigma} \mathrm{H}^2(G_v, Y)$, suppose its image in $\mathrm{H}^2(G_{F,S}, X)^{\vee}$ is γ. Since $\prod_{v \in S - \Sigma} \mathrm{H}^2(G_v, Y) \twoheadrightarrow \mathrm{H}^2(G_{F,S}, X)^{\vee}$, we can find

$$(b_v)_{S-\Sigma} \in \prod_{v \in S - \Sigma} \mathrm{H}^2(G_v, Y)$$

such that the image of $(b_v)_{S-\Sigma}$ in $\mathrm{H}^2(G_{F,S}, X)^{\vee}$ is $-\gamma$. Then

$$(a_v)_{\Sigma} \times (b_v)_{S-\Sigma} \in \ker(\prod_{S} \mathrm{H}^2(G_v, Y) \twoheadrightarrow \mathrm{H}^2(G_{F,S}, X)^{\vee}),$$

whence this tuple is in the image of $\mathrm{H}^2(G_{F,S}, Y)$. Projecting onto the $\prod_{v \in \Sigma}$ factor proves the claim. But the surjectivity of $\mathrm{H}^2(G_{F,S}, Y) \twoheadrightarrow \prod_{v \in \Sigma} \mathrm{H}^2(G_v, Y)$ says precisely that $c^2_{\Sigma} = \dim \operatorname{coker} \theta_2 = 0$.

Consequently we have $h^2_{\Sigma} = h^2(G_{F,S}, Y) - \sum_{v \in \Sigma} h^2(G_v, Y)$. So by the formulas at the end of Theorem 4,

$$s = -h^1_{\Sigma} + c^1_{\Sigma} + h^2_{\Sigma} = -h^1(G_{F,S}, Y) + \sum_{v \in \Sigma} h^2(G_v, Y) + h^2(G_{F,S}, Y) - \sum_{v \in \Sigma} h^2(G_v, Y).$$

Now recall that we have assumed throughout that $\operatorname{End}_{G_{F,S}} V_k = (\operatorname{ad} V_k)^{G_{F,S}} = k$ (since we need this to make sure the unframed deformation ring even exists!). In particular, $(\operatorname{ad}^0 V_k)^{G_{F,S}} = 0$. That is, $h^0(G_{F,S}, Y) = h^0(G_v, Y) = 0$. So we can add $h^0(G_{F,S}, Y) - \sum_{v \in \Sigma} h^0(G_v, Y)$ to s and nothing changes. But now we recognize from the equation above that in fact $s = \chi(G_{F,S}, Y) - \sum_{v \in \Sigma} \chi(G_v, Y)$.

We now invoke the Tate global Euler characteristic formula. [Reference: Milne, *Arithmetic Duality Theorems* Ch. I, Thm. 5.1.] We conclude that

$$\chi(G_{F,S}, Y) = \sum_{v|\infty} h^0(G_v, Y) - [F : \mathbb{Q}] \dim_k Y.$$

We also have for $v < \infty, v \nmid p$, that $\chi(G_v, Y) = 0$. For $v < \infty, v|p$, we have $\chi(G_v, Y) = -[F_v : \mathbb{Q}_p] \dim_k Y$. For $v|\infty$, we have $\chi(G_v, Y) = h^0(G_v, Y)$. One sees that in $s = \chi(G_{F,S}, Y) - \sum_{v \in \Sigma} \chi(G_v, Y)$, the degree contributions all cancel out, so there are no non-archimedean terms. Of the archimedean places, all those in Σ cancel as well, and we are left with the statement of the theorem. □