Lecture 16: Review of representation theory

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In the first (and main) part of these notes, I review the representation theory we have done this semester, highlighting the points that are of most relevance to us. Then I will state a modularity lifting theorem and make a few remarks about how the rerpesentation theory is used in the proof. In my next talk, I will give an outline of the proof of this modularity lifting theorem.

1. Representation theory

The lectures we have had on representation theory centered around these topics:

- The theory of admissible representations of $\mathrm{GL}(2, \mathbf{Q}_p)$ (or more generally, $\mathrm{GL}(2, F)$ with $F/\mathbf{Q}_p$ a finite extension).
- The theory of automorphic representations of GL(2); in particular, the correspondence between Hecke eigenforms in the classical sense and automorphic representations.
- The Jacquet-Langlands correspondence, relating automorphic forms on GL(2) with those on a division algebra.
- Base change, relating automorphic forms on GL(2) over two different fields (one a solvable extension of the other).

I will go through each of these four topics and remind us of the key points for our applications. I will also throw in some material abouth the Langlands correspondence (both local and global) that we may not have covered.

1.1. Admissible representations.

Let $F/\mathbf{Q}_p$ be a finite extension and let G be the group $\mathrm{GL}(2, F)$. Fix an algebraically closed field K of characteristic zero (one always takes K to be the complex numbers or the closue of some $\mathbf{Q}_\ell$). A representation of G on a K-vector space V is *smooth* if the stabilizer of any vector in V is an open subgroup of G; it is *admissible* if it is smooth and for every open subgroup U of G the space V^U is finite dimensional. We are most interested in irreducible admissible representations. Here "irreducible" has its usual sense: the only stable subspaces are 0 and the whole space.

An easy way to construct admissible representations is through induction. Let $\alpha, \beta : F^\times \to K^\times$ be two continuous characters. Continuity amounts to the condition that the restriction of α and β to the group of units U_F should factor through a finite quotient of U_F. Let $V = V(\alpha, \beta)$ be the space of all locally constant functions $f : G \to K$ which satisfy the identity

$$f\left(\begin{pmatrix} a & x \\ & b \end{pmatrix} g\right) = \alpha(a)\beta(b)\left|\frac{a}{b}\right|^{1/2} f(g)$$

for all $a, b \in F^\times$, $x \in F$ and $g \in G$. We let G act on V by right translation: $(gf)(g') = f(g'g)$. It is quite easy to see that this makes V into an admissible representation of V. A more difficult result is the following: if $\alpha\beta^{-1}$ is not equal to $|\cdot|$ or $|\cdot|^{-1}$ then V is irreducible. Here $|\cdot|$ is the norm character of $F^\times$, which takes $a \in F^\times$ to $q^{-\operatorname{val} a}$ where q is the cardinality of the residue field. These irreducible admissible representations are called the *principal series*.

When $\alpha\beta^{-1}$ is equal to $|\cdot|^{\pm 1}$ the representation $V(\alpha, \beta)$ is no longer irreducible. Rather, it is indecomposable and has two Jordan-Holder constituents. One of these constituents is one dimensional while the other is infinite dimensional. Precisley, say $\alpha\beta^{-1} = |\cdot|$ and write $\alpha = \gamma|\cdot|^{1/2}$ and $\beta = \gamma^{-1}|\cdot|^{-1/2}$. Then $V(\alpha, \beta)$ has a unique irreducible subrepresentation $\mathrm{St}(\gamma)$ which is infinite dimensional. The quotient $V(\alpha, \beta)/\mathrm{St}(\gamma)$ is one dimensional and G acts on it through the character $g \mapsto \gamma(\det g)$. Write St in place of $\mathrm{St}(\gamma)$ where γ is the trivial character. The representation St is called the *Steinberg representation*. One has $\mathrm{St}(\gamma) = \mathrm{St} \otimes \gamma$.

We have thus completely analyzed the representations $V(\alpha, \beta)$. There are many irreducible admissible representations of G which do not appear inside of these reperesentations, however; these are called the *supercuspidal representations* of G. We now have the following classification of the irreducible representations of G.

