NÉRON MODELS

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Contents

1. Introduction: Definition and First Properties

§*1.1.* Motivation. The purpose of these notes is to explain the definition and basic properties of the Néron model $\mathcal{A}$ of an abelian variety A over a global or local field K. We also give some idea of the proof that the Néron model exists. In the context of Faltings's proof of Mordell's conjecture, the primary motivation for doing so is that the notion of the Faltings height $h_F(A)$ will be defined in terms of a "Néron differential" $\omega_{\mathcal{A}}$ on $\mathcal{A}$.

An additional (and perhaps more crucial) reason to care is that we will need to use Grothendieck's semistable reduction theorem for abelian varieties. The strategy of the proof of this theorem, to be given in the next two talks by Christian and Brian, is to reduce the case of abelian varieties to Jacobians and then to curves, which can be handled directly. The reduction from Jacobians to curves uses a result of Raynaud to relate the Néron model of the Jacobian of a reasonable curve over a discretely valued field to the relative Picard scheme of a reasonable integral model of the curve.

§*1.2.* Reminder: smooth and étale maps. In order to say what a Néron model is, we need the notion of smoothness. In an effort to make these notes as self-contained as possible, here we briefly state several of the equivalent definitions of a smooth morphism, and present some nice properties of smooth maps. This is meant merely to be a convenient reference for readers unfamiliar with these notions; for proofs see [BLR, §2.2] and [EGA,IV_4,§17]. If you, the reader, know what a smooth morphism is, you should certainly skip this subsection.

1.2.1. A smooth morphism is a "nice" family of nonsingular varieties. This is analogous to a submersion of C^∞ manifolds in differential geometry, which has C^∞ manifolds for fibers and at least locally on the *source* is always a fibration. (Under the additional assumption of properness, a submersion actually is a C^∞-fibration, by a theorem of Ehresmann.) As a reminder, an algebraic variety over a field k is nonsingular if its local rings at every point are **regular**. A Noetherian local ring of Krull dimension n is regular precisely when its maximal ideal has a minimal system of n generators.[1]

Regularity is not generally stable under inseparable extension of the base field, so for a more useful notion is *geometric regularity*: X locally of finite type over k is geometrically regular if X_K is regular for one (or any) algebraically closed extension field K/k. So a good algebro-geometric version of a submersion of manifolds is a "nice" morphism of schemes with regular geometric fibers.

The algebro-geometric translation of a "nice family" is a flat morphism. Because it ends up being crucial to do anything useful, we also throw in a finiteness hypothesis, and are led to the following definition.

[1]Convincing geometric motivation for the notion of regularity can be found in §§III.3-4 of Mumford's red book.

1.2.2. **Definition.** A morphism of schemes $f : X \to Y$ is said to be **smooth at** $x \in X$ if it is locally of finite presentation, flat, and if, setting $y = f(x)$, the fiber $f^{-1}(y)$ of f over y is geometrically regular over the residue field $k(y)$. We say f is smooth **of relative dimension** n at x if it is smooth at x and the dimension of $f^{-1}(y)$ around x is n. We say that f is **smooth** if it is smooth at all $x \in X$, and smooth **of relative dimension** n if it is smooth of relative dimension n at all $x \in X$.

Here are some standard facts about smoothness.

1.2.3. **Proposition.** a. Smoothness is stable under base change and composition and (hence) under fiber products.
   b. Let $f : X \to Y$ be a smooth morphism of S-schemes.
      i. The locus of $x \in X$ such that f is smooth at x is open.
      ii. The sheaf of Kähler differentials $\Omega^1_{X/Y}$ is a locally free $\mathcal{O}_X$ module of rank at $x \in X$ equal to the relative dimension of f at x.
      iii. The relative cotangent sequence $0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$ is exact and locally split. □

Smoothness has a more concrete (but less obviously intrinsic) characterization in terms of equations or differentials:

1.2.4. **Proposition** (Jacobi criterion)**.** Let $j : X \hookrightarrow Z$ be a closed immersion of Y-schemes which are locally of finite presentation. Let $\mathcal{I}$ be the corresponding ideal sheaf of $\mathcal{O}_Z$. Let $x \in X, z = j(x)$, and assume Z is smooth at z of relative dimension n. Then the following are equivalent.

i. $X \to Y$ is smooth at x of relative dimension r.
ii. The conormal sequence $0 \to \mathcal{I}/\mathcal{I}^2 \to j^*\Omega^1_{Z/Y} \to \Omega^1_{X/Y} \to 0$ is split exact at x and $r = \operatorname{rank}_x \Omega^1_{X/Y}$.
iii. If $\mathrm{d}z_1, \dots, \mathrm{d}z_n$ is a basis of the free $\mathcal{O}_{Z,z}$-module $\Omega^1_{Z/S,z}$ and $g_1, \dots, g_N$ are local sections of $\mathcal{O}_Z$ generating $\mathcal{I}_z$, then after relabeling we can arrange that $g_{r+1}, \dots, g_n$ generate $\mathcal{I}_z$ and $\mathrm{d}z_1, \dots, \mathrm{d}z_r, \mathrm{d}g_{r+1}, \dots, \mathrm{d}g_N$ generate $\Omega^1_{Z/S,z}$.
iv. There exist local sections $g_{r+1}, \dots, g_n$ of $\mathcal{O}_Z$ generating $\mathcal{I}_z$ such that $\mathrm{d}\, g_{r+1}(z), \dots, \mathrm{d}\, g_n(z)$ are linearly independent in $\Omega^1_{Z/S} \otimes k(z)$. □

In practice one often takes Z to be an affine space over the base Y. If $Y = \operatorname{Spec} k$, the Jacobi criterion says that a closed subscheme $X \subset \mathbf{A}^n_k$ is smooth at x of dimension r if $\dim_x X = r$ and if the defining ideal $(f_1, \dots, f_N)$ of X is such that the Jacobian $(\frac{\partial f_j}{\partial t_i}(x))$ has rank $n - r$.

1.2.5**.** Accompanying smoothness are the related notions of étale and unramified morphisms.

A morphism $f : X \to Y$ is **étale** (at a point) if it is smooth of relative dimension zero (at that point). We say f is **unramified** at x if it is locally of finite presentation and $\Omega^1_{X/Y,x} = 0$. *Fact*: étale = smooth + unramified = flat + unramified.

Unpacking the Jacobi criterion, étale morphisms are precisely those morphisms which, in a differential geometric context, the implicit function theorem would guarantee to be local isomorphisms. Of course this is not the case in algebraic geometry, but it's a useful heuristic. In these terms, a nice way of thinking about smooth morphisms is via "étale coordinates":

1.2.6. **Proposition.** Let $f : X \to Y$ be a morphism and $x \in X$. Then f is smooth at x of relative dimension n, if and only if there exists an open neighborhood U of x and an étale Y-morphism $g : U \to \mathbf{A}^n_Y$. □

§*1.3.* What is a Néron model?

1.3.1**. Standing Notational Conventions.** In these notes R always denotes a Dedekind domain (for example, a discrete valuation ring) with field of fractions K. We shall follow Melanie's convention of denoting objects over K with $ROMAN$ letters and objects over R with $\mathcal{CALLIGRAPHIC}$ letters. When R is a dvr, its residue field is always denoted k. A subscript k or K on an R-scheme $\mathcal{X}$ always denotes a special or generic fiber, and never indicates an element of an indexed collection of R-scheme $\{\mathcal{X}_i\}_{j\in J}$.

If X is a smooth K-scheme, the Néron model $\mathcal{X}$ of X is, loosely speaking, the "nicest possible" smooth R-scheme which extends X – that is, which satisfies $\mathcal{X}_K \cong X$. We make this notion precise in terms of a universal property.

1.3.2. **Definition.** Let X be a smooth, separated K-scheme of finite type. A **Néron model** of X is a smooth, separated finite type R-scheme $\mathcal{X}$ such that

i. $\mathcal{X}$ is an R-model of X, so the generic fiber $\mathcal{X}_K$ is equipped with an isomorphism to X, which we abusively ignore and write $\mathcal{X}_K = X$, and
ii. $\mathcal{X}$ satisfies the **Néron mapping property** (NMP):
   For each smooth R-scheme $\mathcal{Y}$ and each K-morphism $f : \mathcal{Y}_K \to \mathcal{X}_K = X$, there exists a unique R-morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ which extends f. [2]

[2]This means that $\varphi \otimes_R K = f$ (when $\mathcal{X}_K$ is identified with X via the specified isomorphism).

The situation we will care most about is when X is an abelian variety over K.

1.3.3. **Remark.** In these notes we will generally assume that our Dedekind base $S = \operatorname{Spec} R$ is affine. There is of course nothing crucial about that; in fact (as we will see in Section 2) the real content of the theory of Néron models lies in the case when R is a discrete valuation ring. It can be technically convenient, however, to allow more general "Dedekind schemes" (constructed from spectra of Dedekind domains by gluing) as the base, such as complete curves over finite fields. Even disconnected examples occur naturally, for example when studying descent of properties with respect to étale covers $U \to S$ of a connected Dedekind scheme S, in which case $U \times_S U$ is generally not connected.

1.3.4. **Example.** The first example to keep in mind is abelian schemes. Recall that an **abelian scheme** over R is a smooth proper R-group scheme with connected geometric fibers. An abelian scheme $\mathcal{A}$ over R is a Néron model of its generic fiber $A = \mathcal{A}_K$. We will prove this in Section 2.

1.3.5. **Example.** Consider the case of an elliptic curve E over K. In studying the arithmetic of E, a common device is to contemplate proper R-models of E and their reductions. A particularly nice choice of model (and one that you can really get your hands on for doing computations, which is used extensively in [S1]) is a *minimal Weierstrass model* $\mathcal{W}$ of E. (Note that this might not exist when R has nontrivial Picard group, as there might not be *any* "global" Weierstrass model over R. But in the case of R a dvr, say, it exists.)

If E has bad reduction, $\mathcal{W}$ will not be smooth, but it will at least be proper. In constrast, a Néron model $\mathcal{N}$ of E is by definition smooth, but may not be proper, and very often $\mathcal{N}$ is *not planar*. Nonetheless, in Section 4 we will see that an open subscheme $\mathcal{N}^0$ of $\mathcal{N}$, obtained by removing non-identity components from the special fiber $\mathcal{N}_k$, is isomorphic to the smooth locus $\mathcal{W}^{\mathrm{sm}}$ of $\mathcal{W}$. If E has good reduction then $\mathcal{W} = \mathcal{N}$; that is, the two models are canonically isomorphic. (This follows from the previous example, because in the case of good reduction the minimal Weierstrass model is an abelian scheme.)

1.3.6. **Definition.** The **relative identity component** $\mathcal{A}^0$ of the Néron model $\mathcal{A}$ of an abelian variety A over K is the open subscheme of $\mathcal{A}$ obtained by removing non-identity components of the special fiber $\mathcal{A}_k$.

As we shall see shortly, when A is an abelian variety its Néron model is an R-group scheme. In this case it is an exercise to show that $\mathcal{A}^0$ is an R-subgroup scheme of $\mathcal{A}$.

1.3.7. **Example.** Let $\mathcal{X}$ be a Néron model of its generic fiber X. As a first illustration of the Néron mapping property, consider what it says when we take the smooth R-scheme $\mathcal{Y}$ to be $\operatorname{Spec} R$ itself. In this case the NMP states that each K-point $\operatorname{Spec} K \to X$ extends uniquely to an R-point $\operatorname{Spec} R \to \mathcal{X}$. In other words, the natural map $\mathcal{X}(R) \to X(K)$ is bijective.

1.3.8. **Example.** Let us illustrate with an example how in the case of bad reduction $\mathcal{W}^{\mathrm{sm}}$ can fail to be the Néron model $\mathcal{N}$, for reason of being "too small".

Let R be a dvr with uniformizer π and residue characteristic $\neq 2, 3$. Consider the elliptic curve over K given by

$$E : y^2 = x^3 + \pi^2.$$

This equation defines a Weierstrass model $\mathcal{W}$ of E with discriminant $\Delta = -2^4 3^3 \pi^4$, so $v(\Delta) = 4 < 12$, which implies $\mathcal{W}$ is minimal; see [S1].

