Lecture 8: Hecke algebras and Galois representations

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February, 2010

1. Z-Finiteness of Hecke Algebras

Let S_k denote the complex vector space $\mathrm{S}_k(\Gamma_1(N))$ of cusp forms of weight $k \geq 2$ on $\Gamma_1(N)$. Let $\mathbf{T}$ be the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbf{C}}(\mathrm{S}_k)$ generated by Hecke operators T_p for every prime p and diamond operators $\langle d \rangle$ for every $d \in (\mathbf{Z}/N\mathbf{Z})^{\times}$. In this section our aim is to prove that $\mathbf{T}$ is a finite free $\mathbf{Z}$-module. As it is clear that $\mathbf{T}$ is torsion-free, it is enough to show that $\mathbf{T}$ is a finitely generated $\mathbf{Z}$-module. We show this in Theorem 1.6.

We begin with some general constructions for any congruence subgroup Γ of $\mathrm{SL}_2(\mathbf{Z})$. Let $\{e, e'\}$ be a $\mathbf{C}$-basis for $\mathbf{C}^2$. The group Γ acts on $\mathbf{C}^2$ via the embedding $\mathrm{SL}_2(\mathbf{Z}) \hookrightarrow \mathrm{SL}_2(\mathbf{C})$ with respect to the basis $\{e, e'\}$: for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $c_1 e + c_2 e' \in \mathbf{C}^2$,

$$\gamma \cdot (c_1 e + c_2 e') = (ac_1 + bc_2)e + (cc_1 + dc_2)e'.$$

This action induces an action on $V_k := \mathrm{Sym}^{k-2}(\mathbf{C}^2)$.

Fix any z_0 in the upper half-plane $\mathfrak{h}$. Let f be any element of the $\mathbf{C}$-vector space $\mathrm{M}_k(\Gamma)$ of modular forms of weight k on Γ. We define the function $I_f : \Gamma \longrightarrow V_k$ by

$$I_f(\gamma) = \int_{z_0}^{\gamma z_0} (ze + e')^{k-2} f(z)dz \tag{1.1}$$

for every $\gamma \in \Gamma$.

Proposition 1.1. *The function I_f in (1.1) is a 1-cocycle and its class in $\mathrm{H}^1(\Gamma, V_k)$ is independent of z_0.*

Proof. First, we show that I_f is in $\mathrm{Z}^1(\Gamma, V_k)$. Let $\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and γ_2 be elements of Γ. Since $f|_k \gamma_1 = f$, we have

$$\begin{aligned} \gamma_1 \cdot I_f(\gamma_2) &= \int_{z_0}^{\gamma_2 z_0} ((az+b)e + (cz+d)e')^{k-2} f(z)dz, \\ &= \int_{z_0}^{\gamma_2 z_0} (\gamma_1(z)e + e')^{k-2} f(\gamma_1 z)\frac{dz}{(cz+d)^2}, \\ &= \int_{z_0}^{\gamma_2 z_0} (\gamma_1(z)e + e')^{k-2} f(\gamma_1 z)d(\gamma_1 z), \\ &= \int_{\gamma_1 z_0}^{\gamma_1\gamma_2 z_0} (ze + e')^{k-2} f(z)dz. \end{aligned} \tag{1.2}$$

It follows that

$$\gamma_1 \cdot I_f(\gamma_2) + I_f(\gamma_1) = \int_{\gamma_1 z_0}^{\gamma_1\gamma_2 z_0} (ze + e')^{k-2} f(z)dz + \int_{z_0}^{\gamma_1 z_0} (ze + e')^{k-2} f(z)dz = I_f(\gamma_1\gamma_2),$$

as desired.

Now we show that I_f modulo $\mathrm{B}^1(\Gamma, V_k)$ is independent of z_0. Choose $z_1 \in \mathfrak{h}$. For any $\gamma \in \Gamma$ the difference $\int_{z_0}^{\gamma z_0} (ze + e')^{k-2} f(z)dz - \int_{z_1}^{\gamma z_1} (ze + e')^{k-2} f(z)dz$ is equal to

$$\int_{\gamma z_1}^{\gamma z_0} (ze + e')^{k-2} f(z)dz - \int_{z_1}^{z_0} (ze + e')^{k-2} f(z)dz.$$

The calculations in (1.2) with γz_0 replaced by z_1 show that $\int_{\gamma z_1}^{\gamma z_0} (ze + e')^{k-2} f(z)dz = \gamma \cdot \int_{z_1}^{z_0} (ze + e')^{k-2} f(z)dz$. Hence, we see that the difference is a 1-coboundary.

■

By Proposition 1.1 we can define the $\mathbf{C}$-linear map

$$j : \mathrm{M}_k(\Gamma) \longrightarrow \mathrm{H}^1(\Gamma, V_k) \tag{1.3}$$

by $j(f) = I_f$, where I_f is given in (1.1).

Proposition 1.2. *Choose* $z_0 \in \mathfrak{h}$*. The restriction*

$$\begin{array}{rcl} j : \mathrm{S}_k(\Gamma) & \longrightarrow & \mathrm{H}^1(\Gamma, V_k) \\ f & \mapsto & \big(\gamma \mapsto \int_{z_0}^{\gamma z_0} (ze + e')^{k-2} f dz\big), \end{array}$$

of (1.3) *is injective.*

Proof. For any $h \in \mathrm{S}_k(\Gamma)$ consider the holomorphic map

$$(ze + e')^{k-2} h(z) : \mathfrak{h} \longrightarrow V_k.$$

Since $\mathfrak{h}$ is simply connected, we can choose a holomorphic function $G_h : \mathfrak{h} \longrightarrow V_k$ so that $dG_h = (ze + e')^{k-2} h(z) dz$. For any $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ we see that

$$\begin{aligned} d(G_h \sigma) &= G_h'(\sigma(z)) d\sigma(z)\,, \\ &= \big(\big(\frac{az+b}{cz+d}\big)e + e'\big)^{k-2} h(\sigma(z)) \frac{dz}{(cz+d)^2}\,, \\ &= ((az+b)e + (cz+d)e')^{k-2} (h|_k \sigma)(z) dz\,, \end{aligned}$$

where $(h|_k\sigma)(z) = (cz+d)^{-k} h(\sigma(z))$. Therefore, for every $\sigma \in \mathrm{SL}_2(\mathbf{Z})$ we have

$$G_h \sigma = \sigma \cdot G_{h|_k \sigma} + v_\sigma \tag{1.4}$$

for our fixed choice of antiderivative $G_{h|_k\sigma}$ of $(ze + e')^{k-2}(h|_k\sigma)$ and some $v_\sigma \in V_k$.

Let $\mathrm{SL}_2(\mathbf{Z})$ act on the holomorphic maps $G : \mathfrak{h} \longrightarrow V_k$ as follows:

$$(\sigma * G)(z) = \sigma \cdot (G\sigma^{-1}(z)).$$

For each member $\tilde{h}$ of $\mathrm{SL}_2(\mathbf{Z})$-orbit of h (under $\sigma \mapsto h|_k\sigma$) we choose an antiderivative $G_{\tilde{h}}$ as above, so by (1.4) for every $\sigma \in \mathrm{SL}_2(\mathbf{Z})$ we have

$$\sigma * G_h = G_{h|_k \sigma^{-1}} + c_\sigma \tag{1.5}$$

for some $c_\sigma \in V_k$.

Consider $f \in \mathrm{S}_k(\Gamma)$ in the kernel of j; that is, the 1-cocycle

$$\gamma \mapsto \int_{z_0}^{\gamma z_0} (ze + e')^{k-2} f(z) dz = G_f(\gamma z_0) - G_f(z_0)$$

is a 1-coboundary. Then, for every $\gamma \in \Gamma$ we have

$$G_f(\gamma z_0) - G_f(z_0) = \gamma \cdot v - v \tag{1.6}$$

for some $v \in V_k$. Our aim is to show that $f = 0$.

For $\gamma \in \Gamma$ the equation (1.5) becomes

$$\gamma * G_f = G_f + c_\gamma \tag{1.7}$$

for some $c_\gamma \in V_k$. We evaluate this equation at γz_0 and obtain that $c_\gamma = (\gamma * G_f)(z_0) - G_f(\gamma z_0)$. By using equation (1.6) we see that $c_\gamma = \gamma \cdot (G_f(\gamma^{-1} z_0) - v) - (G_f(z_0) - v)$. We may replace G_f with $G_f - (G_f(z_0) - v_\gamma)$, so (1.7) becomes

$$\gamma * G_f = G_f \tag{1.8}$$

for all $\gamma \in \Gamma$.