Theorem 1.1. *Let V be an irreducible admissible representation of G over K. Then V is equivalent to one and only one of the following:*

- *An irreducible principal series* $V(\alpha, \beta)$ *with* $\alpha\beta^{-1} \neq |\cdot|^{\pm 1}$.
- *A one dimensional representation corresponding to a character* $g \mapsto \gamma(\det g)$.
- *A twist* $\mathrm{St} \otimes \gamma$ *of the Steinberg representation* St.
- *A supercuspidal representation.*

This theorem almost follows by our definition of supercuspidal. The one part that does not is its assertion that the principal series and twists of Steinbergs are inequivalent. The one dimensional representations are often counted as principal series. We will sometimes treat them as such and sometimes not.

An irreducible admissible representation V of G is called *unramified* if it has a vector which is invariant under the maximal compact subgroup $\mathrm{GL}(2, \mathscr{O}_F)$. It is a theorem that V is unramified if and only if it is a principal series of the form $V(\alpha, \beta)$ with α and β unramified characters of $F^\times$ (where here unramified means trivial on U_F), or a one dimensional principal series given by $g \mapsto \gamma(\det g)$ with γ unramified. Note that an unramified character of $F^\times$ is determined by a single number, namely, its value on any uniformizer.

Key points: (1) The irreducible admissible representations of G fall into three classes: principal series, twists of Steinberg and supercuspidal. (2) The unramified representations of G are exactly the principal series representations coming from unramified characters. These are parameterized by (unordered) pairs of numbers (elements of $K^\times$).

1.2. **The local Langlands correspondence.** Keep the notation of the previous section. We have an exact sequence

$$0 \to I_F \to \mathrm{Gal}(\overline{F}/F) \stackrel{\mathrm{val}}{\to} \widehat{\mathbf{Z}} \to 0$$

where I_F is the inertia subgroup of the Galois group. The *Weil group* of F is by definition the subgroup of $\mathrm{Gal}(\overline{F}/F)$ given by $\mathrm{val}^{-1}(\mathbf{Z})$. We call a representation of W_F on a K-vector space V *Frobenius semi-simple* if some fixed Frob in W_F acts semi-simply. Recall that a *Weil-Deligne representation* of F with coefficients in K is a pair (V, N) where:

- V is a K vector space with an action of W_F which is Frobenius semi-simple and under which inertia acts through a finite quotient.
- N is an endomorphism of V which satisfies
$$gNg^{-1} = q^{\mathrm{val}\, g} N$$
where q denotes the cardinality of the residue field of F. Equivalently, N defines a W_F-equivariant map $V(1) \to V$ where $V(1)$ is the twist of V by the character $g \mapsto q^{\mathrm{val}\, g}$.

The collection of all Weil-Deligne representations forms a category and this cateogry is abelian. The following theorem is not difficult:

Theorem 1.2. *Let* $\ell \neq p$ *be a prime number. There is then an equivalence of categories:*

$$\left\{ \begin{array}{l} \textit{Weil-Deligne representations} \\ \textit{with coefficients in } \overline{\mathbf{Q}}_\ell \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \textit{Continuous Frobenius semi-simple representa-} \\ \textit{tions of } W_F \textit{ on } \overline{\mathbf{Q}}_\ell \textit{ vector spaces} \end{array} \right\}$$

Sketch of proof. Let (V, N) be a Weil-Deligne representation. Let ρ denote the action of W_F on V. Define a new representation ρ' of W_F on V by

$$\rho'(\mathrm{Frob}^n g) = \rho(\mathrm{Frob}^n g) \exp(N t_\ell(g)).$$

Here $\mathrm{Frob} \in W_F$ is a fixed Frobenius element, g is an element of the inertia subgroup I_F of W_F and $t_\ell : I_F \to \mathbf{Z}_\ell$ is the tame ℓ-adic character. One easily verifies that ρ' is a continuous Frobenius semi-simple representation. We have thus defined a map of categories. One must then check that it is in fact an equivalence, which is not difficult. □

It is not difficult to classify two dimensional Weil-Deligne representations:

Theorem 1.3. *Let* (V, N) *be a two dimensional Weil-Delgine representation of* F *with coefficients in* K*. Then* (V, N) *falls into exactly one of the following three cases:*

- V *is a direct sum of two characters of* W_F *and* $N = 0$.
- V *is irreducible under* W_F *and* $N = 0$.
- V *is a direct sum* $W \oplus W(1)$ *where* W *is one dimensional (and thus acted on by a character* γ *of* W_F*);* N *kills* $W(1)$ *and maps* W *isomorphically onto* $W(1)$.