By (1.3.7) we have $\mathcal{N}(R) = E(K)$, so we just need to show that $\mathcal{W}^{\mathrm{sm}}(R) \neq E(K)$. By thinking in $\mathbf{P}^2_R$, we see that $\mathcal{W}(R) = E(K)$; one can simply take a K-point of E and rescale its homogeneous coordinates just enough to be in R. So to see that $\mathcal{W}^{\mathrm{sm}}$ cannot be $\mathcal{N}$, it's enough to exhibit an R-point of $\mathcal{W}$ which does not factor through the smooth locus.

The homogeneous coordinates $[0 : \pi : 1]$ define such a point, since it reduces to the singular point $[0 : 0 : 1]$ of the cuspidal special fiber $\mathcal{W}_k$.

1.3.9. **Example** ([BLR], 3.5/5)**.** Let R again be a dvr with uniformizer π. Another *non*-example of a Néron model is $\mathcal{X} = \mathbf{P}^n_R$ as a model of $X = \mathbf{P}^n_K$. The reason is that there can be K-automorphisms f of X which do not extend to R-automorphisms of $\mathcal{X}$, which violates the NMP. This happens for f corresponding to matrices in $\mathrm{GL}_{n+1}(K)$ which when minimally rescaled so as to have coefficients in R, are not invertible over R. For example, we can take $n = 1$ and f defined by $\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In this case (and in general for such examples), the special fiber of the induced map $\varphi : \mathcal{X} \dashrightarrow \mathcal{X}$ is a linear projection, and hence undefined along the center of the projection. For our example, the mod π reduction of the matrix defining f determines the rational map $[x : y] \mapsto [0 : y] = [0 : 1]$ (for $y \neq 0$) of $\mathbf{P}^1_k$, which is undefined at the point $[1 : 0]$.

Here is the basic existence theorem.

1.3.10. **Theorem.** Let R be a discrete valuation ring with field of fractions K, and let A be an abelian variety over K. Then A admits a Néron model $\mathcal{A}/R$.

In Section 3 we will survey some of the ideas which go into the proof.

1.3.11. **Remark.** The condition in (1.3.10) that the generic fiber A be a *proper* K-group can be relaxed. See [BLR, §1.1] for the relevant notion: for a Néron model of a smooth, separated, finite type K-group scheme X to exist, it is necessary and sufficient that the points $X(K^{\mathrm{sh}})$ of X valued in the fraction field of a strict henselization of R are a **bounded** subset of X. When X is affine, this means that $X(K^{\mathrm{sh}})$ is a bounded (for the absolute value determined by the valuation on R) subset of $\mathbf{A}^n_K(K^{\mathrm{sh}})$ under an affine embedding $X \hookrightarrow \mathbf{A}^n_K$; in general a suitable "affine-local" version of this property is required. Of course some work must be done to show that this is a well-defined property of a set of points of X such as $X(K^{\mathrm{sh}})$, and that it holds automatically when X is proper.

1.3.12**.** The Néron mapping property is in some sense a variant of the notion of properness. Returning to the case of a general X, suppose that R is a discrete valuation ring. Let R' be an étale local R-algebra with field of fractions K' and local structural morphism $R \to R'$. For example, we could take $R = \mathbf{Z}_p$ and R' the ring of integers in an unramified finite extension of $\mathbf{Q}_p$. Then $\mathcal{Y} = \operatorname{Spec} R'$ is an example of a smooth R-scheme, as in the Néron mapping property.

Taking this choice of $\mathcal{Y}$, a map $f_K : \operatorname{Spec} K' = \operatorname{Spec}(R' \otimes_R K) = \mathcal{Y}_K \to \mathcal{X}_K = X$ of generic fibers is none other than a K'-point of X, which of course can be viewed as a K'-point x of $\mathcal{X}$. The Néron mapping property says that x extends to an R'-point of $\mathcal{X}$:

$$\begin{tikzcd}
\operatorname{Spec} K' = \mathcal{Y}_K \arrow[r, "x"] \arrow[d] & X \arrow[r] & \mathcal{X} \arrow[d] \\
\operatorname{Spec} R' = \mathcal{Y} \arrow[rr] \arrow[urr, dashed] & & \operatorname{Spec} R
\end{tikzcd}$$

Conversely if a dotted arrow exists making the diagram commute, then it is not hard to see the dotted arrow actually does extend the point x.

Consequently, in the setup above, the Néron mapping property implies a restriction of the valuative criterion of properness to the case of dvr's which are local étale over R. In fact, in the presence of a group structure, this "extension property for étale points" (meaning that $\mathcal{X}(R') \to X(K')$ is bijective for any R' as above) is actually *equivalent* to the Néron mapping property in full generality; cf. (6.1).

From this point of view, a Néron model amounts to a variant of a smooth proper model: modify the condition of properness to ensure Néron models exist in the situations we care about, but retain enough of a condition to ensure these models have good properties.

§*1.4.* Immediate consequences of the definition. To practice applying the Néron mapping property, we now give some simple applications. We return to the generality of a Dedekind domain R with field of fractions K.

1.4.1. **Remark.** The following facts illustrate the important principle that *one can go far with Néron models using merely their existence and universal propery, without knowing explicitly how they are constructed!* Apparently this was the state of affairs for almost all mathematicians until the book [BLR] appeared and the construction was made accessible to the masses.

1.4.2. **Proposition.** Let $\mathcal{X}$ be a smooth and separated R-scheme which is a Néron model of its generic fiber $X = \mathcal{X}_K$, a smooth and separated K-scheme of finite type.

i. If $\mathcal{X}'$ is any other Néron model of X there exists a unique isomorphism $\mathcal{X} \to \mathcal{X}'$ over R inducing the identity on the common generic fiber X. Thus we are justified in calling $\mathcal{X}$ "the" Néron model of X.
ii. If $\mathcal{X}'/R$ is the Néron model of another smooth separated finite type scheme X'/K, then $\mathcal{X} \times_R \mathcal{X}'$ is the Néron model of $X \times_K X'$.
iii. Suppose X is a K-group scheme. Then there is a unique R-group scheme structure on $\mathcal{X}$ extending the group structure on X. Moreover if X is commutative, so is $\mathcal{X}$.

Proof. These are all straightforward applications of the Néron mapping property. □

Let us mention an easy and useful "functorial" property of Néron models (that is false for ramified base change; we'll see an example of such in Section 4 where we discuss Néron models of elliptic curves in more detail):

1.4.3. **Proposition.** The formation of Néron models commutes with étale base change: if $\mathcal{X}$ is a Néron model of $X_{/K}$ and R' is an integral domain étale over R,[3] with field of fractions $K' = R' \otimes_R K$, then $\mathcal{X}' := \mathcal{X}_{R'}$ is a Néron model of $X' := X_{K'}$.

Proof. Clearly $\mathcal{X}'$ is a smooth separated finite type R'-model of X' so it's enough to check the Néron mapping property. So let $\mathcal{Y}'$ be a smooth R'-scheme with K'-fiber Y', and $f : Y' = \mathcal{Y}'_{K'} \to X'$ a K'-morphism. By composition we obtain a K-map

$$g : \mathcal{Y}' \otimes_R K = \mathcal{Y}' \otimes_{R'} K' = Y' \to X' \to X$$

and hence by the NMP for $\mathcal{X}$ there exists a unique R-map $\gamma : \mathcal{Y}' \to \mathcal{X}$ extending g. This invocation of the NMP is valid since $\mathcal{Y}'$ is R-smooth, as the composition of the étale map

[3]This implies that R', when not a field, is again Dedekind: $\dim R' = \dim R = 1$ since $R \to R'$ is flat of relative dimension zero, and from the definitions one checks that "smooth (e.g., étale) over regular is regular", so R' is regular.

$\operatorname{Spec} R' \to \operatorname{Spec} R$ with the smooth map $\mathcal{Y}' \to \operatorname{Spec} R'$ is again smooth. The R-morphism γ induces an R'-map $\varphi : \mathcal{Y}' \to \mathcal{X}'$. Since γ extends g, a diagram chase (or the so-called "pullback lemma") entails that φ extends f. Uniqueness of the R'-map φ follows from the uniqueness of the R-map γ and the mapping property of $\mathcal{X}' = \mathcal{X} \otimes_R R'$. □

§*1.5.* Plan of the rest of the notes. In Section 2 we prove that an abelian scheme over R is a Néron model of its generic fiber, using Weil's theorem on extending rational maps to smooth group schemes. Then, after some sorites on the "local nature" of Néron models, we state a global version of the local existence theorem (1.3.10). In Section 3 we sketch the proof of the main existence theorem (1.3.10). Section 4 discusses Néron models for elliptic curves. In Section 5 we use the Néron model of an abelian variety to prove the criterion of Néron-Ogg-Shafarevich (§5.2), following [ST]. We also define Tamagawa numbers; cf. §5.1. Finally in Section 6 we mention several useful properties of Néron models.

The basic reference for all of the material in these notes is the excellent book [BLR]. The case of elliptic curves is treated in more detail in [L], [S2], [C].

2. First example: abelian schemes

This section has two goals. The first is to verify the example (1.3.4), namely that an abelian scheme is the Néron model of its generic fiber. To do this we must first digress to say a bit about a relative version of the notion of a rational map between varieties. Crucial to our verification will be a result of Weil, which says that in nice circumstances (source and target smooth, target a separated group) such rational maps can actually be extended to honest morphisms.

The second goal is to produce more "global" Néron models than those afforded by (1.3.10). For example, given an abelian variety A over a number field F, we need to prove that it has a Néron model over the ring of integers $\mathfrak{o}_F$. Over the large open piece of $\operatorname{Spec} \mathfrak{o}_F$ where A has "good reduction", this can be accomplished using the result for abelian schemes. At the finitely many bad places of F we have local Néron models by (1.3.10). So the problem will be simply one of "patching" these data together in an appropriate way .

§*2.1.* S-rational maps. We begin our discussion by defining a relative notion of a rational map between two schmes over a base scheme S. This could perhaps be done without imposing so many smoothness hypotheses (see [EGA IV$_4$, §20.2]) but we follow the approach developed in [BLR, 2.5].

2.1.1. **Definition.** An open subscheme U of a smooth S-scheme X is **S-dense** if for each $s \in S$, $U_s = U \times_S k(s)$ is Zariski dense in $X_s = X \times_S k(s)$.

Two S-morphisms $U \to Y$ and $U' \to Y$ from S-dense open subschemes of a smooth S-scheme X to a smooth S-scheme Y are called **equivalent** if they coincide on an S-dense open subscheme of $U \cap U'$. (This is clearly an equivalence relation.) If X and Y are smooth S-schemes, an **S-rational map** $\varphi : X - - \to Y$ is an equivalence class of S-morphisms $U \to Y$ where U is S-dense in X.

The map φ is **defined at** a point $x \in X$ if there is a morphism $U \to Y$ representing φ with $x \in U$; the open set of $x \in X$ such that φ is defined at x is called the **domain of definition** $\operatorname{dom}(\varphi)$ of φ.

If an S-rational map $\varphi : X - - \to Y$ has a representative S-morphism $U \to Y$ which induces an isomorphism from U onto an S-dense open subshceme of Y then φ is called **S-birational**.

Observe that if X, Y, and Z are smooth S-schemes and $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow Z$ are S-rational maps such that the set theoretic image of φ contains an S-dense open subscheme of Y, then the composition $\psi \circ \varphi : X \dashrightarrow Z$ makes sense as an S-rational map.

Rather than listing various properties of S-rational maps, we content ourselves with the following facts, needed below.

2.1.2. **Proposition.** If $X \to S$ is smooth and $Y \to S$ is smooth and separated, then an S-rational map $\varphi : X \dashrightarrow Y$ can be represented by an S-morphism $\mathrm{dom}(\varphi) \to Y$.

Proof. Two representatives for φ agree on an S-dense open subset of the intersection of their domains. Since Y is separated, they therefore agree on the entire intersection. So they can be glued together, and thus φ determines a unique S-morphism $\mathrm{dom}(\varphi) \to Y$. □

2.1.3. **Proposition.** Let X, X', Y be smooth and finitely presented S-schemes with Y separated over S. Let $f : X \dashrightarrow Y$ be an S-rational map and $\pi : X' \to X$ a faithfully flat S-morphism. If $f \circ \pi : X' \dashrightarrow Y$ is defined everywhere on X', then f is defined everywhere on X.

Proof. See [BLR, 2.5/5]. Note that $f \circ \pi$ makes sense as an S-rational map because π is faithfully flat, so in particular surjective. □

Here is Weil's theorem on extending rational maps to smooth groups.