Recall that for the upper half-plane $\mathfrak{h}$, we topologize $\mathfrak{h}^* = \mathfrak{h} \cup \mathbf{P}^1(\mathbf{Q})$ using $\mathrm{SL}_2(\mathbf{Z})$-translates of bounded vertical strips

$$\{z \in \mathfrak{h} |\, \mathrm{Im}(z) > c,\, a < \mathrm{Re}(z) < b\}$$

for a, $b \in \mathbf{R}$ and $c > 0$. Now we prove the following claim.

Claim 1: As we approach any fixed cusp in $\mathfrak{h}^*$, the function G_f remains bounded in V_k.

Proof of Claim 1: Let $s \in \mathfrak{h}^*$ be any cusp and choose $\sigma \in \mathrm{SL}_2(\mathbf{Z})$ such that $\sigma(s) = \infty$. To prove the claim, it is enough to prove that $\sigma * G_f$ is bounded as we approach ∞ in $\mathfrak{h}$. By (1.5), this is just an antiderivative of $f|_k\sigma^{-1}$. Thus, it suffices to prove that each coefficient function of $(ze + e')^{k-2}(f|_k\sigma^{-1})(z)$ has bounded antiderivative as $\mathrm{Im}(z) \to \infty$ in any bounded vertical strip $\{z \in \mathfrak{h} | \, |\mathrm{Re}(z)| < a\}$ where $a \in \mathbf{R}^+$. Since $f \in \mathrm{S}_k(\Gamma)$, we have $(f|_k\sigma^{-1})(z) \in \mathrm{S}_k(\sigma\Gamma\sigma^{-1})$. Let $\tilde{f}(z) := (f|_k\sigma^{-1})(z)$. Since $\tilde{f}$ is a cusp form for $\sigma\Gamma\sigma^{-1}$, for any $a > 0$ there exists $c \in \mathbf{R}^+$ such that

$$|\bar{f}(z)| \ll e^{-c\mathrm{Im}(z)} \quad \text{as } \mathrm{Im}(z) \to \infty$$

uniformly for $|\mathrm{Re}(z)| < a$. Thus, for any $x \in [-a, a]$ and $y_0 \geq M > 0$ the coefficients of $G_{\tilde{f}}(x+iY) - G_{\tilde{f}}(x+iy_0)$ are linear combinations of terms $\int_{y_0}^{Y} y^r \tilde{f}(x+iy)dy$ with uniformly bounded coefficients. This integral is bounded above by $|P_r(Y)|e^{-cY} + |P_r(y_0)|e^{-cy_0}$, where P_r is a fixed polynomial of degree r, and as $Y \to \infty$ this tends to $|P_r(y_0)|e^{-cy_0}$ uniformly in $|x| \leq a$. This shows that each coefficient function of $(ze + e')^{k-2}(\tilde{f}(z))$ has bounded antiderivative as $\mathrm{Im}(z) \to \infty$ in the mentioned vertical strips. Hence, Claim 1 follows.

Using the $\mathrm{SL}_2(\mathbf{Z})$-invariant bilinear pairing $B : \mathbf{C}^2 \times \mathbf{C}^2 \longrightarrow \mathbf{C}$ defined by the determinant, we obtain the induced bilinear pairing

$$B_k : V_k \times V_k \longrightarrow \mathbf{C},$$

which is also $\mathrm{SL}_2(\mathbf{Z})$-invariant. For $\omega_f = (ze + e')^{k-2} f dz$, consider the 2-form

$$\begin{aligned} B_k(\omega_f, \bar{\omega}_f) &= (k-2)!\, |f|^2 \det(ze + e', \bar{z}e + e')^{k-2}\, dz \wedge d\bar{z}, \\ &= (k-2)!\, (2i)^{k-1}\, y^k\, |f|^2 \, \frac{dxdy}{y^2}, \end{aligned} \tag{1.9}$$

where $z = x + iy$. Since f is a cusp form, $B_k(\omega_f, \bar{\omega}_f)$ has finite integral over a fundamental domain F of Γ. Before computing this integral, we compute $B_k(\omega_f, \bar{\omega}_f)$ in another way.

Since $\omega_f = dG_f = g dz$ for $g = (ze + e')^{k-2} f$,

$$B_k(\omega_f, \bar{\omega}_f) = B_k(g, \bar{g})\, dz \wedge d\bar{z}.$$

But g is holomorphic, so $\frac{\partial g}{\partial \bar{z}} = 0$ and hence

$$B_k(g, \bar{g}) = \frac{\partial B_k(G_f, \bar{g})}{\partial z}.$$

Thus, we see that

$$B_k(\omega_f, \bar{\omega}_f) = \frac{\partial B_k(G_f, \bar{g})}{\partial z} dz \wedge d\bar{z} = d(B_k(G_f, \bar{g}) d\bar{z}).$$

By using this equality and Stoke's Theorem we obtain

$$\int_F B_k(\omega_f, \bar{\omega}_f) = \int_{\partial F} B_k(G_f, d\overline{G}_f). \tag{1.10}$$

Now, we want to compute $\int_{\partial F} B_k(G_f, d\overline{G}_f)$. To do this, for each cusp c we choose $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ such that $\gamma(c) = \infty$. We define the "loop" $R_{c,h}$ around c in F to be $\gamma^{-1}(L)$ where L is the horizontal segment joining the two edges at a common "height" h emanating from ∞ in $\gamma(F)$. Define the "closed disc" $D_{c,h} = \gamma^{-1}(U_L)$ where U_L is the closed vertical strip above L including ∞. Then, this integral is equal to

$$\lim_{h \to \infty} \Big(\int_{\partial(F - \cup_c D_{c,h})} B_k(G_f, d\overline{G}_f) + \sum_{c \in \{\text{cusps of } F\}} \int_{R_{c,h}} B_k(G_f, d\overline{G}_f) \Big). \tag{1.11}$$

To calculate the first integral in (1.11) we prove the following claim.

Claim 2: For any $\gamma \in \Gamma$, the pullback $\gamma^*(B_k(G_f, d\overline{G}_f))$ is equal to $B_k(G_f, d\overline{G}_f)$.

Proof of Claim 2: Let $\gamma \in \Gamma$. Since B_k is $\mathrm{SL}_2(\mathbf{Z})$-invariant, we have

$$\gamma^*(B_k(G_f, d\overline{G}_f)) = B_k(G_f\gamma, d(\overline{G_f\gamma})).$$

Since $\gamma \in \Gamma$, by (1.8) we see that $G_f = \gamma^{-1} * G_f$. With this equality we obtain $G_f\gamma = \gamma^{-1} \cdot G_f$. Thus, the above equality gives us

$$\begin{aligned}\gamma^*(B_k(G_f, d\overline{G}_f)) &= B_k(\gamma^{-1}\cdot G_f\,,\, d(\gamma^{-1}\cdot\overline{G}_f)),\\ &= B_k(\gamma^{-1}\cdot G_f\,,\, \gamma^{-1}\cdot d(\overline{G}_f)),\\ &= B_k(G_f, d\overline{G}_f).\end{aligned}$$

The last equality holds because B_k is $\mathrm{SL}_2(\mathbf{Z})$-invariant. Hence, Claim 2 follows.