We can now state a version of the local Langlands correspondence for GL(2).

Theorem 1.4. *There is a natural bijection*

$$\left\{\begin{array}{l}\textit{Irreducible admissible represen-}\\ \textit{tations of } \mathrm{GL}(2,F) \textit{ over } K\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\textit{Two dimensional Weil-Deligne repre-}\\ \textit{sentations with coefficients in } K\end{array}\right\}.$$

Under thie bijection, the principal series correspond to direct sums of characters, the supercuspidals to irreducibles and the twists of Steinberg to the Weil-Deligne representations with non-zero N. More precisely, the principal series $V(\alpha,\beta)$ corresponds to the representation $\alpha' \oplus \beta'$ where α' and β' correspond to α and β by class field theory. One can make a similar statement for twists of Steinberg.

Key points: (1) Two dimensional Weil-Deligne representations fall into three classes. (2) There is a natural bijection between two dimensional Weil-Deligne representations and irreducible admissible representations of $\mathrm{GL}(2,F)$. This bijection preserves the trichotomy on each side and on principal series and twists of Steinberg can be computed in terms of class field theory. (3) Weil-Deligne representations basically correspond to continuous ℓ-adic representations of the Weil group for any $\ell \neq p$, and these are almost the same thing as representations of the absolute Galois group.

1.3. **Automorphic representations.** Now let F be a number field and let $\mathbf{A}_F$ be its adele ring. An *automorphic form* on GL(2) over F is a function $f : \mathrm{GL}(2,\mathbf{A}_F) \to \mathbf{C}$ satisfying a number of properties, the most important of which is that it is invariant on the left under $\mathrm{GL}(2,F)$. The set of all automorphic forms forms a vector space $\mathscr{A}_F$. This vector space carries an action of $\mathrm{GL}(2,\mathbf{A}_F^f)$ by right translation. Furthermore, the Lie algebra and the maximal compact of $\mathrm{GL}(2,F_\infty)$ act on $\mathscr{A}_F$ (that is, $\mathscr{A}_F$ is a Harish-Chandra module for $\mathrm{GL}(2,F_\infty)$). (The full group $\mathrm{GL}(2,F_\infty)$ does not act on $\mathscr{A}_F$ as it destroys the K-finiteness condition.) An *automorphic representation* of $\mathrm{GL}(2,\mathbf{A}_F)$ is something of the form $\pi_f \otimes \pi_\infty$ where π_f is an irreducible admissible representation of $\mathrm{GL}(2,\mathbf{A}_F^f)$ and π_∞ is an irreducible Harish-Chandra module of $\mathrm{GL}(2,F_\infty)$ such that $\pi_f \otimes \pi_\infty$ is equivalent to a submodule of $\mathcal{A}_F$. There is a certain condition called *cuspidal* that one can impose on automorphic forms. The set of all cuspidal forms forms a vector subspace $\mathscr{A}_F^\circ$ of $\mathscr{A}_F$ which is stable under the various actions of pieces of $\mathrm{GL}(2,\mathbf{A}_F)$. An automorphic representation is *cuspidal* if it appears inside this cuspidal space.

Say for the moment that $F = \mathbf{Q}$. As we have discussed earlier in the semester, classical modular eigenforms correspond bijectively to automorphic representations π for which π_∞ is a discrete series representation. More precisely, say f is a newform of level N and weight k and let π be the corresponding automorphic representation. We can then write $\pi = \pi_f \otimes \pi_\infty$ and further decompose π_f as a restricted tensor product $\otimes\pi_p$, where π_p is an irreducible admissible representation of $\mathrm{GL}(2,\mathbf{Q}_p)$. The Harish-Chandra module π_∞ is completely determined by the weight k. For primes p not dividing the level, π_p is an unramified representation of $\mathrm{GL}(2,\mathbf{Q}_p)$. As we have seen, such representations are determined by two numbers; the representation π_p corresponds to the eigenvalues of the Hecke operators T_p and $T_{p,p}$ acting of f. (There is a precise formula to take these two numbers and produce two characters α and β of $\mathbf{Q}_p^\times$ such that π_p is equivalent to $V(\alpha,\beta)$.) For primes p dividing N the representation π_p is *not* unramified. I imagine that it is possible to determine π_p from a classical point of view; however, this is probably a bit complicated. This is one of the main advantages of the formulation in terms of automorphic representations: the information at ramified primes is more readily accessible.