2.1.4. **Theorem** (Weil)**.** Let S be a normal Noetherian scheme and $f : Z \dashrightarrow G$ an S-rational map from a smooth S-scheme Z to a smooth and separated S-group scheme G. If f is defined in codimension 1, meaning that the domain of definition of f contains all points of Z of codimension ≤ 1, then f is defined everywhere.

Sketch. Define $g : Z \times_S Z \dashrightarrow G$ by $(z_1, z_2) \mapsto f(z_1)f(z_2)^{-1}$. What do we mean by this? If $U = \mathrm{dom}(f)$ is the domain of definition of f, then this prescription (regarded as defining a map on T-points for U-schemes T) determines by Yoneda's lemma a morphism $U \times_S U \to G$; as $U \times_S U$ is S-dense in $Z \times_S Z$, this shows that g makes sense as an S-rational map.

Let $V \subset Z \times_S Z$ be the domain of definition of g, which contains $U \times_S U$. The idea is to show that $\Delta \subset V$, i.e. that g is defined near the diagonal of $Z \times_S Z$. Let us see why this is sufficient. By (2.1.3) it's enough to exhibit a smooth, finite type Z' over S and a faithfully flat map $\pi : Z' \to Z$ such that $f \circ \pi : Z' \dashrightarrow G$ is defined everywhere. We take $Z' = V \cap (Z \times_S U)$ and $\pi : Z' \to Z$ the first projection. Since Z is smooth, π is flat. Using that $\Delta \subset V$ one shows that π is surjective, so it is faithfully flat.[4] Now the composition $f \circ \pi : Z' = V \cap (Z \times_S U) \to Z \dashrightarrow G$ is given by $(z_1, z_2) \mapsto z_1 \mapsto f(z_1) = f(z_1)f(z_2)^{-1}f(z_2) = g(z_1, z_2)f(z_2)$, so it agrees with the map given by

$$Z' = V \cap (Z \times_S U) \overset{\mathrm{id}\times\mathrm{pr}_2}{\to} V \times_S U \overset{g\times f}{\to} G \times_S G \overset{\text{mult.}}{\to} G,$$

over the S-dense open subscheme $\pi^{-1}U = U \times_S U \subset Z'$, so they agree as S-rational maps. But the latter is actually a morphism, as V (resp. U) is the domain of defition of g (resp. f)! So $f \circ \pi$ is defined everywhere, and thus f is too, by (2.1.3).

[4] Surjectivity can be checked on geometric fibers. Let Ω be an algeraically closed field and $z : T = \mathrm{Spec}\,\Omega \to Z$ a geometric point of Z. The fiber of V over z is nonempty since (by assumption) $\Delta \subset V$ (so the point (z, z) of V gives a geometric point of $V_z = T \times_Z V$). Since $U \subset Z$ is S-dense, $T \times_S U$ is Zariski dense in $T \times_S Z$. Now $T \times_Z V \subset T \times_Z (Z \times_S Z) = T \times_S Z$ is open and, as we have seen, nonempty. So it meets the dense open $T \times_S U$, which means the fiber $T \times_Z (V \cap (Z \times_S U)) = (T \times_Z V) \cap (T \times_S U)$ of $\pi : Z' \to Z$ over T is nonempty. Thus π is surjective, hence faithfully flat.

We omit the verification that $\Delta \subset V$ (see [BLR, 4.4/1]) except to say that it depends on a dimension argument (for we must use the hypothesis that f is defined in codimension 1), which boils down to the algebraic Hartogs' lemma: a rational function on a normal Noetherian scheme which is regular in codimension 1 is regular everywhere. □

§*2.2.* Abelian schemes.

2.2.1. **Definition.** Let S be a scheme. An **abelian scheme** $\mathcal{A} \to S$ is a smooth proper S-group scheme with connected geometric fibers.

2.2.2. **Example.** Suppose R is a dvr, and $E_{/K}$ is an elliptic curve with good reduction in the sense of [S1], so that the special fiber $\mathcal{W}_k$ of a minimal Weierstrass model $\mathcal{W}$ for E over R is smooth.[5] It follows that $\mathcal{W}$ is smooth over $S = \operatorname{Spec} R$ and it is easy to see, therefore, that all the geometric fibers of $\mathcal{W}$ are (connected) smooth curves of genus 1. Moreover the identity of E gives rise to a section ϵ of $\mathcal{W}$ (the point $[0:1:0]$ in $\mathbf{P}^2(R)$). Thus $(\mathcal{W}, \epsilon)$ is an elliptic curve over S in the sense of [KM]. By [KM, 2.1.2] it follows that $\mathcal{W}$ has a unique structure of S-group scheme compatible with the "geometric" group law on the generic fiber E. Thus $\mathcal{W}$ is an abelian S-scheme.

2.2.3. **Definition.** Motivated by this example, we *define* the notion of good reduction for an abelian variety over K as follows: we say A over K has **good reduction** if there exists an abelian scheme $\mathcal{A}$ over $\operatorname{Spec} R$ with generic fiber A. Otherwise we say A has **bad reduction**.

Here is the main result of the section.

2.2.4. **Proposition.** Let R be a Dedekind domain with fraction field K. Let $\mathcal{A}$ be an abelian scheme over $\operatorname{Spec} R$ with generic fiber A. Then $\mathcal{A}$ is a Néron model of A.

To prove this proposition we will use Weil's extension theorem (2.1.4), plus the following technical device which will also be used in §2.3 to extend the existence theorem (1.3.10) to a global setting.

2.2.5. **Lemma.** Let S be a scheme and $s \in S$.

i. If X and Y are finitely presented S-schemes, then the natural morphism
$$\varinjlim_{U \ni s} \operatorname{Hom}_U(X \times_S U, Y \times_S U) \to \operatorname{Hom}_{\mathcal{O}_{X,s}}(X \times_S \mathcal{O}_{S,s}, Y \times_S, \mathcal{O}_{S,s})$$
(limit taken over open neighborhoods of s in S) is bijective.

ii. If $X_{(s)}$ is a finitely presented $\mathcal{O}_{S,s}$-scheme, then there is an open neighborhood U of s in S and a U-scheme X' of finite presentation such that $X' \times_S \mathcal{O}_{S,s} \cong X_{(s)}$.

Proof. See [EGA IV$_3$, 8.8.2]. □

2.2.6. *Proof of* (2.2.4). Let $\mathcal{Y}$ be a smooth $S = \operatorname{Spec} R$-scheme with generic fiber Y and $f : Y \to A$ a K-morphism. We must show that there is a unique R-morphism $\varphi : \mathcal{Y} \to \mathcal{A}$ extending f. Since $\mathcal{A}$ is separated we can work locally on $\mathcal{Y}$, as two extensions of f defined locally on $\mathcal{Y}$ must agree where both defined and hence glue.

So we may assume that $\mathcal{Y}$ is of finite presentation, and by smoothness even that Y is irreducible. By (2.2.5), f spreads out to a morphism $\varphi_0 : \mathcal{Y} \times_S S' \to \mathcal{A}$ defined over an open neighborhood S' of the generic point of S.

[5]Silverman requires only that $\mathcal{W}_k$ is regular, but this is wrong when k is imperfect.

The next step is to extend φ_0 to an R-rational map $\varphi : \mathcal{Y} - - \to \mathcal{A}$. Set $Y = \mathcal{Y}_K$. Let $s \in S$ be a closed point in $S - S'$, $\mathcal{Y}_s$ the fiber over s, and η the generic point of an irreducible component $\mathcal{Y}_{s,i}$ of $\mathcal{Y}_s$. Since R is 1-dimensional and $\mathcal{Y}$ is R-flat, $\mathcal{Y}_s$ is codimension 1 in $\mathcal{Y}$. Since R is regular and $\mathcal{Y}$ is smooth, $\mathcal{O}_{\mathcal{Y},\eta}$ is regular; thus the local ring $\mathcal{O}_{\mathcal{Y},\eta}$ is a discrete valuation ring, with fraction field the function field $K(Y)$ of Y. The morphism f induces a diagram

$$\begin{tikzcd}
\operatorname{Spec} K(Y) \arrow[r] \arrow[d] & Y \arrow[r] & A \arrow[r] & \mathcal{A} \arrow[d] \\
\operatorname{Spec} \mathcal{O}_{\mathcal{Y},\eta} \arrow[rrr] \arrow[rrru, dashed] & & & \operatorname{Spec} R
\end{tikzcd}$$

By the valuative criterion of properness for the abelian R-scheme $\mathcal{A}$, a dotted arrow exists; i.e. f extends uniquely to a morphism $\operatorname{Spec} \mathcal{O}_{\mathcal{Y},\eta} \to \mathcal{A}$. By (2.2.5), this means f extends to a morphism $U_\eta \to \mathcal{A}$ from a neighborhood U_η of η in $\mathcal{Y}$. This can process can be repeated for all the generic points η of all the closed fibers $\mathcal{Y}_s$ over $s \in S - S'$. By the separatedness of $\mathcal{A}$, the resulting morphisms can be glued together with $\varphi_0 : \mathcal{U}_0 \to \mathcal{A}$ to give a map $\varphi : \mathcal{U} \to \mathcal{A}$ on the open subset $\mathcal{U} = \mathcal{U}_0 \cup \bigcup_\eta U_\eta \subset \mathcal{Y}$ extending f. Moreover $\mathcal{U}$ is S-dense in $\mathcal{Y}$ since it contains a neighborhood of the generic point of each irreducible component of each fiber $\mathcal{Y}_s$. Thus φ is an S-rational map $\mathcal{Y} - - \to \mathcal{A}$ extending f.

We now invoke (2.1.4), which applies because the abelian scheme $\mathcal{A}$ is a smooth and separated (because proper) group scheme and because $\mathcal{Y}$ is smooth. By construction the domain of definition $\mathcal{U}$ of φ contains all the codimension-one points of $\mathcal{Y}$ contained in the closed fibers; since S is Dedekind, all the other codimension-one points of $\mathcal{Y}$ are contained in the generic fiber Y, where φ is defined *a priori*. So the hypotheses of (2.1.4) are satisfied, and thus φ is defined everywhere. This verifies the NMP for $\mathcal{A}$ and proves the claim. □

2.2.7. **Corollary.** Let A be an abelian variety over fraction field K of a discrete valuation ring R. Let $\mathcal{A}$ be its Néron model over R. The following are equivalent.

i. A has good reduction.
ii. $\mathcal{A}$ is an abelian scheme over R.
iii. The identity component $\mathcal{A}_k^0$ of the special fiber of $\mathcal{A}$ is proper (hence an abelian variety over k).

The proof of the corollary requires an algebro-geometric input.

2.2.8. **Lemma.** Let $\mathcal{X}$ be a smooth R-scheme with geometrically connected generic fiber X and proper special fiber $\mathcal{X}_k$. Then $\mathcal{X}$ is proper. (Bonus: also, $\mathcal{X}_k$ is geometrically connected.)

Proof. See [ST, §1, Lemma 3]. □

2.2.9. *Proof of* (2.2.7). (ii)⇒(i) and (ii)⇒(iii) are trivial. For (i)⇒(ii), note that by (2.2.4) any abelian R-scheme which is a model for A is in fact a Néron model and hence isomorphic to $\mathcal{A}$. For (iii)⇒(ii), the properness of $\mathcal{A}_k^0$ entails by (2.2.8) that the open subscheme $\mathcal{A}^0$ obtained from $\mathcal{A}$ by removing non-identity components from the special fiber, is itself proper over R. It is easy to see that $\mathcal{A}^0$ is always a subgroup scheme of $\mathcal{A}$, so in this case it is therefore an abelian scheme. Hence by (2.2.4) it is the Néron model of A, which implies that the inclusion $\mathcal{A}^0 \hookrightarrow \mathcal{A}$ is an isomorphism, proving (ii). □

2.2.10. **Remark.** We will add another item to this list of criteria for good reduction in §5.2.

2.2.11. **Remark.** Additive and multiplicative reduction for elliptic curves can also be characterized in terms of the special fiber of the Néron model. We will show in §4.3 that these notions coincide with those in [S1].

§*2.3.* **Local nature of Néron models.** We have stated the basic existence result, Theorem 1.3.10, but this says nothing about finding Néron models over a non-local Dedekind domain. The question is whether the Néron models $\mathcal{X}_{\mathfrak{p}}$ of a K-scheme X over the localizations $R_{\mathfrak{p}}$ of R (if they exist!) "patch together" to form a global Néron model $\mathcal{X}$ over R. The corresponding patching problem on honest open sets is straightforward.