By Claim 2, the integrals on edges L_1 and L_2 of F such that $L_1 = \gamma L_2$ for some $\gamma \in \Gamma$ cancel. That gives us

$$\int_{\partial(F-\cup_c D_{c,h})} B_k(G_f, d\overline{G}_f) = 0 \tag{1.12}$$

for any h. Now, consider any cusp c of F and the loop $R_{c,h}$ around it. We want to compute $\lim_{h\to\infty}\int_{R_{c,h}} B_k(G_f, d\overline{G}_f)$. Choose $\sigma \in \mathrm{SL}_2(\mathbf{Z})$ such that $\sigma(\infty) = c$. We have

$$\begin{aligned}\int_{R_{c,h}} B_k(G_f, d\overline{G}_f) &= \int_{\sigma^{-1}(R_{c,h})} \sigma^*(B_k(G_f, d\overline{G}_f)),\\ &= \int_{\sigma^{-1}(R_{c,h})} B_k(G_f\sigma, d\overline{G_f\sigma});\end{aligned} \tag{1.13}$$

the last equality holds because B_k is $\mathrm{SL}_2(\mathbf{Z})$-invariant. The loop $\sigma^{-1}(R_{c,h})$ is a loop $R_{\infty,h}$ around ∞ at height h. By equation (1.4), the function $G_f\sigma$ is just $\sigma\cdot G_{f|_k\sigma}$ up to translation by a constant in V_k. Thus, as B_k is $\mathrm{SL}_2(\mathbf{Z})$-invariant, instead of computing the limit with integral (1.13), we may compute it with $\int_{R_{\infty,h}} B_k(G_{f|_k\sigma}, d\overline{G}_{f|_k\sigma})$ with any choice of antiderivative $G_{f|_k}\sigma$. We do this by calculating the integrals of the $\{e, e'\}$-coefficients of the integrand.

By Claim 1, any antiderivative $G_{f|_k\sigma}$ is bounded in V_k as we approach ∞ in a bounded vertical strip, and $d\overline{G}_{f|_k\sigma}$ has an explicit formula in terms of the cusp form $\bar{f}|_k\sigma$. Thus, for any $a > 0$ there exists $b > 0$ such that

$$|\bar{f}|_k(z)| \ll e^{-b\mathrm{Im}(z)} \quad \text{as } \mathrm{Im}(z)\to\infty$$

uniformly for $|\mathrm{Re}(z)| < a$, so $\lim_{h\to\infty}\int_{R_{\infty,h}} B_k(G_f, d\overline{G_f}) = 0$. As a result, for each cusp c of F and the loop $R_{c,h}$ around it $\lim_{h\to\infty}\int_{R_{c,h}} B_k(G_f, d\overline{G}_f) = 0$. Hence,

$$\lim_{h\to\infty}\sum_{c\in\{\text{cusps of } F\}}\int_{R_{c,h}} B_k(G_f, d\overline{G}_f) = 0. \tag{1.14}$$

By (1.12) and (1.14), we see that the integral (1.10) becomes

$$\int_F B_k(\omega_f, \bar{\omega}_f) = 0.$$

In (1.9), we computed $B_k(\omega_f, \bar{\omega}_f)$ explicitly. Thus, this gives us

$$(k-2)!\,(2i)^{k-1}\int_F y^k\,|f|^2\,\frac{dxdy}{y^2} = 0.$$

The function inside the integral is nonnegative, so $f = 0$, as promised. ■

From now on, we assume that $\Gamma = \Gamma_1(N)$. By Proposition 1.2, we have injective $\mathbf{C}$-linear map

$$j : \mathrm{S}_k \hookrightarrow \mathrm{H}^1(\Gamma, V_k). \tag{1.15}$$

Now, we want to construct operators acting on $\mathrm{H}^1(\Gamma, V_k)$ compatible via j with the Hecke operators acting on S_k and preserving the $\mathbf{Z}$-structure on $\mathrm{H}^1(\Gamma, V_k)$. To do this we view Hecke operators acting on S_k as double cosets $\Gamma\alpha\Gamma$ where α is an element of

$$\Delta = \{\beta \in \mathrm{M}_2(\mathbf{Z}) \,|\, \det(\beta) > 0,\ \beta \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \bmod N\}. \tag{1.16}$$

It suffices to construct some T_α acting on $\mathrm{H}^1(\Gamma, V_k)$ for every $\alpha \in \Delta$ such that

(i) the map j in (1.15) carries $[\Gamma\alpha\Gamma]$-action on the left to T_α-action on the right,
(ii) T_α preserves the $\mathbf{Z}$-structure on $\mathrm{H}^1(\Gamma, V_k)$ coming from the one on V_k.

The following three lemmas give such T_α.

Lemma 1.3. *Choose $\alpha \in \Delta$ and coset representatives $\{\alpha_i\}$ for the left multiplication action of Γ in $\Gamma\alpha\Gamma$, so that $\Gamma\alpha\Gamma = \coprod_{i=1}^n \Gamma\alpha_i$. For every i and $\gamma \in \Gamma$, define $j[i]$ uniquely via $\alpha_i\gamma = \gamma_i\alpha_{j[i]}$. There is a well-defined operator*

$$\begin{aligned} T_\alpha : \mathrm{H}^1(\Gamma, V_k) &\longrightarrow \mathrm{H}^1(\Gamma, V_k). \\ c &\longmapsto (\gamma \mapsto \sum_{i=1}^n (\det \alpha)^{k-1} \alpha_i^{-1} \cdot c(\gamma_i)), \end{aligned}$$

which does not depend on the coset representatives.

Let $\Gamma_\alpha := \alpha^{-1}\Gamma\alpha \cap \Gamma$. Using the natural finite-index inclusion $\iota_1 : \Gamma_\alpha \hookrightarrow \Gamma$ and the finite-index inclusion $\iota_2 : \Gamma_\alpha \hookrightarrow \Gamma$ defined by $\iota_2(\beta) = \alpha\beta\alpha^{-1}$, the resulting composite map of the restriction and corestriction maps

$$\mathrm{H}^1(\Gamma, V_k) \xrightarrow[\text{along } \iota_2]{\text{Res}} \mathrm{H}^1(\Gamma_\alpha, V_k) \xrightarrow[\text{along } \iota_1]{\text{Cor}} \mathrm{H}^1(\Gamma, V_k)$$

is the operation T_α.

Proof. We first show that if we use another choice of coset representatives $\{\alpha_i'\}$ for Γ in $\Gamma\alpha\Gamma$, then the operator T_α on 1-cocycles (valued in 1-cochains) changes by 1-coboundaries. Consider

$$\alpha_i' = \tilde{\gamma}_i\alpha_i$$

where $\tilde{\gamma}_i \in \Gamma$ for every i. Since we have $\alpha_i\gamma = \gamma_i\alpha_{j[i]}$ for every i and $\gamma \in \Gamma$, with the new choice of coset representatives we obtain $\tilde{\gamma}_i^{-1}\alpha_i'\gamma = \gamma_i\tilde{\gamma}_{j[i]}^{-1}\alpha_{j[i]}'$. Writing $\gamma_i' := \tilde{\gamma}_i\gamma_i\tilde{\gamma}_{j[i]}^{-1}$, we get

$$\alpha_i'\gamma = \gamma_i'\alpha_{j[i]}'$$

for every i and $\gamma \in \Gamma$. With the new choice of coset representatives $\{\alpha_i'\}$, for $c \in \mathrm{Z}^1(\Gamma, V_k)$ and $\gamma \in \Gamma$ we have the equalities

$$
\begin{aligned}
\sum_{i=1}^{n}(\det\alpha)^{k-1}\alpha_i'^{-1}\cdot c(\gamma_i') &= \sum_{i=1}^{n}(\det\alpha)^{k-1}\alpha_i{}^{-1}\tilde{\gamma}_i^{-1}\cdot c(\tilde{\gamma}_i\gamma_i\tilde{\gamma}_{j[i]}^{-1}),\\
&= \sum_{i=1}^{n}(\det\alpha)^{k-1}\alpha_i^{-1}\tilde{\gamma}_i^{-1}\cdot c(\tilde{\gamma}_i)+\sum_{i=1}^{n}(\det\alpha)^{k-1}\alpha_i^{-1}\cdot c(\gamma_i\tilde{\gamma}_{j[i]}^{-1}),\\
&= \sum_{i=1}^{n}(\det\alpha)^{k-1}\alpha_i^{-1}\tilde{\gamma}_i^{-1}\cdot c(\tilde{\gamma}_i)+\sum_{i=1}^{n}(\det\alpha)^{k-1}\alpha_i^{-1}\gamma_i\cdot c(\tilde{\gamma}_{j[i]}^{-1})\\
&\quad+\sum_{i=1}^{n}(\det\alpha)^{k-1}\alpha_i^{-1}\cdot c(\gamma_i),\\
&= -\sum_{i=1}^{n}(\det\alpha)^{k-1}\alpha_i^{-1}\cdot c(\tilde{\gamma}_i^{-1})+\sum_{i=1}^{n}(\det\alpha)^{k-1}\gamma\,\alpha_{j[i]}^{-1}\cdot c(\tilde{\gamma}_{j[i]}^{-1})\\
&\quad+\sum_{i=1}^{n}(\det\alpha)^{k-1}\alpha_i^{-1}\cdot c(\gamma_i),\\
&= \sum_{i=1}^{n}(\det\alpha)^{k-1}\alpha_i^{-1}\cdot c(\gamma_i)+(\gamma\cdot v_0-v_0),
\end{aligned}
$$

where $v_0=\sum_{i=1}^{n}(\det\alpha)^{k-1}\alpha_i^{-1}\cdot c(\tilde{\gamma}_i^{-1})$. Hence, we have shown that the operator T_α on 1-cocycles does not depend on the chosen coset representatives if we view its values modulo $\mathrm{B}^1(\Gamma, V_k)$. Now, we want to show that it is a well-defined operator.