When $F \neq \mathbf{Q}$ the discussion of the previous paragraph carries over but is a bit more complicated. The reason that it becomes more complicated is that the corresponding classical picture becomes more complicated. For example, in the setting of Hillbert modular forms the space which plays the role of the modular curve can be disconnected: it will be a disjoint union of spaces of the form $\mathfrak{h}^n/\Gamma_i$ where $\mathfrak{h}$ is the upper half plane and the Γ_i are certain arithmetic groups. The proper analogue of a modular form is then a tuple (f_i) where f_i is a function on $\mathfrak{h}^n$ invariant under Γ_i. The Hecke operators then permute the f_i in addition to acting in the usual fashion. This additional bookkeeping required makes the classical point of view much more cumbersome to deal with. It is another reason for swithcing to the representation theoretic perspective.

Key points: (1) Classical modular forms correspond to automorphic representations of $\mathrm{GL}(2,\mathbf{A}_\mathbf{Q})$ satisfying a certain condition at infinity. (2) Automorphic representations are built out of irreducible admissible representations at each finite place and a Harish-Chandra module at infinity. Almost all of these irreducible admissible representations are unramified and the two parameters that determine them correspond to the two Hecke eigenvalues in the classical picture. (3) Automorphic representations are much better to deal with for certain applications: even in the most basic case of classical modular forms they give easier access to information at ramified primes; in more complicated situations, they remove the cumbersome bookkeeping that is present in the classical picture.

1.4. **The global Langlands correspondence.** Let f be a modular form on the upper half plane of weight k and level N which is an eigenform for the Hecke operators T_p and $T_{p,p}$ away from N. As we discussed in the first semester, there is a Galois representation

$$\rho_{f,\ell} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

which satisfies and is uniquely determined by the following property: if $p \neq \ell$ is a prime not dividing N then $\rho_{f,\ell}$ is unramified at p and the characteristic polynomial of $\rho_{f,\ell}(\mathrm{Frob}_p)$ is given by $T^2 - a_pT + a_{p,p}$ where a_p and $a_{p,p}$ are the eigenvalues of f under T_p and $T_{p,p}$. The representation $\rho_{f,\ell}$ is "odd," that is, its determinant on a complex conjugation is -1.

As we have seen, in certain situations it is better to use automorphic representations in place of modular forms. This is one of those situations! The above result can be generalized and refined, and to state the improved version it is better to use automorphic representations. Let F be a totally real number field and let π be an automorphic representation of $\mathrm{GL}(2, \mathbf{A}_F)$ such that π_∞ is a discrete series representation. Then there is a Galois representation

$$\rho_{\pi,\ell} : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$$

which satisfies and is uniquely determined by the following property: if v is a place of F which does not lie above ℓ then $\rho_{\pi,\ell}|G_{F,v}$ corresponds to π_v under the local Langlands correspondence. The representation $\rho_{\pi,\ell}$ is also odd: its determinant on any complex conjugation is -1. (Note that the condition that π_∞ be discrete series is equivalent to the condition that the corresponding classical modular form be holomorphic.)

The above result is clearly more general than the first one since it permits F to be a totally real field rather than just $\mathbf{Q}$. However, even for $F = \mathbf{Q}$ it is a stronger result: it specified the local Galois representation everywhere except at ℓ in terms of the corresponding local component of the automorphic representation. The local Galois representation at ℓ is much more subtle: it is not determined by the corresponding component of the automorphic representation.

It is expected that the $\rho_{\pi,\ell}$ give all the Galois representations which are odd, ramified at finitely many places and satisfy some local condition at ℓ (coming from ℓ-adic Hodge theory). This has basically been proved for $F = \mathbf{Q}$ but is still open for all other F. The most critical intermediate result in the proof for $F = \mathbf{Q}$ is a modular lifting theorem; we will prove such a theorem in this seminar.