2.3.1. **Exercise.** Let X be a K-scheme and let $\{S_i\}$ be an (affine, if you wish) open cover of $S = \operatorname{Spec} R$. Use (1.4.2)(i) and (1.4.3) to show that Néron models $\mathcal{X}_i$ of X over S_i glue to give a Néron model $\mathcal{X}$ of X over R.

On the other hand, "stalk-locality" of Néron models is slightly subtle.

2.3.2. **Proposition.** Let $S = \operatorname{Spec} R$ and let X be a smooth and separated K-scheme of finite type. There exists a Néron R-model $\mathcal{X}$ of X if and only if there exists a nonempty (= dense) open subscheme $S' \subset S$, a Néron model $\mathcal{X}'$ of X over S', and Néron models $\mathcal{X}_{(s)}$ of X over $\mathcal{O}_{S,s}$ for each of the finitely many closed points $s \in S - S'$.

Sketch. One direction is obvious. For the reverse, one must use (2.2.5) to spread out the finitely presented $\mathcal{O}_{S,s_i}$-scheme $\mathcal{X}_{(s_i)}$ to a smooth finite type scheme $\mathcal{X}_i$ over an open neighborhood S_i of s_i, for each $s_i \in S - S'$. Since $\mathcal{X}_i$ and $\mathcal{X}'$ have the same generic fiber X, by shrinking S_i one can ensure $\mathcal{X}_i \times_{S_i} (S_i \cap S') = \mathcal{X}' \times_{S'} (S_i \cap S')$. Thus the $\mathcal{X}_i$ and $\mathcal{X}$ glue to give a smooth separated finite type R-model $\mathcal{X}$ of X. Moreover for each closed point $s \in S$, $\mathcal{X} \times_S \operatorname{Spec} \mathcal{O}_{S,s}$ coincides with either $\mathcal{X}_{(s_i)}$ if $s = s_i \in S - S'$ or $\mathcal{X}' \times_S \operatorname{Spec} \mathcal{O}_{S,s}$. The claim is then reduced to the lemma which follows. □

2.3.3. **Lemma.** Let $\mathcal{X}$ be a scheme of finite type over $S = \operatorname{Spec} R$ with generic fiber X. Then $\mathcal{X}$ is a Néron model of X over R if and only if for each closed point $s \in S$, the $\mathcal{O}_{S,s}$-scheme $\mathcal{X}_{(s)} := \mathcal{X} \times_S \operatorname{Spec} \mathcal{O}_{S,s}$ is a Néron model of X over $\mathcal{O}_{S,s}$.

Proof. See [BLR, 1.2/4]. The issue is that in verifying the Néron mapping property in each direction, one must again make use of (2.2.5) to spread out morphisms of finite presentation of $\mathcal{O}_{S,s}$-schemes into morphisms of schemes over an open neighborhood of s. □

Combining the preceding results with the local existence theorem (1.3.10) (whose proof we sketch in Section 3), we obtain the following global existence theorem.

2.3.4. **Theorem.** Let R be any dedekind domain, with field of fractions K. Let A be an abelian variety over K. Then A admits a "global" Néron model $\mathcal{A}$ over $S = \operatorname{Spec} R$. Moreover, if $S \subset \operatorname{Spec} R$ is obtained by deleting the primes where A had bad reduction, then $\mathcal{A} \times_S S'$ is an abelian scheme over S'.

Sketch. Using (2.2.5) one spreads out A to a finite type scheme $\mathcal{A}_0$ over a neighborhood U of the generic point of $S = \operatorname{Spec} R$. By shrinking U, we can arrange that $\mathcal{A}_0$ is smooth and proper and admits an R-group law extending the one on A, so that $\mathcal{A}_0$ is actually an abelian scheme compatibly with the group law on A (cf. [BLR, 1.4/2]). So $\mathcal{A}_0$ is the Néron model of A over U, by (2.2.4). Now by (1.3.10) there are Néron models of A over the local rings of the finitely many points of $S - U$. Hence the theorem follows from (2.3.2). □

3. Sketch of proof of Theorem 1.3.10

In this section we'll present an overview of the ideas which go into the proof of the existence theorem (1.3.10).

§*3.1.* Outline. Let us begin by outlining the construction, which proceeds in 5 steps. We begin with our abelian variety A over the fraction field K of a discrete valuation ring R.

3.1.1. Since A is projective, fix a projective embedding $A \hookrightarrow \mathbf{P}^n_K \subset \mathbf{P}^n_R$. Let $\mathcal{A}_0$ be the schematic closure of A in $\mathbf{P}^n_R$ (in the sense of §3 of Melanie's talk). This is a proper flat R-model of A, but is very likely non-smooth.

3.1.2. To remedy this, we smooth $\mathcal{A}_0$ out. This is accomplished by an algorithm which specifies a sequence of blow-ups centered in the special fiber of $\mathcal{A}_0$, and results in a projective R-morphism $f : \mathcal{A}_1 \to \mathcal{A}_0$ such that

i. f_K is an isomorphism, and
ii. if $\mathcal{A}_1^{\mathrm{sm}}$ denotes the R-smooth locus, then for any étale R-algebra R', the canonical map $\mathcal{A}_1^{\mathrm{sm}}(R') \to \mathcal{A}_0(R')$ is bijective.

Condition (ii) says that R'-valued points of $\mathcal{A}_0$, which lift uniquely to $\mathcal{A}_1$ due to the properness of $\mathcal{A}_1$, factor through $\mathcal{A}_1^{\mathrm{sm}}$. While $\mathcal{A}_1$ is not necessarily smooth, it is nonetheless called the **smoothening** of $\mathcal{A}_0$.

3.1.3. We regard the smooth locus $\mathcal{A}^{\mathrm{weak}} := \mathcal{A}_1^{\mathrm{sm}}$ as satisfying a weak version of the NMP. Indeed, $\mathcal{A}^{\mathrm{weak}}$ is what's known as a weak Néron model of A. A **weak Néron model** $\mathcal{X}$ of a smooth and separated finite type K-scheme X is a smooth and separated R-model of X of finite type, satisfying the extension property for étale points: for any étale R-algebra R' with field of fractions K', the canonical map $\mathcal{X}(R') \to X(K')$ is bijective. [6] (For $\mathcal{A}^{\mathrm{weak}}$, this follows from property (ii) of the previous step.) This is useful because said extension property can actually be souped up to a "rational" version of the NMP. If $\mathcal{X}$ is a weak Néron model of X, we say it satisfies the **weak Néron mapping property** if the following holds.

(WNMP) Given a smooth R-scheme $\mathcal{Z}$ with irreducible special fiber $\mathcal{Z}_k$
and any K-rational map $f : \mathcal{Z}_K - - \to X$,
there exists an extension of f to an R-rational map $\varphi : \mathcal{Z} - - \to \mathcal{X}$.

It turns out that any weak Néron model satisfies the WNMP; cf. §3.3. In particular, our smooth but non-proper R-model $\mathcal{A}^{\mathrm{weak}}$ of A does so.

3.1.4. Ultimately the Néron model $\mathcal{A}$ of A must have an R-group scheme structure extending the group law on A. As a step towards this, we next shrink $\mathcal{A}^{\mathrm{weak}}$ to another R-model $\mathcal{A}^{\mathrm{bg}}$ of A, by throwing out certain components from the special fiber $\mathcal{A}^{\mathrm{weak}}_k$. By choosing the right components, we can use the WNMP to ensure that the K-group law on A extends to an R-*birational* group law on $\mathcal{A}^{\mathrm{bg}}$. This notion is defined in §3.4. (The superscript "bg" stands for "birational group".) The construction of the birational group law is sketched in §3.5.

3.1.5. Finally, the Néron model $\mathcal{A}$ is constructed from $\mathcal{A}^{\mathrm{bg}}$ by a theorem of Weil (3.4.2), which says roughly that any "R-birational group" (such as $\mathcal{A}^{\mathrm{bg}}$) can be uniquely enlarged to a smooth, separated, finite type R-scheme with a compatible ("honest", not birational)

[6] Another way of phrasing this property, in terms of the strict henselization R^{sh} of R and its fraction field K^{sh}, is to say that $X(K^{\mathrm{sh}}) = \mathcal{X}(R^{\mathrm{sh}})$.

R-group law. Of course one must then verify that this $\mathcal{A}$ actually satisfies the (strong) Néron mapping property; see §3.6.

In the rest of this section we give a few more details (to be omitted from the talk) concerning each of the previous steps. These details can be skipped, but be assured that the construction is actually very cool, so you are kind of missing out if you are content to use the existence of Néron models purely as a black box.

§*3.2.* Smoothening process. The idea behind the smoothening process is to define a non-negative integer $\delta(\mathcal{X})$ which measures how far an R-scheme $\mathcal{X}$ is from being smooth ($\delta(\mathcal{X}) = 0$ if and only if $\mathcal{X}$ is R-smooth); one then identifies certain "permissible" closed subschemes of the special fiber $\mathcal{X}_k$ and shows that blowing these up causes δ to decrease.

We omit any real description of how this works, other than to state precisely the general theorem which results from the process. The reader is refered to [BLR, Ch. 3] for this theory, which also plays a role in the proof of the important Artin approximation theorem.

3.2.1. **Theorem** ([BLR], 3.1/3)**.** Let $\mathcal{X}$ be an R-scheme of finite type with generic fiber X smooth over K. Then there is a finite sequence of blow-ups centered in the non-smooth loci of the successive special fibers resulting in an R-morphism $f : \mathcal{X}' \to \mathcal{X}$ which is projective, an isomorphism on generic fibers, and such that the induced map $\mathcal{X}'^{\mathrm{sm}}(R') \to \mathcal{X}(R')$ is an isomorphism for any étale R-algebra R'. $\square$

This justifies (3.1.2).

§*3.3.* The Weak Néron Mapping Property.

3.3.1. **Proposition.** Let $\mathcal{X}$ be a weak Néron model of X. Then $\mathcal{X}$ satisfies the WNMP.

Proof. We fix a smooth R-scheme $\mathcal{Z}$ with irreducible special fiber $\mathcal{Z}_k$ and a K-rational map $f : \mathcal{Z}_K - - \to X$. We need to extend f to an R-rational map $\varphi : \mathcal{Z} - - \to \mathcal{X}$.

Let $U \subset \mathcal{Z}_K$ be an open dense subscheme upon which f is defined, and let Y be its complement. Let $\mathcal{Y}$ be the schematic closure of Y in $\mathcal{Z}$. Using the definition of schematic closure and the fact that R is a dvr, it's easy to check that $\mathcal{Z}_k - \mathcal{Y}_k$ is Zariski dense in $\mathcal{Z}_k$.[7] Thus $\mathcal{U} = \mathcal{Z} - \mathcal{Y}$ is R-dense in $\mathcal{Z}$. If we can find an R-rational map $\mathcal{U} - - \to \mathcal{X}$ extending the *morphism* $f : \mathcal{U}_K = U \to X$, we will be done, since its domain of definition will then be R-dense in $\mathcal{Z}$ as well. So we can replace $\mathcal{Z}$ with $\mathcal{U}$ and assume f is a morphism $\mathcal{Z}_K \to X$.

If we can solve the problem locally on $\mathcal{Z}$ then we can glue the resulting R-rational maps. This is because $\mathcal{X}$ is separated, so R-rational extensions of maps to the generic fiber of X are unique; cf. (2.1.2). Thus we can also assume $\mathcal{Z}$ is of finite presentation.

Let $\mathit{\Gamma}$ be the schematic closure in $\mathcal{Z} \times_R \mathcal{X}$ of the graph of f, with projections

$$\mathcal{Z} \xleftarrow{p} \mathit{\Gamma} \xrightarrow{q} \mathcal{X}.$$

The idea is to show that p is R-birational, i.e. invertible on an R-dense open of $\mathcal{Z}$. Then $\varphi = q \circ p^{-1} : \mathcal{Z} - - \to \mathcal{X}$ satisfies the conclusions of the claim.