We choose coset representatives $\{\alpha_i\}$ for $\Gamma\backslash\Gamma\alpha\Gamma$ so that $\Gamma=\coprod\Gamma_\alpha(\alpha^{-1}\alpha_i)$. We can do this by [1, Lemma 5.1.2]. Since we have $\alpha_i\gamma=\gamma_i\alpha_{j[i]}$ for every $\gamma\in\Gamma$, we see that $(\alpha^{-1}\alpha_i)\gamma=(\alpha^{-1}\gamma_i\alpha)\alpha^{-1}\alpha_{j[i]}$. Since $\alpha^{-1}\alpha_i\in\Gamma$ for every i, we have $(\alpha^{-1}\alpha_i)\gamma(\alpha^{-1}\alpha_{j[i]})^{-1}\in\Gamma$. Thus, it follows from [2, p. 45] that

$$
\begin{aligned}
\mathrm{Cor}:\mathrm{H}^1(\Gamma,V_k) &\longrightarrow \mathrm{H}^1(\Gamma_\alpha,V_k),\\
c &\mapsto \Big(\gamma\mapsto\sum_{i=1}^{n}(\alpha^{-1}\alpha_i)^{-1}\cdot c((\alpha^{-1}\alpha_i)\gamma(\alpha^{-1}\alpha_{j[i]})^{-1}\\
&\qquad\qquad=\sum_{i=1}^{n}\alpha_i^{-1}\alpha\cdot c(\alpha^{-1}\gamma_i\alpha)\Big)
\end{aligned}
$$

where $\alpha_i\gamma=\gamma_i\alpha_{j[i]}$. To compute the restriction map along ι_2, observe that the isomorphism

$$
\begin{aligned}
V_k &\longrightarrow V_k\\
v &\mapsto \alpha\cdot v
\end{aligned}
$$

is equivariant for the Γ_α-action on the left-side and Γ-action on the right-side via the embedding ι_2. Thus, the restriction map is computed as follows

$$
\begin{aligned}
\mathrm{Res}:\mathrm{H}^1(\Gamma_\alpha,V_k) &\longrightarrow \mathrm{H}^1(\Gamma,V_k)\\
c &\mapsto \big(\gamma\mapsto\alpha^{-1}\cdot c(\alpha\,\gamma\,\alpha^{-1})\big).
\end{aligned}
$$

As a result, we see that the composite map Cor $\circ$ Res is the desired map. Hence, T_α is a well-defined action $\mathrm{H}^1(\Gamma, V_k)$.

■

Lemma 1.4. *The T_α-action on $\mathrm{H}^1(\Gamma, V_k)$ is induced by scalar extension of the analogous operation on $\mathrm{H}^1(\Gamma,\mathrm{Sym}^{k-2}(\mathbf{Z}^2))$.*

Proof. Since $k\geq 2$, we have $(\det\alpha)^{k-1}\alpha_i^{-1}=(\det\alpha)^{k-2}((\det\alpha)\,\alpha_i^{-1})$, with $(\det\alpha)\,\alpha_i^{-1}$ a matrix having $\mathbf{Z}$ entries. The result then follows from the cocycle formula for Γ_α.

■

Lemma 1.5. *Consider the action of T_α on $\mathrm{H}^1(\Gamma, V_k)$ that we defined in Lemma 1.3. The injective map j in (1.15) carries the $[\Gamma\alpha\Gamma]$-action on S_k over to the T_α-action on $\mathrm{H}^1(\Gamma, V_k)$ for every α in Δ as in (1.16).*

Proof. Choose $\alpha \in \Delta$ and coset representatives $\{\alpha_i\}$ for $\Gamma\backslash\Gamma\alpha\Gamma$, so $\Gamma\alpha\Gamma = \coprod_{i=1}^n \Gamma\alpha_i$. For $f \in \mathrm{S}_k$ we have $f|_k[\Gamma\alpha\Gamma] = \sum_{i=1}^n f|_k\alpha_i$. Now for each i and $\gamma \in \Gamma$, we compute $I_{f|_k\alpha_i}(\gamma)$ via (1.1):

$$\begin{aligned} I_{f|_k\alpha_i}(\gamma) &= \int_{z_0}^{\gamma z_0} (ze+e')^{k-2}(f|_k\alpha_i)dz, \\ &= \alpha_i^{-1} \cdot \int_{z_0}^{\gamma z_0} \alpha_i \cdot (ze+e')^{k-2}(f|_k\alpha_i)dz, \\ &= \alpha_i^{-1} \cdot (\det \alpha_i)^{k-1} \int_{\alpha_i z_0}^{\alpha_i\gamma z_0} (ze+e')^{k-2} f\, dz. \end{aligned}$$

The last equality follows by the calculations that are similar to the ones that we did in (1.2). Since for $\gamma \in \Gamma$ right multiplication by γ permutes $\Gamma\alpha_i$, for every i and $\gamma \in \Gamma$ there exists a unique $j[i]$ and $\gamma_i \in \Gamma$ such that $\alpha_i\gamma = \gamma_i\alpha_{j[i]}$. By using this equality we compute

$$\begin{aligned} I_{f|_k[\Gamma\alpha\Gamma]}(\gamma) &= (\det\alpha)^{k-1}\sum_{i=1}^n {\alpha_i}^{-1} \cdot \int_{\alpha_i z_0}^{\gamma_i\alpha_{j[i]}z_0} (ze+e')^{k-2} f\, dz, \\ &= (\det\alpha)^{k-1}\sum_{i=1}^n {\alpha_i}^{-1} \cdot \big(\int_{z_0}^{\gamma_i\alpha_{j[i]}z_0} (ze+e') f\, dz - \int_{z_0}^{\alpha_i z_0} (ze+e')^{k-2} f\, dz \big), \\ &= (\det\alpha)^{k-1}\sum_{i=1}^n {\alpha_i}^{-1} \cdot \big(\int_{\gamma_i z_0}^{\gamma_i\alpha_{j[i]}z_0} (ze+e') f\, dz + \int_{z_0}^{\gamma_i z_0} (ze+e') f\, dz \\ &\quad - \int_{z_0}^{\alpha_i z_0} (ze+e') f\, dz \big), \\ &= (\det\alpha)^{k-1}\sum_{i=1}^n {\alpha_i}^{-1} \cdot (\gamma_i \cdot \int_{z_0}^{\alpha_{j[i]}z_0} (ze+e') f\, dz + \int_{z_0}^{\gamma_i z_0} (ze+e') f\, dz \\ &\quad - \int_{z_0}^{\alpha_i z_0} (ze+e') f\, dz \,) \qquad \text{by similar calculations done in (1.2)}, \\ &= (\det\alpha)^{k-1}\big(\sum_{i=1}^n \gamma\, \alpha_{j[i]}^{-1} \cdot \int_{z_0}^{\alpha_{j[i]}z_0} (ze+e') f\, dz + \sum_{i=1}^n \alpha_i^{-1} \cdot \int_{z_0}^{\gamma_i z_0} (ze+e') f\, dz \\ &\quad - \sum_{i=1}^n \alpha_i^{-1} \cdot \int_{z_0}^{\alpha_i z_0} (ze+e') f\, dz \,\big) \qquad \text{since } \alpha_i^{-1}\gamma_i = \gamma\, \alpha_{j[i]}^{-1}, \\ &= (\det\alpha)^{k-1}\big(\sum_{i=1}^n \alpha_i^{-1} \cdot \int_{z_0}^{\gamma_i z_0} (ze+e') f\, dz + (\gamma\cdot v_1 - v_1) \big), \end{aligned}$$

where $v_1 = \sum_{i=1}^n \alpha_i^{-1} \cdot \int_{z_0}^{\alpha_i z_0} (ze+e') f\, dz$. Therefore, we see that for every $\alpha \in \Delta$ and $f \in \mathrm{S}_k$ we have the quality $j(f|_k[\Gamma\alpha\Gamma]) = T_\alpha(j(f))$ in $\mathrm{H}^1(\Gamma, V_k)$. Hence, the lemma follows.