Key point: Given an automorphic representation π of a toally real number field which is discrete series at infinity, there is a corresponding Galois representation $\rho_{\pi,\ell}$. (Or rather, one for each ℓ.) The restriction of $\rho_{\pi,\ell}$ to a decomposition group away from ℓ corresponds to the local component of π under the local Langlands correspondence. Furthermore, $\rho_{\pi,\ell}$ is an odd representation.

1.5. **The Jacquet-Langlands correspondence.** Let F be a number field. Let G be the algebraic group $\mathrm{GL}(2)$ over F. Let D be a quaternion algebra over F and let G' be its unit group, regarded as an algebraic group (so $G'(A) = (D \otimes_F A)^\times$). One then has the notion of an automorphic representation of G'. The global Jacquet-Langlands correspondence is the following theorem:

Theorem 1.5. *The is a natural bijection:*

$$\{\text{Automorphic representations of } G'\} \leftrightarrow \left\{\begin{array}{l}\text{Automorphic representations of } G\\ \text{which are essentially square integrable}\\ \text{at all places where } D \text{ ramifies}\end{array}\right\}$$

(An irreducible admissible representation of $\mathrm{GL}(2, F_v)$ *is* essentially square integrable *if it is a twist of the Steinberg or supercuspidal, i.e., not principal series.) Furthermore, if π' is an automorphic representation of G' and π the corresponding automorphic representation of G then π_v is determined completely by π'_v. Two special cases: (1) if D splits at v and we identify D_v with $M_2(F_v)$ then π'_v is identified with π_v; (2) if π'_v is the trivial representation then π_v is the Steinberg representation.*

Assume that D is ramified at all infinite places; this is the case we care most about. For a compact open subgroup U of $(D \otimes \mathbf{A}_F^f)^\times$ let $S_2(U)$ denote the space of all functions

$$D^\times \backslash (D \otimes \mathbf{A}_F^f)^\times / U \to \mathbf{C}.$$

Note that the double quotient above is a finite set; we really do mean *all* possible functions, there is no possible continuity condition to impose. For a place v of F at which U is maximal compact and D is split there is a natural Hecke operator T_v that acts on $S_2(U)$. The Jacquet-Langlands correspondence implies that if f is a parallel weight 2 holomorphic cuspidal Hilbert eigenform whose associated automorphic representation is essentially square integrable at the places where D is ramified then there is an element g of $S_2(U)$ which is an eigenvector for all the Hecke operators and has the same eigenvalues as f. (Here U is determined from the level of f.) Therefore, as long as we are in a situation where the appropriate local conditions are in place, we can work with $S_2(U)$ instead of the space of Hilbert modular forms. This is advantageous because functions on a finite set are very easy to think about! For instance, there is an obvious integral structure on $S_2(U)$ (take integral valued functions) and so the notion of a mod p modular form on D is evident.

Key points: (1) One can move automorphic forms and representations between GL(2) and quaternion algebras; the only obstructions are local and fairly simple. (2) By taking D to be ramified at infinity, automorphic forms on D can be thought of as functions on a finite set.

1.6. **Base change.** Let π be an automorphic representation of $\mathrm{GL}(2, \mathbf{A}_F)$ with F a number field, such that π_∞ is discrete series. As we have seen, there is then an associated Galois representation $\rho_{\pi,\ell}$. Given an extension F'/F we can restrict $\rho_{\pi,\ell}$ to $G_{F'}$. This is the sort of Galois representation that we expect is of the form $\rho_{\pi',\ell}$ for some automorphic representation π' of $\mathrm{GL}(2, \mathbf{A}_{F'})$. The automorphic representation π' has been proven to exist when the extension F'/F is solvable. Precisely we have the following:

Theorem 1.6. *Let F'/F be a solvable extension of number fields. There is a natural map of sets*

$$\mathrm{BC} : \left\{ \begin{matrix} \text{Automorphic representations} \\ \text{of } \mathrm{GL}(2, \mathbf{A}_F) \end{matrix} \right\} \to \left\{ \begin{matrix} \text{Automorphic representations} \\ \text{of } \mathrm{GL}(2, \mathbf{A}'_F) \end{matrix} \right\}$$

such that if $\pi' = \mathrm{BC}(\pi)$ then: (1) the local component π'_v can be computed in terms of π_v; (2) if π_∞ is discrete series then so is π' and $\rho_{\pi',\ell} = \rho_{\pi,\ell}|_{G_{F'}}$.