To invert p, we first observe that by Chevalley, the image $\mathcal{T} = p(\mathit{\Gamma}) \subset \mathcal{Z}$ is constructible. Suppose we knew that $\mathcal{T}_k$ contains a nonempty (i.e. dense) open of $\mathcal{Z}_k$. Let η be the generic point of $\mathcal{Z}_k$; then there is some $\xi \in p^{-1}(\eta) \subset \mathit{\Gamma}$. Then $\mathcal{O}_{\mathcal{Z},\eta}$ is a dvr dominated by $\mathcal{O}_{\mathit{\Gamma},\xi}$. But the induced extension of fraction fields is trivial, since p is the first projection from the schematic closure of a *graph*, and hence p_K is an isomorphism. So $\mathcal{O}_{Z,\eta} \to \mathcal{O}_{\mathit{\Gamma},\xi}$ is also an

[7] Since the schematic closure $\mathcal{Y}$ is necessarily R-flat, we have $\dim \mathcal{Y}_k = \dim Y < \dim \mathcal{Z}_K = \dim \mathcal{Z}_k$.

isomorphism, and hence by (2.2.5) and the fact that $\mathcal{Z}$ and Γ are of finite presentation, p is an isomorphism between open neighborhoods of η and ξ, so it is R-birational.

So the problem boils down to showing that p_k is dominant, i.e. that the image $\mathcal{T}_k$ is dense in $\mathcal{Z}_k$; this is where we use that $\mathcal{X}$ is a weak Néron model. A bit of thought[8] will convince you that it suffices to check this after base change from R to its strict henselization. In other words, we may replace R by R^{sh} and assume that R is henselian and k is separably closed. This is useful because (one can show) that for a smooth scheme over a separably closed field, rational points are dense. Applying this to $\mathcal{Z}_k$, we have that $\mathcal{Z}_k(k) \subset \mathcal{Z}_k$ is dense. Since R is henselian and $\mathcal{Z}$ is smooth, we also have that each $z_k \in \mathcal{Z}_k(k)$ lifts to a point $z \in \mathcal{Z}(R)$. Let $x_K = f(z_K) \in X(K)$. Since $\mathcal{X}$ is a weak Néron model, x_K extends to some $x \in \mathcal{X}(R)$.[9] Thus (z, x) is in the (closure of the) graph of f, i.e. $\Gamma(R)$, which means that $z \in \mathcal{T}(R)$, so $z_k \in \mathcal{T}_k(k)$. Putting this together, this says that $\mathcal{Z}_k(k) \subset \mathcal{T}_k(k) \subset \mathcal{T}_k$. Thus $\mathcal{T}_k$ is dense in $\mathcal{Z}_k$, so by constructibility it contains a dense open. □

§*3.4.* Birational Group laws and Weil's theorem. The next step is to use the WNMP to construct from the weak Néron model $\mathcal{A}^{\text{weak}}$ a birational group $\mathcal{A}^{\text{bg}} \subset \mathcal{A}^{\text{weak}}$. First we must say what such is.

3.4.1. **Definition.** If $\mathcal{X}$ is a smooth, separated, faithfully flat R-model of a K-group X with multiplication $m : X \times X \to X$, an **R-birational group law** on $\mathcal{X}$ is an extension of m to an R-rational map $\mu : \mathcal{X} \times_R \mathcal{X} \dashrightarrow \mathcal{X}$ such that the following properties are satisfied.

i. μ is associative, in the sense of the commutativity of the obvious diagrams of R-rational maps.[10]
ii. The universal translations $(\mathrm{pr}_1, \mu) : ((x, y) \mapsto (x, xy)$ and $(\mu, \mathrm{pr}_2) : (x, y) \mapsto (xy, y)$ are both R-birational maps $\mathcal{X} \times_R \mathcal{X} \overset{\sim}{\dashrightarrow} \mathcal{X} \times_R \mathcal{X}$.

Note that an R-birational group law is *not* required to have an "identity section" or an "inversion" map.

A **solution** of the R-birational group law μ is a smooth, separated, finite type R-group $\overline{\mathcal{X}}$ with multiplication $\overline{\mu}$, together with an R-dense open subscheme $\mathcal{X}' \subset \mathcal{X}$ and an R-dense open immersion $\mathcal{X}' \hookrightarrow \overline{\mathcal{X}}$ such that $\overline{\mu}$ restricts to $\mu|_{X'}$.

In other words, a solution of a birational group law is a way of enlarging an R-dense open to an honest group scheme.

3.4.2. **Theorem** (Weil; see [BLR], 5.1/5)**.** Let R be a dvr, and $\mathcal{X}$ a smooth, separated, finite type, faithfully flat R-model of a K-group X, with an R-birational group law μ extending the group law on X. Then there exists a unique solution $(\overline{\mathcal{X}}, \overline{\mu}, \mathcal{X}')$ of μ, and moreover one does not need to shrink $\mathcal{X}$. That is, $\mathcal{X} = \mathcal{X}'$ is an R-dense open subscheme of an R-group $\overline{\mathcal{X}}$ with multiplication $\overline{\mu}$ which restricts to μ on $\mathcal{X}$. □

This theorem is serious business, and we say nothing about the proof (which is spread out in [BLR, 5.1/3, 5.2/2, 5.2/3, 6.5/2]). It will be used to extend the birational group $\mathcal{A}^{\text{bg}}$ to an R-group $\mathcal{A} = \overline{\mathcal{A}^{\text{bg}}}$, the Néron model of A. But first we must show that $\mathcal{A}^{\text{weak}}$ can actually

[8] Or see *The Geometry of Schemes*, V.8.

[9] See the footnote in (3.1.3).

[10] Some care is required to make sense of this, since these diagrams require composing R-rational maps, and it must be verified that these compositions make sense. This rests upon the image of $\mathrm{dom}(\mu)$ containing an R-dense open of $\mathcal{X}$, which is satisfied due to condition (ii) of the definition.

be made into a birational group $\mathcal{A}^{\mathrm{bg}}$. This is kind of tricky, and is taken up in the next subsection.

§*3.5.* ω-minimal components and construction of the birational group $\mathcal{A}^{\mathrm{bg}}$. To make a birational R-group model $\mathcal{A}^{\mathrm{bg}}$ of A out of the weak Néron model $\mathcal{A}^{\mathrm{weak}}$, we remove certain "non-ω-minimal" components from $\mathcal{A}^{\mathrm{weak}}_k$. Let us say what we mean by this. For the present purposes, we can ignore how the weak Néron model was constructed. So suppose that $\mathcal{B}$ over R is a weak Néron model of an abelian variety B over K.

First we recall the notion of invariant differential forms on group schemes (such as the K-group B). In general, if G is an S-group scheme for a base scheme S, a Kähler differential $\omega \in \mathrm{H}^0(\Omega^i_{G/S})$ is called **left-invariant** if $t_g^*\omega_T = \omega_T$ in $\Omega^i_{G_T/T}$ for all S-schemes T and all $g \in G(T)$, where t_g the translation-by-g map $G_T \to G_T$.

3.5.1. **Exercise.** If $S = \operatorname{Spec} k$ for a field k and G is a smooth S-group, then a differential ω is left-invariant if and only if the differential $\omega_{\overline{k}}$ on $G_{\overline{k}}$ is invariant under left $G(\overline{k})$-translation.

3.5.2. **Proposition** ([BLR], 4.2/1&3)**.** Fix an S-group scheme G with identity section $e \in G(S)$. For each $\omega_0 \in \mathrm{H}^0(e^*\Omega^i_{G/S})$, there exists a unique left-invariant different $\omega \in \mathrm{H}^0(\Omega^i_{G/S})$ such that $e^*\omega = \omega_0$. If G is smooth of relative dimension d over a *local* base S, then $\Omega^i_{G/S}$ is $\mathcal{O}_G$-free of rank $\binom{d}{i}$ with a basis of left-invariant i-forms. □

In particular, for our (d-dimensional, say) abelian variety $B_{/K}$, $\Omega^d_{B/K}$ has a generating B-invariant global section ω, unique up to $K^\times$-scaling since $\mathrm{H}^0(\mathcal{O}_B^\times) = K^\times$.

Let η be the generic point of some component C of $\mathcal{B}_k$. Since $\mathcal{B}$ is smooth, the local ring $\mathcal{O}_{\mathcal{B},\eta}$ is a dvr uniformized by the uniformizer π of R, and $\Omega^d_{\mathcal{B}/R,\eta}$ is a free $\mathcal{O}_{\mathcal{B},\eta}$-module of rank 1. The invariant differential ω on B can be viewed as a nonzero rational differential form on $\mathcal{B}$, and so there is a unique integer n – the order of the zero (or pole) of said rational differential along the divisor C – such that $\pi^{-n}\omega$ (belongs to and) generates $\Omega^d_{\mathcal{B}/R,\eta}$.We denote this n by $\mathrm{ord}_\eta(\omega)$.

From now on we enumerate the components of $\mathcal{B}_k$ as $\{C_i\}_{i\in I}$.

3.5.3. **Definition.** Let $n_i = \mathrm{ord}_{\eta_i}(\omega)$ and $n_0 = \min_i\{n_i\}$, where η_i is the generic point of the component C_i. We say C_i is **ω-minimal** if $n_i = n_0$.

3.5.4**. Key Fact**. Let $\mathcal{B}_i$ denote the open subscheme of $\mathcal{B}$ where all components of $\mathcal{B}_k$ except C_i have been removed. If there exists an R-rational map $\varphi : \mathcal{B}_i - - \to \mathcal{B}_j$ which is an isomorphism on generic fibers, we have $n_i \geq n_j$, and the restriction of φ to its domain of definition is an open immersion if and only if $n_i = n_j$.

(**Caveat**. Actually this holds only for φ satisfying a certain technical condition that we do not mention here, but that holds for the group translations to which we shall apply the fact in (3.5.6). For the proof of the fact and its applicability, see [BLR, 4.2/5, 4.3/1].)

3.5.5**.** We say $\mathcal{B}_i$ and $\mathcal{B}_j$ are **equivalent** if there exists an R-birational map $\mathcal{B}_i - - \to \mathcal{B}_j$ inducing the identity on the common generic fiber B. This is manifestly an equivalence relation on the set of R-models $\{\mathcal{B}_i\}_{i\in I}$ of B, and by (3.5.4) also on the subset $\{\mathcal{B}_i\}_{i\in I_0\subset I}$ of those $\mathcal{B}_i$ such that the special fiber C_i is ω-minimal. Let $\{\mathcal{B}_i\}_{i\in I_1\subset I_0}$ denote a set of representatives for the equivalence classes of said $\mathcal{B}_i$ with ω-minimal C_i. Thus the sets $I \supset I_0 \supset I_1$ index - respectively - all the models $\mathcal{B}_i$, those $\mathcal{B}_i$ with ω-minimal special fiber, and a set of equivalence class representatives for the $\mathcal{B}_i$ with ω-minimal special fiber.

We denote by $\mathcal{B}^{\mathrm{bg}}$ the subscheme of $\mathcal{B}$ obtained by removing all components C_j for $j \in I - I_1$ from the special fiber $\mathcal{B}_k$, or (what is the same) by gluing along their generic fibers the $\mathcal{B}_i$ for $i \in I_1$.

3.5.6. **Proposition.** The scheme $\mathcal{B}^{\mathrm{bg}}$ has an R-birational group law μ extending the K-group law on B.

3.5.7. *Sketch of the proof.* Let ξ be a generic point of $\mathcal{B}^{\mathrm{bg}}_k$ and $S' = \operatorname{Spec} R'$ for $R' = \mathcal{O}_{\mathcal{B}^{\mathrm{bg}},\xi}$. If $t_0 : B \times B \to B \times B$ is the universal translation $(b_1, b_2) \mapsto (b_1, b_1 b_2)$ we can consider the base change $t : S'_K \times B \to S'_K \times B$ along $\xi_K : S'_K \to B$. The crux of the problem is to extend t to an S'-birational map $\tau : S' \times_R \mathcal{B}^{\mathrm{bg}} - - \to S' \times_R \mathcal{B}^{\mathrm{bg}}$.

3.5.8. To attack this, one reduces to the case where $R = R'$ by proving that both the formation of weak Néron models and the notion of ω-minimal components of a weak Néron model commute (an an appropriate sense) with the change-of-base $R \to R'$ above; cf. [BLR, 3.5/4, 4.3/3] for these lemmas. In this manner the problem is reduced to showing that an arbitrary translation $t : B \to B$ extends to an R-birational map $\tau : \mathcal{B}^{\mathrm{bg}} - - \to \mathcal{B}^{\mathrm{bg}}$. It is here that we use the inputs into the proposition, namely that $\mathcal{B}$ is a weak Néron model and the definition of $\mathcal{B}^{\mathrm{bg}}$ in terms of gluing representatives for the equivalence classes of the $\mathcal{B}_i$ with ω-minimal special fiber.