■

Theorem 1.6. *Let $\mathbf{T}$ be the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbf{C}}(\mathrm{S}_k)$ generated by Hecke operators T_p for every prime p and diamond operators $\langle d\rangle$ for every $d \in (\mathbf{Z}/N\mathbf{Z})^\times$. Then $\mathbf{T}$ is finitely generated as a $\mathbf{Z}$-module.*

Proof. By Proposition 1.2, we have $\mathbf{C}$-linear injection

$$j : S_k \longrightarrow \mathrm{H}^1(\Gamma, V_k)$$

for $\Gamma = \Gamma_1(N)$. By Lemma 1.3, for every $\alpha \in \Delta$ (see (1.16)) we have a well-defined action T_α on $\mathrm{H}^1(\Gamma, V_k)$. By Lemma 1.5, the action T_α on $\mathrm{H}^1(\Gamma, V_k)$ is compatible with the action of $[\Gamma\alpha\Gamma]$ on S_k.

Let $\mathbf{T}'$ be the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbf{C}}(\mathrm{H}^1(\Gamma, V_k))$ generated by the T_α for every $\alpha \in \Delta$. Then, by Lemma 1.4, the $\mathbf{Z}$-algebra $\mathbf{T}'$ is in the image of the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbf{Z}}(\mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\mathbf{Z}^2)))$. Since $\mathrm{H}^1(\Gamma, \mathrm{Sym}^{k-2}(\mathbf{Z}^2))$ is a finitely generated $\mathbf{Z}$-module, $\mathbf{T}'$ is also a finitely generated $\mathbf{Z}$-module. By construction, the $\mathbf{T}'$-action on $\mathrm{H}^1(\Gamma, V_k)$ preserves S_k, so we get a restriction map

$$\nu : \mathbf{T}' \longrightarrow \mathrm{End}_{\mathbf{C}}(\mathrm{S}_k)$$

defined by $\nu(T) = T|_{\mathrm{S}_k}$ for every $T \in \mathbf{T}'$. The image of ν in $\mathrm{End}_{\mathbf{C}}(\mathrm{S}_k)$ is $\mathbf{T}$. Therefore, since $\mathbf{T}'$ is finitely generated $\mathbf{Z}$-module, $\mathbf{T}$ is finitely generated $\mathbf{Z}$-module.

■

2. Some Commutative Algebra

In this section we again assume that $\Gamma = \Gamma_1(N)$. Remember that we denote the $\mathbf{C}$-vector space $\mathrm{S}_k(\Gamma_1(N))$ of cusp forms of weight k on Γ by S_k. Let $\mathrm{S}_k(\Gamma, \mathbf{Q})$ be the space of cusp forms with in S_k with Fourier coefficients in $\mathbf{Q}$. By [4, Thm. 3.52], we know that S_k has a $\mathbf{C}$ basis that comes from $\mathrm{S}_k(\Gamma, \mathbf{Q})$ and so we have a surjection

$$\mathrm{S}_k(\Gamma, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C} \longrightarrow \mathrm{S}_k.$$

Actually, this basis also spans the $\mathbf{Q}$-vector space $\mathrm{S}_k(\Gamma, \mathbf{Q})$ and so this surjection is in fact an isomorphism. This “justifies” the following two definitions.

Definition 2.1. For any field F with characteristic 0,

$$\mathrm{S}_k(\Gamma, F) := \mathrm{S}_k(\Gamma, \mathbf{Q}) \otimes_{\mathbf{Q}} F.$$

Remember that $\mathbf{T}$ is the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbf{C}}(\mathrm{S}_k)$ generated by Hecke operators T_p for every prime p and diamond operators $\langle d \rangle$ for every $d \in (\mathbf{Z}/N\mathbf{Z})^\times$.

Definition 2.2. For any domain R with characteristic 0, we define

$$\mathbf{T}_R := \mathbf{T} \otimes_{\mathbf{Z}} R$$

acting on $\mathrm{S}_k(\Gamma, \mathrm{Frac}(R))$.

Remark 2.3. By Theorem 1.6 we know that $\mathbf{T}_R$ is a finite free R-module.

Let ℓ be a prime number. Fix an embedding $\overline{\mathbf{Q}} \subset \overline{\mathbf{Q}_\ell}$. Let K be a finite extension of $\mathbf{Q}_\ell$ in $\overline{\mathbf{Q}_\ell}$. Let $\mathcal{O}$ be its ring of integers and λ be its maximal ideal. Consider the finite flat $\mathcal{O}$-algebra $\mathbf{T}_{\mathcal{O}}$.

Proposition 2.4. *The minimal prime ideals of* $\mathbf{T}_{\mathcal{O}}$ *are those lying over the prime ideal* (0) *of* $\mathcal{O}$.

Proof. Let P be a minimal prime ideal of $\mathbf{T}_{\mathcal{O}}$. Since $\mathbf{T}_{\mathcal{O}}$ is a flat $\mathcal{O}$-algebra, the going down theorem holds between $T_{\mathcal{O}}$ and $\mathcal{O}$ (see [3, Thm. 9.5]). Therefore, $P \cap \mathcal{O} = (0)$. Now, suppose that P' is a prime ideal of $\mathbf{T}_{\mathcal{O}}$ such that $P' \subset P$ and $P' \cap \mathcal{O} = (0)$. As $\mathbf{T}_{\mathcal{O}}$ is an integral extension of $\mathcal{O}$, there are no strict inclusions between prime ideals lying over (0). Thus, $P' = P$. Hence, the proposition follows. ■

The K-algebra $\mathbf{T}_K$ is Artinian. Hence, it has only a finite number of prime ideals, all of which are maximal. By Proposition 2.4, the natural map

$$\mathbf{T}_{\mathcal{O}} \hookrightarrow \mathbf{T}_{\mathcal{O}} \otimes_{\mathcal{O}} K \cong \mathbf{T}_K$$

induces a bijection

$$\{\text{minimal prime ideals of } \mathbf{T}_{\mathcal{O}}\} \leftrightarrow \{\text{prime ideals of } \mathbf{T}_K\}. \tag{2.1}$$

Moreover, since $\mathcal{O}$ is complete, $T_{\mathcal{O}}$ is λ-adically complete and by [3, Thm. 8.15] there is an isomorphism

$$\mathbf{T}_{\mathcal{O}} \cong \prod_{\mathfrak{m}} \mathbf{T}_{\mathfrak{m}}.$$

The product is taken over the finite set of maximal ideals $\mathfrak{m}$ of $\mathbf{T}_{\mathcal{O}}$ and $\mathbf{T}_{\mathfrak{m}}$ denotes the localization of $\mathbf{T}_{\mathcal{O}}$ at $\mathfrak{m}$. Each $\mathbf{T}_{\mathfrak{m}}$ is a complete local $\mathcal{O}$-algebra which is finite free as an

$\mathcal{O}$-module. With this isomorphism we see that every prime ideal of $\mathbf{T}_{\mathcal{O}}$ is contained in the unique maximal ideal of $\mathbf{T}_{\mathcal{O}}$. Hence, we have a surjection

$$\{\text{minimal prime ideals of } \mathbf{T}_{\mathcal{O}}\} \twoheadrightarrow \{\text{maximal ideals of } \mathbf{T}_{\mathcal{O}}\}. \tag{2.2}$$

Let G_K be the absolute Galois group of K. Suppose $f = \sum a_n q^n$ is a normalized eigenform in $\mathrm{S}_k(\Gamma, \overline{K})$. Then $T \mapsto (T\text{−eigenvalue of } f)$ defines a ring map $\mathbf{T} \longrightarrow \overline{K}$ and so induces a K-algebra homomorphism $\Theta_f : \mathbf{T}_K \longrightarrow \overline{K}$. The image is the finite extension of K generated by the a_n and the kernel is a maximal ideal of $\mathbf{T}_K$ which depends only on the G_K-conjugacy class of f. Thus, we have the map

$$\varphi : \left\{ \begin{array}{c} \text{normalized eigenforms in} \\ \mathrm{S}_k(\Gamma, \overline{K}) \text{ modulo } G_K\text{−conjugacy} \end{array} \right\} \longrightarrow \{\text{maximal ideals of } \mathbf{T}_K\} \tag{2.3}$$

defined by $\varphi(f) = \mathrm{Ker}(\Theta_f)$.