There is a local base change map also: if F'_v is a finite extension of F_v then there is a base change map BC from irreducible admissible representations of $\mathrm{GL}(2, F_v)$ to those of $\mathrm{GL}(2, F'_v)$. In fact, the meaning of (1) in the above theorem is precisely that $\pi'_v = \mathrm{BC}(\pi_{\mathrm{v}})$. Thus local and global base change are compatible. The local base change map satisfies a property analogous to (2) above, namely, it commutes with the local Langlands correspondence.

From the above properties of local base change, and what we know about local Langlands, it is easy to see some examples of how local base change works. For example, the principal series $V(\alpha, \beta)$ corresponds under local Langlands to the Galois representation $\alpha' \oplus \beta'$ where α' and β' correspond to α and β under class field theory. Restricting this to $G_{F'_v}$ we simply get $\alpha'|_{G_{F'_v}} \oplus \beta'|_{G_{F'_v}}$. Going the other way under local Langlands, this corresponds to the principal series $V(\alpha'', \beta'')$ where α'' and β'' correspond to $\alpha'|_{G_{F'_v}}$ and $\beta|_{G_{F'_v}}$ under class field theory. Now, class field theory turns restriction to a larger number field into composition with the norm. Thus $\alpha'' = N^*\alpha$ and $\beta'' = N^*\beta$, where $N : (F')^\times \to F^\times$ is the norm map. We thus find

$$\mathrm{BC}(V(\alpha, \beta)) = V(N^*\alpha, N^*\beta).$$

The base change of a principal series is always a principal series. Similarly, the base change of a twist of Steinberg is again a twist of Steinberg — restricting to a bigger field will never turn a non-zero N zero or vice versa. By this reasoning, the base change of a supercuspidal will never be a twist of Steinberg. However, an irreducible Galois representation can certainly restrict to a reducible one. Thus it is possible for the base change of a supercuspidal to be principal series. In fact, if π is any irreducible admissible representation of $\mathrm{GL}(2, F_v)$ then one can find an extension F'_v/F_v such that $\mathrm{BC}(\pi)$ is either unramified or Steinberg. Any base change of Steinberg is still Steinberg, however.

The above local discussion has the following global application (when compbined with some global class field theory). Given an automorphic representation π of $\mathrm{GL}(2, \mathbf{A}_F)$ there exists a finite solvable Galois extension F'/F such that the base change of π to F' is everywhere unramified or Steinberg. In fact, if F is totally real (as it will be in our applications) then F' can be taken to be totally real as well.

There is a sort of converse to base change that will be useful for us, which we refer to as *solvable descent.*

Theorem 1.7. *Let F be a totally real number field and let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be a Galois representation. Assume that there exists a finite, totally real, solvable extension F'/F and a parallel weight 2 automorphic representation π' of $\mathrm{GL}(2, \mathbf{A}_{F'})$ such that $\rho|_{G_{F'}} = \rho_{\pi',p}$ and both are irreducible. Then there exists a parallel weight 2 automorphic representation π of $\mathrm{GL}(2, \mathbf{A}_F)$ such that $\rho = \rho_{\pi,p}$.*

In other words: if ρ becomes modular over a solvable extension then ρ is modular.

Key points: (1) There is an operation ("base change") on automorphic representations and local representations which corresponds to restriction on the Galois side, at least for solvable extensions. (2) Given an automorphic representation, one can always make a solvable base change such that the result is either unramified or Steinberg at all places. One cannot get rid of Steinbergs through base change, however. (3) Given a Galois representation, one can check if it comes from an automorphic form by checkings over a solvable extension (subject to some technicalities).

2. Modularity lifting

We will now state a modularity lifting theorem that we will later use and indicate how base change and the Jacquet-Langlands correspondence are used in the proof. We must first make some Galois theoretic definitions.