3.5.9. For any $i \in I_1$, we can consider t as a map from the generic fiber of $\mathcal{B}_i$ to that of $\mathcal{B}$. Since $\mathcal{B}$ is a weak Néron model, it satisfies the WNMP by (3.3.1); this says that t extends to an R-rational map $\tau_i : \mathcal{B}_i - - \to \mathcal{B}$. Since $(\mathcal{B}_i)_k$ is irreducible, in fact τ_i lands in some $\mathcal{B}_j$ where C_j is *a priori* possibly non-ω-minimal, so j might not be in I_0, let alone I_1. But since C_i *is* ω-minimal, by (3.5.4) in fact the inequality $n_0 = n_i \geq n_j \geq n_0$ holds, so we see that $n_i = n_j = n_0$ and thus C_j is also ω-minimal. Since $n_i = n_j$, we also have by (3.5.4) that τ_i is an open immersion on its domain of definition. So the image of $\mathrm{dom}(\tau_i)$ in $\mathcal{B}_j$ is actually an R-dense open subscheme, where density in the closed fiber $(\mathcal{B}_j)_k = C_j$ follows from the irreducibility of C_j and the fact that $\mathrm{dom}(\tau_i)_k$ is dense in C_i, and thus nonempty. Hence $\tau_i : \mathcal{B}_i - - \to \mathcal{B}_j$ is R-birational.

Note that there is no uniqueness in the WNMP, so the index $j \in I_0$ is not necessarily uniquely determined by i. Nonetheless for each i we can choose such a j, which we denote by $\alpha(i) \in I_0$. Since $C_{\alpha(i)}$ is ω-minimal, $\mathcal{B}_{\alpha(i)}$ is equivalent to $\mathcal{B}_\ell$ for a unique index $\ell \in I_1$, which denote by $\beta(i)$. Thus there exists an R-birational map $\sigma_i : \mathcal{B}_{\alpha(i)} - - \to \mathcal{B}_{\beta(i)}$ inducing the identity on generic fibers. Since R-*bi*rational maps always compose, we have therefore produced an R-birational extension $\sigma_i \circ \tau_i : \mathcal{B}_i - - \to \mathcal{B}_{\beta(i)} \subset \mathcal{B}^{\mathrm{bg}}$ of t for each $i \in I_1$. Gluing the $\sigma_i \circ \tau_i$ along the generic fibers of the $\mathcal{B}_i$ for $i \in I_1$, we obtain the desired R-rational extension $\tau : \mathcal{B}^{\mathrm{bg}} - - \to \mathcal{B}^{\mathrm{bg}}$ of t.

We still need to show that τ is R-*bi*rational. Applying the procedure above to the translation $t' : B \to B$ inverse to t, for any i there is an R-birational extension $\rho : \mathcal{B}_{\beta(i)} - - \to \mathcal{B}_\ell$ of t' for some index $\ell \in I_1$. The composition $\rho \circ \sigma_i \circ \tau_i : \mathcal{B}_i - - \to \mathcal{B}_\ell$ is R-birational and induces the identity map $t' \circ t$ on B. So $\mathcal{B}_i$ and $\mathcal{B}_\ell$ are equivalent, and hence $i = \ell$ as the $\{\mathcal{B}_j\}_{j \in I_1}$ are a set of equivalence class representatives. In particular i is determined by $\beta(i)$ as the unique $\ell \in I_1$ such that t' extends to an R-birational map $\mathcal{B}_{\beta(i)} - - \to \mathcal{B}_\ell$. Thus the set map $\beta : I_1 \to I_1$ is injective and therefore bijective. It follows that the image of $\mathrm{dom}(\tau)$ contains a dense open subset of each component C_i $(i \in I_1)$ of $\mathcal{B}^{\mathrm{bg}}_k$, namely $\mathrm{dom}(\tau_{\beta^{-1}(i)})_k$. So the image of $\mathrm{dom}(\tau)$ is R-dense in $\mathcal{B}^{\mathrm{bg}}$.

We next construct an R-rational extension $\tau' : \mathcal{B}^{\mathrm{bg}} --\to \mathcal{B}^{\mathrm{bg}}$ of t' in the same manner as we constructed τ. Then the image of $\mathrm{dom}(\tau')$ is also R-dense in $\mathcal{B}^{\mathrm{bg}}$, and hence τ and τ' are composable in the sense of R-rational self-maps of $\mathcal{B}^{\mathrm{bg}}$ in either order. By the argument at the beginning of the previous paragraph, the restriction of $\tau' \circ \tau$ to any $\mathcal{B}_i \subset \mathcal{B}^{\mathrm{bg}}$ again lands in $\mathcal{B}_i$ and $(\tau' \circ \tau)|_{\mathcal{B}_i} : \mathcal{B}_i --\to \mathcal{B}_i$ is an R-birational map inducing the identity on B. By the uniqueness of R-rational extensions of maps to the generic fiber of a separated R-scheme, this means that $(\tau' \circ \tau)|_{\mathcal{B}_i}$ is equivalent as an R-rational map to the identity map on $\mathcal{B}_i$. Since this holds for each i, it follows that $\tau' \circ \tau : \mathcal{B}^{\mathrm{bg}} --\to \mathcal{B}^{\mathrm{bg}}$ is equivalent as an R-rational map to the identity map on $\mathcal{B}^{\mathrm{bg}}$. Similar reasoning holds for the composition $\tau \circ \tau'$, and hence τ and τ' invert one another in the sense of R-rational maps. Therefore τ is R-birational, as desired.

3.5.10. We now grant the existence of an S'-birational extension $\tau : S' \times_R \mathcal{B}^{\mathrm{bg}} --\to S' \times_R \mathcal{B}^{\mathrm{bg}}$ of the universal left translation $t_0 : B \times B \to B \times B$ with $S' = \mathrm{Spec}\,\mathcal{O}_{\mathcal{B}^{\mathrm{bg}},\xi}$ for any generic point of $\mathcal{B}_k^{\mathrm{bg}}$, and explain (somewhat sketchily) how to produce the desired birational group law on $\mathcal{B}^{\mathrm{bg}}$. Consider the partially-defined universal translation map $\Phi : \mathcal{B}^{\mathrm{bg}} \times_R \mathcal{B}^{\mathrm{bg}} --\to \mathcal{B}^{\mathrm{bg}} \times_R \mathcal{B}^{\mathrm{bg}}$ induced from multiplication m on B. The R'-birationality of τ implies that Φ is defined at the generic points of $(\mathcal{B}^{\mathrm{bg}} \times_R \mathcal{B}^{\mathrm{bg}})_k$ lying over ξ via the first projection. Varying ξ among all the generic points of $\mathcal{B}_k^{\mathrm{bg}}$ and applying the same reasoning, it follows that Φ is defined at all the generic points of $(\mathcal{B}^{\mathrm{bg}} \times_R \mathcal{B}^{\mathrm{bg}})_k$. Thus Φ is actually R-rational, i.e. its domain of definition is R-dense. Likewise the R'-*bi*rationality of τ entails that the image of $\mathrm{dom}(\Phi)$ contains all the generic points of $(\mathcal{B}^{\mathrm{bg}} \times_R \mathcal{B}^{\mathrm{bg}})_k$, and one deduces that Φ is R-birational. Composing Φ with a projection defines μ. The remainder of what must be shown, follows more or less formally from the fact that (B, m) is a group, since the R-dominance of Φ ensures that the required compositions for studying associativity of μ are defined. □

§*3.6.* Verification of the NMP for the solution $\mathcal{A}$ of $\mathcal{A}^{\mathrm{bg}}$. In light of (3.5.6), the removal of non-ω-minimal components from $\mathcal{A}_k^{\mathrm{weak}}$ yields a smooth, separated, finite type model $\mathcal{A}^{\mathrm{bg}}$ of A equipped with an R-birational group law extending the multiplication on A. By (3.4.2), this has a solution $\mathcal{A} := \overline{\mathcal{A}^{\mathrm{bg}}}$.

3.6.1. **Proposition.** $\mathcal{A}$ is a Néron model of A.

Sketch. Let $\pi : \mathcal{Z} \to \mathrm{Spec}\,R$ be a smooth R-scheme and $f : Z = \mathcal{Z}_K \to A$ a K-morphism. Note that since $\mathcal{Z}_k$ is smooth, its irreducible components are its conencted components. Removing all components but one from $\mathcal{Z}_k$ (without affecting the other components at all!) we can assume $\mathcal{Z}_k$ is irreducible, since if we produce a (unique) R-morphism $\varphi : \mathcal{Z} \to \mathcal{A}$ extending f in this case, they can be glued to obtain the desired R-extension of f in the general case. Arguing via a reduction similar to that in the proof of (3.5.6), one shows that there exists an R-rational map $\tau : \mathcal{Z} \times_R \mathcal{A} --\to \mathcal{A}$ extending $(z, a) \mapsto f(z)a$ on the generic fiber. It is defined on the generic fiber and hence in codimension 1, so by (2.1.4) it is defined everywhere. If $e : \mathrm{Spec}\,R \to \mathcal{A}$ is the identity section of the group scheme $\mathcal{A}$, then $\tau \circ (\mathbf{1}_{\mathcal{Z}}, e \circ \pi) : \mathcal{Z} \to \mathcal{Z} \times_R \mathcal{A} \to \mathcal{A}$ is a morphism extending $z \mapsto (z, e_K) \mapsto f(z)e_K = f(z)$, which coincides with f, on the generic fiber. Since $\mathcal{A}$ is separated, such an extension is easily seen to be unique. □

4. Elliptic curves

In this section we study the example of Néron models of elliptic curves in more detail. The purpose is to get a feel for what Néron models actually look like, since elliptic curves are the only abelian varieties for which it is practical to look at explicit equations. In this section R is a discrete valuation ring and K its fraction field.

§4.1. **Models of elliptic curves.** To contextualize the Néron model $\mathcal{N}$ of an elliptic curve E over K it is useful to compare it with other "canonical" R-models for E, which we now introduce.

4.1.1. **Definition.** An **R-model** for a separated K-scheme X is a separated flat R-scheme $\mathcal{X}$ such that $\mathcal{X}_K = X$. A **morphism** of R-models $\mathcal{X} \to \mathcal{X}'$ is a morphism of R-schemes extending the identity on $\mathcal{X}_K = \mathcal{X}'_K = X$ (with respect to the chosen, fixed identifications of $\mathcal{X}_K$ and $\mathcal{X}'_K$ with X). If there exists a morphism of R-models $\mathcal{X} \to \mathcal{X}'$ we say that $\mathcal{X}$ **dominates** $\mathcal{X}'$; this is a partial order (as by separatedness, a morphism of R-models from $\mathcal{X}$ to itself must be the identity).

The prototype of a domination map between R-models of X is the blow-up of a closed subscheme supported in the special fiber of an R-model $\mathcal{X}$.

4.1.2. **Definition.** A **minimal regular proper model** of a smooth K-curve (= smooth, proper, geometrically connected, 1-dimensional K-scheme) C is a regular proper R-model $\mathcal{C}$ of C which is minimal for the relation of domination among all regular proper R-models of C, in the sense that for any such model $\mathcal{C}'$, any domination map $\mathcal{C} \to \mathcal{C}'$ is an isomorphism.

4.1.3. **Theorem.** If C is a smooth K-curve of positive genus, there exists a unique minimal regular proper model $\mathcal{C}$ of C; moreover $\mathcal{C}$ enjoys the universal property that for every regular proper R-model $\mathcal{C}'$ of C, there exists a unique domination map $\mathcal{C}' \to \mathcal{C}$. □

This is a hard theorem; Christian may say something about its proof in his talk. By the theorem, an elliptic curve E over K has a minimal regular proper R-model $\mathcal{E}$. Roughly speaking, this is a regular proper R-model such that the special fiber has "as few components as possible": none can be blown down without losing regularity.

4.1.4. **Example.** Let $K = \mathbf{Q}_p$ and $R = \mathbf{Z}_p$. For the elliptic curves $X_0(11)$ and $X_1(11)$ over K, the respective minimal regular proper models $\mathcal{X}_0(11)$ and $\mathcal{X}_1(11)$ have respectively 2 and 1 irreducible components in their special fibers. The canonical isogeny $X_1(11) \to X_0(11)$ does not extend to an R-morphism $\mathcal{X}_1(11) \to \mathcal{X}_0(11)$. Indeed, if it did, the map would have to be dominant, since it is surjective on the generic fibers. But it would also have to be proper, and therefore surjective. And this is is impossible in light of the number of components of the two special fibers.