Proposition 2.5. *The map φ in (2.3) is a bijection.*

Proof. For any maximal ideal $\mathfrak{m}$ of $\mathbf{T}_K$, all K-algebra embeddings $\mathbf{T}_K/\mathfrak{m} \hookrightarrow \overline{K}$ are obtained from a single one by composing with an element of G_K. Thus, we can make the identification

$$\{\text{maximal ideals of } \mathbf{T}_K\} = \mathrm{Hom}_{K-\mathrm{alg}}(\mathbf{T}_K, \overline{K})\big/(G_K\text{−action}).$$

Thus, to prove the proposition it is enough to show that the G_K-equivariant map

$$\psi : \{\text{normalized eigenforms in } \mathrm{S}_k(\Gamma, \overline{K})\} \longrightarrow \mathrm{Hom}_{K-\mathrm{alg.}}(\mathbf{T}_K, \overline{K})$$

defined by $\psi(f)(T) = (T\text{−eigenvalue of } f)$ is bijective. To do this, consider the $\overline{K}$-linear map

$$\begin{array}{rcl} \delta : \mathrm{S}_k(\Gamma, \overline{K}) & \longrightarrow & \mathrm{Hom}_{K-\mathrm{vsp}}(\mathbf{T}_K, \overline{K}) \\ f & \mapsto & \big(\alpha_f : T \mapsto a_1(Tf)\big). \end{array} \tag{2.4}$$

If we can show that δ is an isomorphism of $\overline{K}$-vector spaces, then we claim we are done. Because in (2.4) we claim that $f \in \mathrm{S}_k(\Gamma, \overline{K})$ is a normalized eigenform if and only if α_f is a ring homomorphism. To see this, suppose $f \in \mathrm{S}_k(\Gamma, \overline{K})$ is a normalized eigenform, so there exists a K-algebra homomorphism $\Theta_f : \mathbf{T}_K \to \overline{K}$ defined by $Tf = \Theta_f(T)f$ for every $T \in \mathbf{T}_K$. Clearly $\delta(f) = \alpha_f$ where

$$\alpha_f(T) = a_1(Tf) = a_1(\Theta_f(T)f) = \Theta_f(T)a_1(f) = \Theta_f(T)$$

for every $T \in \mathbf{T}_K$. Thus, α_f is a K-algebra homomorphism. Conversely, consider any K-algebra homomorphism $\alpha : \mathbf{T}_K \longrightarrow \overline{K}$, so $\alpha(T) = a_1(Tf)$ for some unique $f \in \mathrm{S}_k(\Gamma, \overline{K})$. Let $\lambda_n = \alpha(T_n)$ for every $T_n \in \mathbf{T}_K$. Then we have

$$a_1(T\,T_n f) = \alpha(T\,T_n) = \alpha(T)\alpha(T_n) = \lambda_n\, a_1(Tf) = a_1(T\,\lambda_n f)$$

for every $T \in \mathbf{T}_K$ and $n \geq 1$. Taking $T = T_m$ for every $m \geq 1$ gives $T_n f = \lambda_n f$ for every $n \geq 1$, proving that f is an eigenform. Moreover, as α is a K-algebra map, $1 = \alpha(\mathrm{id}) = a_1(f)$. Hence, f is a normalized eigenform in $\mathrm{S}_k(\Gamma, \overline{K})$.

Now, we will show that δ is an isomorphism of $\overline{K}$-vector spaces. For injectivity, suppose $\delta(f) = \alpha_f$ is the zero map, so $a_1(Tf) = 0$ for every $T \in \mathbf{T}_K$. In particular, $a_n(f) = a_1(T_n f) = 0$ for every $n \geq 1$, which implies that $f = 0$. To prove surjectivity of δ, it is enough to show that

$$\dim_{\overline{K}} \mathrm{Hom}_{K-\mathrm{vsp}}(\mathbf{T}_K, \overline{K}) \;\leq\; \dim_{\overline{K}} \mathrm{S}_k(\Gamma, \overline{K}). \tag{2.5}$$

Since $\mathrm{Hom}_{K-\mathrm{vsp}}(\mathbf{T}_K, \overline{K}) \cong \mathrm{Hom}_{\overline{K}}(\mathbf{T}_{\overline{K}-\mathrm{vsp}}, \overline{K})$, we can work with $\mathrm{Hom}_{\overline{K}}(\mathbf{T}_{\overline{K}-\mathrm{vsp}}, \overline{K})$. Actually, with this identification, studying the map δ is the same as studying the $\overline{K}$-bilinear mapping

$$\begin{array}{rcl} \mathrm{S}_k(\Gamma, \overline{K}) \times \mathbf{T}_K & \longrightarrow & \overline{K} \\ (f \quad , \quad T) & \mapsto & a_1(Tf) \end{array}$$

between finite-dimensional $\overline{K}$-vector spaces. Thus, to prove (2.5), it is enough to show that the map

$$\begin{array}{rcl} \epsilon : \mathbf{T}_{\overline{K}} & \longrightarrow & \mathrm{Hom}_{\overline{K}}(\mathrm{S}_k(\Gamma, \overline{K}), \overline{K}) \\ T & \mapsto & (f \to a_1(Tf)) \end{array}$$

is injective. Suppose $\epsilon(T)$ vanishes for some T. Thus, for every $f \in \mathrm{S}_k(\Gamma, \overline{K})$ and for every integer $n \geq 1$ we have $a_1(T_n T f) = a_1(T T_n f) = 0$. Therefore, $Tf = 0$ for every $f \in \mathrm{S}_k(\Gamma, \overline{K})$. Since $\mathbf{T}_{\overline{K}}$ acts faithfully on $\mathrm{S}_k(\Gamma, \overline{K})$, we get $T = 0$, proving that the map ϵ is injective. Hence, the proposition follows.

■

Combining the bijections (2.1) and (2.3) and the surjection (2.2), we have the following diagram.

$$\begin{array}{ccc} \{\text{minimal prime ideals of } \mathbf{T}_{\mathcal{O}}\} & \twoheadrightarrow & \{\text{maximal ideals of } \mathbf{T}_{\mathcal{O}}\} \\ \updownarrow & & \\ \{\text{prime ideals of } \mathbf{T}_K\} & & \\ \updownarrow & & \\ E = \left\{ \begin{array}{c} \text{normalized eigenforms in} \\ \mathrm{S}_k(\Gamma, \overline{K}) \text{ modulo } G_K\text{−conjugacy} \end{array} \right\} & & \end{array} \tag{2.6}$$

Let $\mathfrak{m}$ be any maximal ideal of $\mathbf{T}_{\mathcal{O}}$, so $\mathfrak{m}$ is the kernel of a map $\Phi : \mathbf{T}_{\mathcal{O}} \longrightarrow \overline{\mathbf{F}}_\ell$. We want to attach a residual representation $\bar{\rho}_{\mathfrak{m}}$ over $\overline{\mathbf{F}}_\ell$ to $\mathfrak{m}$ using the diagram (2.6). Let $\{f_1, ..., f_r\}$ be a set of representatives of all normalized eigenforms in E such that in the diagram (2.6) their corresponding minimal prime ideals $\wp_{f_i}$ in $\mathbf{T}_{\mathcal{O}}$ are inside the maximal ideal $\mathfrak{m}$. For each i, let $\wp'_{f_i}$ be the corresponding prime ideal in $\mathbf{T}_K$, so $\wp'_{f_i} \cap \mathbf{T}_{\mathcal{O}} = \wp_{f_i}$. Thus, for each i, we have a map

$$\begin{array}{rcl} \Theta_{f_i} : \mathbf{T}_{\mathcal{O}} & \longrightarrow & \overline{\mathcal{O}} \\ T_n & \mapsto & a_n(f_i) \end{array}$$

with kernel $\wp_{f_i}$. Since each $\wp_{f_i} \subset \mathfrak{m}$, the map $\Phi : \mathbf{T}_{\mathcal{O}} \longrightarrow \overline{\mathbf{F}}_\ell$ factors through $\mathrm{Im}\,\Theta_{f_i}$ for each i as follows,

$$\begin{array}{ccccc} & & \mathrm{Im}\,\Theta_{f_1} & & \\ & \nearrow & \vdots & \searrow & \\ \mathbf{T}_{\mathcal{O}} & & \longrightarrow & & \overline{\mathbf{F}}_\ell. \\ & \searrow & \vdots & \nearrow & \\ & & \mathrm{Im}\,\Theta_{f_r} & & \end{array}$$