Let $F/\mathbf{Q}_p$ be a finite extension. We say that a Galois representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is *ordinary* if it is of the form

$$\begin{pmatrix} \alpha\chi_p & * \\ & \beta \end{pmatrix}$$

where α and β are finitely ramified characters, and, as always, χ_p denotes the p-adic cycloctomic character. (One could allow for more general definitions of ordinary, replacing χ_p by χ_p^n; for now we will stick with this one.) Let E/F be an extension over which α and β become unramified. The representation $\rho|_{I_E}$ is an extension of the trivial representation by χ_p and so defines an element of $H^1(I_E, \overline{\mathbf{Q}}_p(\chi_p))$, which is identified with $\overline{\mathbf{Q}}_p \otimes (E^{\mathrm{un}})^\times$ by Kummer theory. (Here E^{un} is the maximal unramified extension of E and I_E is the inertia subgroup of G_E.) We say that ρ is *potentially crystalline* if this class belongs to $\overline{\mathbf{Q}}_p \otimes \mathscr{O}_{E^{\mathrm{un}}}^\times$. This is idendependent of the choice of E.

Now let $F/\mathbf{Q}$ be a finite totally real extension. Recall that a representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is *odd* if $\det \rho(c) = -1$ for all complex conjugations $c \in G_F$. We can now state a modular lifting theorem.

Theorem 2.1. *Let $p > 5$. Let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be an odd, finitely ramified representation such that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible and ρ is potentially crystalline and ordinary at all places above p. Assume that there exists an automorphic representation π of $\mathrm{GL}(2, \mathbf{A}_F)$ such that $\rho_{\pi,p}$ is potentially crystalline and ordinary at all places above p and $\overline{\rho}_{\pi,p} = \overline{\rho}$. Then there exists an automorphic rerpesentation π' such that $\rho = \rho_{\pi',p}$.*

We will now indicate some ways in which base change and the Jacquet-Langlands correspondence come up in the proof of this theorem. To begin with, we can use base change to make some immediate reductions that simplify the situation. For example, our representation ρ is of the form

$$\begin{pmatrix} \alpha\chi_p & * \\ & \beta \end{pmatrix}$$

at each place above p. By making a solvable base change, we can reduce to the case where α and β are unramified. Even more drastically, we can make a solvable base change to reduce to the case where $\overline{\rho}|_{G_{F_v}}$ is *trivial* at any given finite set of places. Moving to such a situation can make some of the local deformation theory easier. Two other things we can do with base change: we can reduce to the case where $\det \rho$ is the cyclotomic character (our hypotheses imply that it is a finite twist of the cyclotomic character); and we can reduce to the case that $F/\mathbf{Q}$ has even degree, which is useful for finding quaternion algebras with prescribed ramification.

The above applications of base change are very useful but fairly superficial. We now describe a more serious application. In the hypotheses of the theorem, we have been given an automorphic rerpesentation

π such that $\overline{\rho} = \overline{\rho}_{\pi,p}$. In the proof, however, we need it to be the case that ρ and $\rho_{\pi,p}$ are potentially unramified at the same set of places. This need not be the case for the π we have. Of course, we are free to replace π with another form π' such that $\overline{\rho}_{\pi,p} = \overline{\rho}_{\pi',p}$, that is, one that is congruent to π modulo p (while still maintaining the other hypotheses). So the question is: given π as in the theorem, can we find a congruent π' such that $\rho_{\pi',p}$ and ρ are potentially unramified at the same set of places? Alternatively, we know that $\rho_{\pi',p}$ is potentially unramified precisely at the places where it is not Steinberg, so we could also ask if we can replace π by a congruent form and prescribe the set of places at which this new form is Steinberg.

Clearly, this issue cannot be resolved with base change; in fact, it requires some real work. In the early days of the modularity lifting theorem, these congruences were found using the geometry of the modular curves. These proofs were difficult and fairly specific. Since then, new proofs have been found which are easier and more general. The common theme of these proofs is to use the Jacquet-Langlands correspondence and then do some computations with modular forms on quaternion algebras — which are just functions on a finite set. It is much easier to manipulate these functions than forms on the modular curve!

To prove the theorem we identify a cerain universal deformation ring of the Galois representation ρ with a certain Hecke algebra. Originally, this Hecke algebra was one for GL(2). However, by Jacquet-Langlands, we can find the same Hecke algebra on a quaternion algebra, and as we have explained, it is often easier to prove things in that setting. So we will in fact use a Hecke algebra on a quaternion algebra. Thus the Jacquet-Langlands correspondence will be built into our proof at a very fundamental level.