Therefore we see that the formation of the minimal regular proper model $\mathcal{C}$ of a K-curve C is not functorial with respect to finite maps, although it is functorial with respect to isomorphisms.

While it is not such a concrete thing, $\mathcal{E}$ can be quite directly related to the Néron model $\mathcal{N}$ of E.

4.1.5. **Theorem.** Write $\mathcal{E}^{\mathrm{sm}}$ for the smooth locus of $\mathcal{E}$. The canonoical map $\mathcal{E}^{\mathrm{sm}} \to \mathcal{N}$ induced by the NMP is an isomorphism.

Proof. Essentially one goes through the construction of the Néron model starting with $\mathcal{E}$ as an initial proper model. One observes that minimality of $\mathcal{E}$ implies that $\mathcal{E}^{\mathrm{sm}}$ remains unchanged as the construction proceeds. (I.e., all components of $\mathcal{E}^{\mathrm{sm}}_k$ are ω-minimal, and the solution of the induced birational group law cannot be bigger than $\mathcal{E}^{\mathrm{sm}}$.) See [BLR, 1.5/1] for the details. Alternately one can prove directly from the minimality of $\mathcal{E}$ that $\mathcal{E}^{\mathrm{sm}}$ has an R-group structure extending the group law on E; see [L, Lemma 10.2.12]. Using (2.1.4), this R-group scheme structure is enough to verify the NMP for $\mathcal{E}^{\mathrm{sm}}$. □

So far as it goes, this theorem is very nice. It can even be used to classify the possible special fibers of the Néron model, simply using the combinatorics of the intersection pairing on $\mathcal{E}$, viewed as an arithmetic "surface". For this theory see [L, §10.2.1].

However for a *concrete* description of $\mathcal{N}$ – say, in the form of an easy-to-understand morphism $\mathcal{N} \to \mathcal{N}'$ where $\mathcal{N}'$ is an R-model of E given by explicit equations – it is more convenient to work with Weierstrass models. We assume for convenience that the residue characteristic is $\neq 2, 3$.

4.1.6. **Definition.** A Weierstrass equation $y^2z = x^3 + \beta xz^2 + \gamma z^3$ for E in $\mathbf{P}^2_K$ defines a **minimal Weierstrass model** $\mathcal{W}$ of E over R if $\beta, \gamma \in R$ are such that the valuation $v(\Delta) = v(4\beta^3 + 27\gamma^2)$ is minimized among the valuations of the discriminants of Weierstrass equations for E with R-coefficients.

Such models are unique up to an explicit set of automorphisms of $\mathbf{P}^2_R$, and can be recognized, for example, by the sufficient condition $v(\Delta) < 12$; see [S1, §VII.1], or [C, §2] for a more "intrinsic" approach.

A minimal Weierstrass model can be obtained by blowing down the finitely many components of $\mathcal{E}_k$ which are disjoint from the closure in $\mathcal{E}$ of the identity in E. (See [L, Thm. 9.4.35].) Together with (4.1.5), this essentially proves the following.

4.1.7. **Theorem.** The smooth locus $\mathcal{W}^{\mathrm{sm}}$ of a minimal Weierstrass model for E is isomorphic to the relative identity component $\mathcal{N}^0$ the the Néron model of E. □

4.1.8. **Example.** As mentioned, the possibilities for $\mathcal{E}_k$ can be classified combinatorially. Even better, an algorithm of Tate lets one compute $\mathcal{E}$ starting from a Weierstrass equation for E. Ultimately one ends up with a list of possible tuples of values for the valuations of Δ, j, β and γ for a minimal Weierstrass equation, and an explicit description of the special fiber of $\mathcal{N}$ in each case. See [S2, §IV.9] for this algorithm in all of its excruciating glory (assuming the residue field k is perfect).

For example, the case of split multiplicative reduction is Néron's "type b_n" (Kodaira's "type I_n"), which corresponds to $v(j) = -n < 0, v(\beta) = v(\gamma) = 0$. In this case the special fiber $\mathcal{N}_k$ is the smooth part of a chain of n rational curves glued along rational closed points into a loop (or when $n = 1$, a single nodal rational curve). In particular $\mathcal{N}^0_k = \mathbf{P}^1 - \{\text{two points}\} \cong \mathbf{G}_m$.

4.1.9. **Example.** Using (4.1.7) it is easy to give examples of the failure of the formation of Néron models to commute with ramified base change. Consider any elliptic curve with bad reduction, but which acquires good reduction afer a ramified extension of the field K. This situation can be read off of a minimal Weierstrass equation over K. For example, we can take the elliptic curve

$$E : y^2 = x^3 + p$$

over $K = \mathbf{Q}_p$. This Weierstrass equation is minimal, as $v(\Delta) = v(27p^2) = 2 < 12$, assuming $p \nmid 6$. Over the totally ramified extension $K' = \mathbf{Q}_p(\pi)$ where $\pi = \sqrt[6]{p}$, E acquires good reduction: we can make the change of variables $x \to \pi^2 x, y \to \pi^3 y$, and the equation becomes $E_{K'} : \pi^6 y^2 = \pi^6 x^3 + \pi^6$, which is K'-isomorphic to $y^2 = x^3 + 1$, which has good reduction. In particular, if $R' = \mathfrak{o}_{K'}$ is the ring of integers of K', the Néron model $\mathcal{N}'$ of $E_{K'}$ over R' cannot coincide with $\mathcal{N} \otimes_R R'$ where $\mathcal{N}$ is the Néron model of E over $R = \mathbf{Z}_p$. For the extension $R \to R'$ is faithfully flat and properness descends along such maps; since $\mathcal{N}'$ must be proper, this would contradict the fact that $\mathcal{N}$ is non-proper.

In the rest of this section we relate various natural properties and quantities concerning the Néron model $\mathcal{N}$ of E to their analogues as defined in [S1] without mentioning Néron models. For the proofs we refer to [L], since these proceed via (4.1.5) and use information about the intersection pairing and special fiber of the various possibilities for the minimal regular proper model $\mathcal{E}$ of E.

§*4.2.* **The identity component and the filtration.** Here we mention how to translate between the language of Néron models and the filtration of $E(K)$ defined in Silverman when K is the fraction field of a complete dvr R with finite residue field k. Let $\mathcal{N}$ be the Néron model of E.

Observe that by the NMP, $E(K) = \mathcal{N}(R)$. By completeness of R and smoothness of $\mathcal{N}$ we also have a surjective group homomorphism $\mathcal{N}(R) \to \mathcal{N}_k(k)$. Composing these, we obtain Silverman's "reduction map"

$$r : E(K) \to \mathcal{N}_k(k) \supset \mathcal{N}_k^0(k) = \mathcal{W}^{\mathrm{sm}}(k).$$

We now define a filtration of abelian groups $E(K)^1 \subset E(K)^0 \subset E(K)$ by

$$E(K)^1 := \ker(r) \subset E(K)^0 := r^{-1}(\mathcal{N}_k^0(k)) \subset E(K).$$

We define the **component group** of E to be the finite étale k-group $\Phi_E := \mathcal{N}_k/\mathcal{N}_k^0$.

4.2.1. **Proposition** ([L], Prop. 10.2.26)**.** The reduction map r induces isomorphisms

$$E(K)^0/E(K)^1 \cong \mathcal{N}_k^0(k), \qquad E(K)/E(K)^0 \cong \Phi_E(k). \quad \square$$

§*4.3.* **Reduction types.** Here we compare the notions of multiplicative and additive reduction from [S1] with alternative definitions in terms of a Néron model.

4.3.1. **Definition.** Let $E_{/K}$ be an elliptic curve and $\mathcal{N}$ its Néron R-model. We say E has **multiplicative** (resp. **additive**) reduction if the connected component of the geometric special fiber $\mathcal{N}_{\overline{k}}^0$ is $\overline{k}$-isomorphic to $\mathbf{G}_m$ (resp. $\mathbf{G}_a$).

Recall that in [S1], E is said to have multiplicative (resp. additive) bad reduction if the geometric special fiber $\mathcal{W}_{\overline{k}}$ of a minimal Weierstrass R-model $\mathcal{W}$ of E is a nodal (resp. cuspidal) cubic in $\mathbf{P}^2_{\overline{k}}$, and that E is said to have good reduction if $\mathcal{W}_k$ is smooth.

4.3.2. **Proposition.** Good, multiplicative, additive reduction a la Silverman coincide with our definitions of these notions.

Proof. If E has good reduction then $\mathcal{N} = \mathcal{N}^0$ is an elliptic curve (that is, an abelian scheme of relative dimension 1) over R. In particular $\mathcal{W}^{\mathrm{sm}}$ is proper, which implies that $\mathcal{W} = \mathcal{W}^{\mathrm{sm}}$ is smooth, so $\mathcal{W}_k$ is too. Conversely if $\mathcal{W}_k$ is k-smooth then $\mathcal{W}$, being flat and finite presentation, is R-smooth, so $\mathcal{N} = \mathcal{W}^{\mathrm{sm}} = \mathcal{W}$ is proper, and thus E has good reduction.

So assume E has bad reduction. Now $\mathcal{N}_k^0 = \mathcal{W}_k^{\mathrm{sm}}$ and this is nonproper by (2.2.7). But we know what the possible special fibers of $\mathcal{W}$ are: $\mathcal{W}_{\overline{k}}^{\mathrm{sm}}$ is an affine plane cubic with a smooth compactification that adds either 1 or 2 points at infinity, in the case when $\mathcal{W}_k$ is cuspidal or nodal, respectively. So $\mathcal{N}_{\overline{k}}^0$ is a smooth connected 1-dimensional affine algebraic $\overline{k}$-group with either 1 or 2 points at infinity in its smooth compactification $\mathbf{P}^1$, depending on whether

E has additive or multiplicative reduction in the sense of [S1]. This implies that $\mathcal{N}^0_{\overline{k}}$ with the identity is isomorphic as a pointed curve to either $\mathbf{G}_a$ with 0 or $\mathbf{G}_m$ with 1, respectively. Now a straightforward analysis of the automorphism groups of the pointed schemes $(\mathbf{G}_a, 0)$ and $(\mathbf{G}_m, 1)$ shows that the only $\overline{k}$-group structures they admit are the usual ones, which completes the proof. □

5. Some more examples

§*5.1.* The Tamagawa number of an abelian variety.

5.1.1. **Definition.** Let A be an abelian variety over the fraction field K of a dvr R with residue field k. The **Tamagawa number** $c(A)$ is number $\#\Phi(k)$ of k-rational points of the (finite étale) **component group** $\Phi = \mathcal{A}_k/\mathcal{A}^0_k$.

Given an abelian variety over a global field F, its Tamagawa numbers (one for each place of F) arise in the formulation of the Birch and Swinnerton-Dyer Conjecture.

In order to compare with other definitions in the literature, it is worth mentining that $c(A)$ is also the number of geometrically connected components of $\mathcal{A}_k$ [L, Cor. 10.2.21(a)], and when k is finite also the number of connected components with a k-rational point.

In the case of an elliptic curve E, it turns out (cf. [L, Rem. 10.2.24]) that this number $c(E)$ is equal to the number of geometrically integral components occuring with multiplicity 1 in the special fiber of the minimal regular proper model $\mathcal{E}$, which contains the Néron model $\mathcal{N}$ of E as the smooth locus. As the possibilities for the minimal regular proper models can be classified and their special fibers read off from a minimal Weierstrass equation for E, it follows that the Tamagawa number of E can also be so computed. Even better, one actually gets the group structure of $\Phi(\overline{k}) = \Phi(k_s)$ and $\Phi(k)$; cf. *loc. cit.*

§*5.2.* The criterion of Néron-Ogg-Shafarevich. As an illustration of the utility of Néron models, we can give another proof of the Néron-Ogg-Shafarevich criterion (in the case of perfect residue field).

Let R be a dvr with fraction field K and residue field k, with $p = \mathrm{char}(k)$. Let $G_K = \mathrm{Gal}(K_s/K)$. Fix an extension v' of the valuation v on R to K_s and write I and D for the corresponding inertia and decomposition subgroups of G_K, so that $D/I = \mathrm{Gal}(k_s/k)$ is the Galois group of the separable algebraic closure of k obtained as the residue field of the valuation ring of v' in K_s.