For each i, the quotient $\mathbf{T}_K/\wp'_{f_i}$ is a finite extension K_{f_i} of K. Let $\mathcal{O}_{f_i}$ be its ring of integers and k_{f_i} be its residue field. Each map $\mathrm{Im}\,\Theta_{f_i} \longrightarrow \overline{\mathbf{F}}_\ell$ lifts to $\mathcal{O}_{f_i}$, lifting the embedding of the residue field of $\mathrm{Im}\,\Theta_{f_i}$ to an embedding of k_{f_i} into $\overline{\mathbf{F}}_\ell$. The above commutative diagram tells us that for every integer $n \geq 1$, we have

$$\overline{a_n(f_1)} = \ldots = \overline{a_n(f_r)}$$

in $\overline{\mathbf{F}}_\ell$. Consider the semisimplified residual representation $\bar{\rho}_{f_i}$ associated to each f_i; it is defined over k_{f_i}. For every prime p such that $p \not| N\ell$ we have

$$\mathrm{tr}(\bar{\rho}_{f_1}(\mathrm{Frob}_p)) = \ldots = \mathrm{tr}(\bar{\rho}_{f_r}(\mathrm{Frob}_p))$$

over $\overline{\mathbf{F}}_\ell$. We obtain a similar result for the determinants of $\bar{\rho}_{f_i}(\mathrm{Frob}_p)$'s when we compare the characters $\bar{\chi}_{f_i}$ associated to f_i's. Therefore, we obtain

$$\bar{\rho}_{f_1} \cong \ldots \cong \bar{\rho}_{f_r}$$

over $\overline{\mathbf{F}}_\ell$. We let $\bar{\rho}_{\mathfrak{m}}$ denote this common residual representation.

3. The Main Theorem

In this section we prove the following theorem.

Theorem 3.1. *Let K be a finite extension of $\mathbf{Q}_\ell$ such that its ring of integers $\mathcal{O}$ is big enough to contain all Hecke eigenvalues at level N. Let λ be its maximal ideal, k its residue field and $\mathfrak{m}$ a maximal ideal of $\mathbf{T}_{\mathcal{O}}$. Consider the associated residual representation*

$$\bar{\rho}_{\mathfrak{m}} : \mathrm{G}_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(k)$$

over k. Assume $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible. Then there exists a unique deformation

$$\rho_{\mathfrak{m}} : \mathrm{G}_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2((\mathbf{T}_{\mathfrak{m}})_{\mathrm{red}})$$

such that

(1) *$\rho_{\mathfrak{m}}$ is unramified at every prime p such that $p \nmid N\ell$,*
(2) *For every prime p such that $p \nmid N\ell$, the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_p)$ is $x^2 - \mathbf{T}_p x + p^{k-1}\langle p\rangle$.*

Before proving this theorem, consider the following theorem which was proved by Akshay in his talk. The corollary of this theorem will be the main ingredient while proving Theorem 3.1.

Theorem 3.2. *Let R be a complete local Noetherian ring and let $\rho : \mathrm{G}_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(R)$ be a residually absolutely irreducible representation. If S is a complete local Noetherian subring of R which contains all the traces of ρ, then the Galois representation ρ is conjugate to a representation $\mathrm{G}_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(S)$.*

Corollary 3.3. *Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbf{Q}_\ell$, with maximal ideal λ and residue field k. Let Σ be a finite set of places of $\mathbf{Q}$ containing ℓ. Let $\rho : \mathrm{G}_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(R)$ be the universal deformation unramified outside Σ for an absolutely irreducible representation $\bar{\rho} : \mathrm{G}_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(k)$ unramified outside Σ, taken on the category of complete local Noetherian $\mathcal{O}$-algebras with residue field k. The traces $\mathrm{tr}(\rho(\mathrm{Frob}_p))$ for all but finitely many primes $p \notin \Sigma$ generate a dense $\mathcal{O}$-subalgebra of R.*

Proof. Let M_R be the maximal ideal of R. By succesive approximation, it is enough to show that such $\mathrm{tr}(\rho(\mathrm{Frob}_p))$ generate $R/(\lambda, M_R^2)$ as k-algebras. Let $R_1 := R/(\lambda, M_R^2)$. The ring R_1 is the universal deformation ring for $\bar{\rho}$ for k-algebras with residue field k such that the square of the maximal ideal is zero. Let S be a k-subalgebra of R_1 generated by $\mathrm{tr}(\rho(\mathrm{Frob}_p))$ for almost all primes $p \notin \Sigma$. Being a subring of R_1, the square of the maximal ideal of S is also zero. If we can show that $R_1 = S$, then we're done.

By Theorem 3.2 we have the following commutative diagram (up to conjugation) which lifts $\bar{\rho}$

$$\begin{array}{ccc} \mathrm{G}_{\mathbf{Q}} & \longrightarrow & \mathrm{GL}_2(S) \\ & \overset{\rho_1}{\searrow} & \downarrow \\ & & \mathrm{GL}_2(R_1) \end{array}$$

Also, since R_1 is the universal deformation ring of $\bar{\rho}$ we have the following commutative diagram (up to conjugation) which lifts $\bar{\rho}$

$$\begin{array}{ccc} \mathrm{G}_{\mathbf{Q}} & \xrightarrow{\rho_1} & \mathrm{GL}_2(R_1) \\ & \searrow & \downarrow \\ & & \mathrm{GL}_2(S) \end{array}$$

As a result we have the following composition of maps

$$R_1 \longrightarrow S \hookrightarrow R_1$$

which carries ρ_1 to itself and hence is the identity map. Thus, $S = R_1$. ■

Proof of Theorem 3.1. Let f be a normalized eigenform in $S_k(\Gamma, \overline{K})$ such that the corresponding minimal prime ideal $\mathfrak{p}_f$ in $\mathbf{T}_{\mathcal{O}}$ is contained in $\mathfrak{m}$ (see diagram (2.6)). By Deligne, we have a Galois representation ρ_f over $\mathcal{O}$ associated to f whose residual reduction is $\bar{\rho}_{\mathfrak{m}}$:

$$\begin{array}{ccc} \mathrm{G}_{\mathbf{Q}} & \xrightarrow{\rho_f} & \mathrm{GL}_2(\mathcal{O}) \\ & \searrow^{\bar{\rho}_{\mathfrak{m}}} & \downarrow \\ & & \mathrm{GL}_2(k) \end{array}$$

Let $(R, \rho : \mathrm{G} \longrightarrow \mathrm{GL}_2(R))$ be the universal deformation of $\bar{\rho}_{\mathfrak{m}}$ unramified outside $N\ell$. Then ρ_f corresponds to an $\mathcal{O}$-algebra map $R \longrightarrow \mathcal{O}$, so the diagram

$$\begin{array}{ccc} \mathrm{G}_{\mathbf{Q}} & \xrightarrow{\rho} & \mathrm{GL}_2(R) \\ & \searrow^{\rho_f} & \downarrow \\ & & \mathrm{GL}_2(\mathcal{O}) \end{array}$$

commutes up to conjugation by $1 + \mathrm{M}_2(\lambda)$ in $\mathrm{GL}_2(\mathcal{O})$. By Corollary 3.3, we see that the set of $\mathrm{tr}(\rho(\mathrm{Frob}_q))$ for every prime $q \nmid N\ell$ generates a dense $\mathcal{O}$-subalgebra in R.