5.2.1. **Theorem.** Let A be an abelian variety over K and Fix a prime $\ell \neq p$. The following are equivalent.

i. A has good reduction.
ii. The ℓ-adic Tate module $T_\ell(A)$ is an unramified representation of G_K; i.e. I acts upon it trivially.

5.2.2. Let $\mathcal{A}$ be the Néron R-model of A. To prove that (i)⇒(ii) in (5.2.1) we relate the inertial invariants $A[\ell^n](K_s)^I$ of the ℓ^n-torsion of A to the ℓ^n-torsion $\mathcal{A}_k[\ell^n](k_s)$ of the special fiber. Let K^{nr} be the maximal unramified extension of K in K_s, the fixed field of I. Then the valuation ring R' of the valuation v' on K^{nr} is strictly henselian with residue field k_s. Since R' is henselian and $\mathcal{A}$ is smooth, the "reduction map" $\mathcal{A}(R') \to \mathcal{A}_k(k_s)$ is surjective. Since R' is a union of étale R-algebras, the Néron mapping property says that $\mathcal{A}(R') = A(K^{\mathrm{nr}})$. As the prime ℓ is distinct from p, multiplication by ℓ^n is an étale endomorphism of $\mathcal{A}$. Since

R' is henselian, this implies (cf. [BLR, 7.3/3]) that the surjection $\mathcal{A}(R') \to \mathcal{A}_k(k_s)$ induces an *isomorphism* on ℓ^n-torsion.[11] Putting this all together, we have shown the following.

5.2.3. **Lemma.** There are bijections $A[\ell^n](K_s)^I = A[\ell^n](K^{\mathrm{nr}}) = \mathcal{A}[\ell^n](R') = \mathcal{A}_k[\ell^n](k_s)$. □

5.2.4. *Proof that (i)⇒(ii) in* (5.2.1). If A has good reduction then by (2.2.7) $\mathcal{A}_k$ is an abelian variety of dimension $d = \dim \mathcal{A}_k = \dim A$. So $\mathcal{A}_k[\ell^n](k_s)$ is a free $\mathbf{Z}/\ell^n\mathbf{Z}$-module of rank $2d$. By (5.2.3) the same is therefore true of $A[\ell^n](K_s)^I$. But $A[\ell^n](K_s)$ is also a free $\mathbf{Z}/\ell^n\mathbf{Z}$-module of rank $2d$. So $A[\ell^n](K_s)^I = A[\ell^n](K_s)$ is unramified. Passing to the inverse limit over n proves (ii). □

For the deeper implication (ii)⇒(i) – for which the proof in [S1] is not self-contained, relying implicitly on the finiteness of $\mathcal{A}_k/\mathcal{A}_k^0$ – we need some facts about algebraic groups.

5.2.5. **Theorem** (Chevalley)**.** Let k be a perfect field and G a smooth connected k-group scheme. Then there is a unique short exact sequence of algebraic groups

$$1 \to H \to G \to B \to 1$$

with H linear algebraic and B an abelian variety. □

5.2.6. **Theorem** (Structure of commutative linear algebraic groups)**.** Let k be a perfect field and G a smooth connected affine k-group. There exists a decomposition $G = T \times U$ of G as a product of smooth closed k-subgroups, where U is unipotent and T is a torus. □

5.2.7. The connected component $\mathcal{A}_k^0$ of the special fiber is a smooth connected commutative k-group. So by (5.2.5) and (5.2.6), $\mathcal{A}_{\overline{k}}^0$ is an extension of an abelian variety B over $\overline{k}$ by a linear algebraic $\overline{k}$-group $H = T \times U$ for T a torus and U unipotent.

Let $c = c(A)$. We first claim that for a prime $\ell \neq p$, the $\mathbf{Z}/\ell^n\mathbf{Z}$-module $\mathcal{A}_k[\ell^n](k_s) = \mathcal{A}_k[\ell^n](\overline{k})$ is an extension of a group of order dividing c by a free $\mathbf{Z}/\ell^n\mathbf{Z}$-module of rank $\dim T + 2\dim B$. For this we observe that the sequence of $\mathbf{Z}/\ell^n\mathbf{Z}$-modules

$$0 \to H[\ell^n](\overline{k}) \to \mathcal{A}_{\overline{k}}^0[\ell^n](\overline{k}) \to B[\ell^n](\overline{k}) \to 0$$

is exact since as $\ell \neq p$, the group $H(\overline{k})$ is ℓ^n-divisble. Since $U(\overline{k})$ has no ℓ-power torsion, $H[\ell^n](\overline{k}) = T[\ell^n](\overline{k})$ is $\mathbf{Z}/\ell^n\mathbf{Z}$-free of rank $\dim T$, while $B[\ell^n](\overline{k})$ is $\mathbf{Z}/\ell^n\mathbf{Z}$-free of rank $2\dim B$. So $\mathcal{A}_k^0(k_s)$ is $\mathbf{Z}/\ell^n\mathbf{Z}$-free of rank $\dim T + 2\dim B$, and this group sits in $\mathcal{A}_k(k_s)$ with finite index dividing $c(A)$.

5.2.8. *Proof that (ii)⇒(i) in* (5.2.1). Since $T_\ell(A)$ is unramified, there are arbitrarily large n such that $A[\ell^n](K_s)$ is unramified. Hence by (5.2.3) $\mathcal{A}_k[\ell^n](k_s)$ is $\mathbf{Z}/\ell^n\mathbf{Z}$-free of rank $2\dim A$ for arbitrarily large n. So there are arbitrarily large n such that this group has order $\ell^{2n\dim A}$. On the other hand by (5.2.7) this group has order $\ell^{n(\dim T + 2\dim B)}c'$ for some $c'|c(A)$. Thus

$$2n\dim A = n(\dim T + 2\dim B) + \log_\ell c'.$$

So allowing n to grow large and examining this formula asymptotically, we find $2\dim A = \dim T + 2\dim B$. But we also have the relationship $\dim T + \dim U + \dim B = \dim \mathcal{A}_{\overline{k}} = \dim \mathcal{A}_K = \dim A$. Rearranging gives $2\dim U + \dim T = 0$, and hence $H = 0$ and $\mathcal{A}_{\overline{k}}^0 = B$ is an abelian variety. In particular $\mathcal{A}_k^0$ is proper and we conclude from (2.2.7) that A has good reduction. □

[11] It is instructive to compare this with the argument in [S1] for injectivity of the reduction map on prime-to-p-torsion; Silverman identifies the kernel of the reduction map with the formal group of the elliptic curve or abelian variety in question.

5.2.9. **Corollary.** If $0 \to A' \to A \to A'' \to 0$ is a short exact sequence of abelian varieties over K and if A has good reduction, then A' and A'' also have good reduction.

Proof. The Tate modules of A' and A'' are subquotients of that of A. $\square$

§*5.3.* Grothendieck's p-adic version of Néron-Ogg-Shafarevich.

5.3.1. **Theorem** (Grothendieck)**.** Let A be an abelian variety over K. Assume R has mixed characteristic $(0,p)$.[12] Then A has good reduction if and only if its p-divisible group $A(p)$ extends to a p-divisible group over R. Thus – by (5.2.1) and the relationship Mike explained between étale ℓ-divisible groups and unramified Galois representation – for *any* prime ℓ (including p), the existence of an extension of $A(\ell)$ to an ℓ-divisible group over R is a necessary and sufficient condition for A to have good reduction.

Proof. See [SGA 7_I, 5.10]. The proof uses the results of Tate on p-divisible groups discussed in Brandon's talk. Néron models come in via the semistable reduction theorem, to be discussed later in the seminar. $\square$

6. More properties of the Néron model

In this section we list some useful facts about Néron models, with reference to [BLR] for proofs.

§*6.1.* A criterion for a group scheme to be a Néron model. Sometimes it's nice to be able to check that an R-scheme is a Néron model without having to verify the Néron mapping property for arbitrary smooth points. In the presence of a group structure it's enough to verify this for points valued in étale R-algebras.

6.1.1. **Proposition.** A smooth R-group scheme of finite type $\mathcal{G}$ is the Néron model of its generic fiber G if and only if $\mathcal{G}(R^{\mathrm{sh}}) \to \mathcal{G}(K^{\mathrm{sh}})$ is an isomorphism (and if and only if the latter is surjective and $\mathcal{G}$ is separated).

Proof. The proof, using the weak Néron mapping property (3.3) and Weil's extension theorem (2.1.4), can be found in [BLR, 7.1/1]. $\square$

§*6.2.* Base change and descent. Over a discrete valuation ring R, the formation of Néron models is compatible with extensions of the base R'/R of **ramification index 1**; this means that a uniformizer for R also uniformizes R' and that the residue extension k'/k is separable (but possibly non-algebraic). The key device for proving this is also interesting in its own right: Néron models descend from the strict henselization R^{sh}. This is sort of a converse to the compatibility of the formation of Néron models with étale base change:

6.2.1. **Proposition.** Let $R \subset R' \subset R^{\mathrm{sh}}$ be a local extension of discrete valuation rings contained in the strict henselization of R, and let $K \subset K' \subset K^{\mathrm{sh}}$ be the respective fraction fields. Let G be a K-smooth group scheme of finite type and assume $G' = G_{K'}$ has a Néron model $\mathcal{G}'$ over R'. Then $\mathcal{G}'$ descends to a Néron model $\mathcal{G}$ of G over R; that is, $\mathcal{G}' = \mathcal{G} \otimes_R R'$.

Proof. See [BLR, 6.5/4]. $\square$

[12] By later work of de Jong, equicharacteristic p is OK, too.

6.2.2. **Proposition.** Let $R \subset R'$ be a local extension of discrete valuation rings with respective fraction fields $K \subset K'$, and let G be a smooth K-group scheme of finite type. Suppose R'/R has ramification index 1. Then G admits a Néron model $\mathcal{G}$ over R if and only if $G_{K'}$ admits a Néron model $\mathcal{G}'$ over R', in which case $\mathcal{G}' = \mathcal{G} \otimes_R R'$.

Proof. See [BLR, 7.2/1]. After reducing to the case where R and R' are strictly henselian via (6.2.1), one verifies that $\mathcal{G} \otimes_R R'$ is a Néron model of $G_{K'}$ by checking the criterion of (6.1.1). For the latter one must make use of the smoothening construction of (3.1.2). Suppose on the other hand that one has $\mathcal{G}'$ but not $\mathcal{G}$. The existence of $\mathcal{G}'$ turns out to be equivalent to the boundedness of $G_{K'}(K'^{\mathrm{sh}})$ in $G_{K'}$ by [BLR, 6.5/4]; see the remark (1.3.11) for this notion. This in turn implies the boundedness of $G(K^{\mathrm{sh}})$ in G and hence the existence of $\mathcal{G}$. $\square$

An important case of the preceding proposition is when $R' = \widehat{R}$ is the completion of R.

§***6.3.*** **Exactness.** We end by mentioning a result concerning the exactness properties of the formation of Néron models. One interesting aspect of this is its connection to what we covered in the seminar during the Fall.

6.3.1. **Theorem** ([BLR], 7.5/4)**.** Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence of abelian varieties over K. Assume R has characteristic $(0, p)$ and absolute ramification index $e < p - 1$. If A has good reduction, then the sequence of Néron R-models $0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0$, which by (5.2.9) consists of abelian schemes, is exact – in the sense that $\pi : \mathcal{A} \to \mathcal{A}''$ is a smooth surjection and $\iota : \mathcal{A}' \to \mathcal{A}$ is a closed immersion identifying $\mathcal{A}'$ with the kernel of π, i.e. fiber over the identity section of $\mathcal{A}''$.

The proof uses Raynaud's results on group schemes of type $(p, p, \ldots, p)$, of course, as the hypotheses indicate. In fact if A merely has **semi-abelian reduction** – meaning that $\mathcal{A}^0_k$ is an extension of an abelian variety by a torus, rather than by a commutative linear algebraic group with nontrivial unipotent part – the sequence of Néron models is still left exact, and by a criterion of Grothendieck to be discussed later, $\mathcal{A}'$ and $\mathcal{A}''$ ave semi-abelian reduction. In particular $\mathcal{A}'$ is a closed subgroup scheme of $\mathcal{A}$.

References

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[ST] Serre & Tate, Good reduction of abelian varieties.