Consider the map

$$\begin{array}{rcl} \eta : \ R & \longrightarrow & \prod_{\mathfrak{p}_f \subset \mathfrak{m}} \mathcal{O} \\ \mathrm{tr}(\rho(\mathrm{Frob}_q)) & \mapsto & \prod_{\mathfrak{p}_f} a_q(f) \end{array}$$

where the product is taken over minimal primes $\mathfrak{p}_f$ contained in $\mathfrak{m}$, with f the corresponding normalized eigenform in $\mathrm{S}_k(\Gamma, \overline{K})$. Consider the embedding

$$\begin{array}{rcl} (\mathbf{T}_{\mathfrak{m}})_{\mathrm{red}} & \hookrightarrow & \prod_{\mathfrak{p}_f \subset \mathfrak{m}} \mathbf{T}_{\mathcal{O}}/\mathfrak{p}_f \\ T_q & \mapsto & \prod_{\mathfrak{p}_f} T_q \pmod{\mathfrak{p}_f}. \end{array}$$

With the identification

$$\begin{array}{rcl} \prod_{\mathfrak{p}_f \subset \mathfrak{m}} \mathcal{O} & = & \prod_{\mathfrak{p}_f \subset \mathfrak{m}} \mathbf{T}_{\mathcal{O}}/\mathfrak{p}_f \\ \prod_{\mathfrak{p}_f} a_q(f) & \mapsto & \prod T_q \pmod{\mathfrak{p}_f}, \end{array}$$

we see that all $\mathrm{tr}(\rho(\mathrm{Frob}_q))$ for $q \nmid N\ell$ land in the closed subalgebra $(\mathbf{T}_{\mathfrak{m}})_{\mathrm{red}}$. Since they generate dense algebra in R, the ring R also lands in there under η, say inducing $h : R \longrightarrow (\mathbf{T}_{\mathfrak{m}})_{\mathrm{red}}$. Thus, we get

$$\rho_{\mathfrak{m}} : \mathrm{G}_{\mathbf{Q}} \xrightarrow{\rho} \mathrm{GL}_2(R) \xrightarrow{h} \mathrm{GL}_2((\mathbf{T}_{\mathfrak{m}})_{\mathrm{red}}).$$

This gives existence and also uniqueness since any other $\rho'_{\mathfrak{m}}$ would give another map $h' : R \longrightarrow (\mathbf{T}_{\mathfrak{m}})_{\mathrm{red}}$ and compatibility with traces of representations then forces $\mathrm{tr}(\rho(\mathrm{Frob}_q)) \mapsto T_q$. Thus, h and h' coincide on a dense set, hence $h = h'$. By checking in each $\mathbf{T}_{\mathcal{O}}/\mathfrak{p}_f = \mathcal{O}$, we see that $\rho_{\mathfrak{m}}(\mathrm{Frob}_q)$ has the expected characteristic polynomial for every $q \nmid N\ell$.

4. Reduced Hecke Algebras

In this section, let K be a finite extension of $\mathbf{Q}_\ell$ and $\mathcal{O}$ its ring of integers. For any ring A, let $\widetilde{\mathbf{T}}_A$ be the A-subalgebra of $\mathbf{T}_A$ generated by the Hecke operators T_p for $p \nmid N\ell$ and diamond operators $\langle d \rangle$ for every $d \in (\mathbf{Z}/N\mathbf{Z})^\times$. Fix a maximal ideal $\mathfrak{m}$ of $\widetilde{\mathbf{T}}_\mathcal{O}$. We have a map $\widetilde{\mathbf{T}}_\mathcal{O} \longrightarrow \overline{\mathbf{F}}_\ell$ with kernel $\mathfrak{m}$. Since $\mathbf{T}_\mathcal{O}$ is an integral extension of $\widetilde{\mathbf{T}}_\mathcal{O}$ and $\overline{\mathbf{F}}_\ell$ is algebraically closed, this map can be extended to $\mathbf{T}_\mathcal{O}$. Let $\mathfrak{m}'$ be the kernel of this extended map, so it is a maximal ideal of $\mathbf{T}_\mathcal{O}$. Consider common (up to isomorphism) residual representation $\bar{\rho}_f$ for all normalized eigenforms f whose corresponding minimal primes $\mathfrak{p}_f$ (see (2.6)) are contained in $\mathfrak{m}'$. Call it $\bar{\rho}_\mathfrak{m}$. In this section we prove the following theorem.

Theorem 4.1. *If the Serre conductor $\mathcal{N}(\bar{\rho}_\mathfrak{m})$ is equal to N then the $\mathcal{O}$-algebra $(\widetilde{\mathbf{T}}_\mathcal{O})_\mathfrak{m}$ is reduced.*

Proof. Since the Serre conductor $\mathcal{N}(\bar{\rho}_\mathfrak{m})$ is equal to N, the minimal possible level of a normalized eigenform f such that $\bar{\rho}_f \simeq \bar{\rho}_\mathfrak{m}$ over $\overline{\mathbf{F}}_\ell$ is N. Thus, such f are newforms. To prove the theorem, we will show that $(\widetilde{\mathbf{T}}_\mathcal{O})_\mathfrak{m} \otimes_\mathcal{O} K$, which contains $(\widetilde{\mathbf{T}}_\mathcal{O})_\mathfrak{m}$, is reduced. We have the equality

$$(\widetilde{\mathbf{T}}_\mathcal{O})_\mathfrak{m} \otimes_\mathcal{O} K = \prod_{\mathfrak{p}_K} (\widetilde{\mathbf{T}}_K)_{\mathfrak{p}_K}$$

where the product is taken over all prime ideals $\mathfrak{p}_K$ of the Artinian ring $\widetilde{\mathbf{T}}_K$ such that $\mathfrak{p}_K \cap \widetilde{\mathbf{T}}_\mathcal{O} \subset \mathfrak{m}$ and $(\widetilde{\mathbf{T}}_K)_{\mathfrak{p}_K}$ denotes the localization of $\widetilde{\mathbf{T}}_K$ at $\mathfrak{p}_K$. Thus, each $\mathfrak{p}_K$ in the product corresponds to a newform. To prove the theorem it is therefore enough to show that $(\widetilde{\mathbf{T}}_K)_\mathfrak{p}$ is a field when $\mathfrak{p}$ corresponds to a newform.

Assume the prime ideal $\mathfrak{p}$ of $\widetilde{\mathbf{T}}_K$ corresponds to a newform $f \in \mathrm{S}_k(\Gamma, K)$ of level N. We can increase K to a finite extension. Thus, without loss of generality we can assume that K is big enough to contain the Hecke eigenvalues of all normalized eigenforms at level N. Since $\mathrm{S}_k(\Gamma, K)$ is faithful $\widetilde{\mathbf{T}}_K$-module, the localization $(\mathrm{S}_k(\Gamma, K))_\mathfrak{p}$ at $\mathfrak{p}$ is faithful $(\widetilde{\mathbf{T}}_K)_\mathfrak{p}$-module. If we can prove that $(\mathrm{S}_k(\Gamma, K))_\mathfrak{p}$ is one dimensional as a vector space over K then we are done, because this would force $(\widetilde{\mathbf{T}}_K)_\mathfrak{p}$ to be equal to K.

We have

$$\mathrm{S}_k(\Gamma, K) = K\,f \oplus \big(\bigoplus_g \mathrm{S}_g(\Gamma, K)\big)$$

where the direct sum is taken over all newforms g of level N_g and $\mathrm{S}_g(\Gamma, K)$ is spanned by $g(vz)$ for the divisors v of N/N_g. By multiplicity one, for every g which is different from f, there exists a prime $q \nmid N\ell$ such that

$$a_q(g(vz)) = a_q(g(z)) \neq a_q(f(z))$$

for every $v|(N/N_g)$. We know that $(T_q - a_q(f)) \in \mathfrak{p}$ and it acts on $g(vz)$ as

$$\begin{aligned}(T_q - a_q(f))g(vz) &= T_q(g(vz)) - a_q(f)g(vz)\\ &= (a_q(g) - a_q(f))g(vz).\end{aligned}$$

By the above argument, $(a_q(g) - a_q(f)) \in K^\times$. But $(\mathbf{T}_K)_\mathfrak{p}$ is Artin local, so its maximal ideal is nilpotent. This forces $(\bigoplus_{g\neq f} \mathrm{S}_g(\Gamma, K))_\mathfrak{p} = 0$. As a result, $(\mathrm{S}_k(\Gamma, K))_\mathfrak{p} = Kf$ and the theorem follows. ■

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