SEMI-STABLE REDUCTION FOR CURVES

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1. Arithmetic Surfaces

Let S be a connected Dedekind scheme. We denote by $K = K(S)$ its function field. For example, S could be the spectrum of a DVR, and then K is its field of fractions.

Question 1. Suppose we are given a curve C, say normal and proper and geometrically connected over $K(S)$ (so C is automatically projective). Does there exist a "good" model over S?

We start with the following:

Definition 2. An *arithmetic surface* is an integral, projective, and flat S-scheme $\pi : X \to S$ of relative dimension 1, where S is as before.

Clearly, the generic fiber X_η of an arithmetic surface is an integral and projective curve over the function field $K(S)$. Conversely, if $S = \operatorname{Spec} R$ for a DVR R, then an arithmetic surface over S is a model of its generic fiber.

Given a normal proper and geometrically connected curve C over the function field $K(S)$ of S, we seek "good" models over S. For instance, if we choose a projective embedding of C into some $\mathbb{P}^n_{K(S)}$ then its Zariski closure in $\mathbb{P}^n_S$ is an arithmetic surface (where the S-flatness uses that S is Dedekind). Hence, we can always find an arithmetic surface with generic fiber C. Since S is normal and C is reduced and geometrically connected over $K(S)$, so $\mathrm{H}^0(C, \mathcal{O})$ is a finite purely inseparable extension field of $K(S)$, it follows from considerations with Stein factorization (whose fibers are always geometrically connected) that any such arithmetic surface has geometrically connected fibers over S.

For such an arithmetic surface, we need to resolve its singularities.

Resolution of Singularities. Given an arithmetic surface, there exists a resolution of singularities:

Theorem 3 (Lipman)**.** *Let $X \to S$ be an arithmetic surface, where S is an excellent Dedekind scheme. Then, successively blowing up the non-regular locus and normalizing, we eventually arrive (after a finite number of steps) at a regular arithmetic surface $\pi' : X' \to S$.*

This is highly non-trivial, and we refer to [Liu, Section 8.3.4] for references, discussion, and further results.

Exceptional Divisors. So, we now have a regular arithmetic surface $\pi : X \to S$ whose generic fiber is our given curve C/K. Now, blowing up a closed point in a fiber of our arithmetic surface do not change the generic fiber. However, it introduces a special curve in this fibers, and a "good" model should have so such curves. More precisely,

Definition 4. Let $\pi : X \to S$ be a regular arithmetic surface. A prime divisor E on X is called *exceptional divisor* (or, (-1)-*curve*), if there exists a regular arithmetic surface $Y \to S$ and a morphism $f : X \to Y$ of S-schemes, such that $f : (X - E) \to (Y - f(E))$ is an isomorphism, and $f(E)$ is a point.

Thus, $f : X \to Y$ is the blow-up of Y in the closed point $f(E)$ with exceptional divisor E.

Proposition 5. *Let $f : X \to Y$ be an S-morphism of regular arithmetic surfaces that is the blow-up in a closed point $y \in Y_s$ for some $s \in S$. Then, the exceptional divisor $E := X_y$ satisfies*

$$E \cong \mathbb{P}^1_{\kappa(y)}, \quad \text{and} \quad \kappa(y) = H^0(E, \mathcal{O}_E).$$

Moreover, we have

$$\mathcal{O}_X(E)|_E \cong \mathcal{O}_E(-1) \quad \text{and} \quad E^2 = -[\kappa(y) : \kappa(s)],$$

where E^2 denotes the self-intersection number.

Proof. This follows from the definition and standard properties of blow-ups, see [Liu, Proposition 9.2.5]. □

The converse of this result is more difficult to prove, and goes back to Castelnuovo, who proved it for complex projective surfaces:

Theorem 6 (Lichtenbaum–Shafarevich, Castelnuovo's Criterion)**.** *Let $\pi : X \to S$ be a regular arithmetic surface. Let $E \subset X_s$ be a vertical prime divisor and set $k' := H^0(E, \mathcal{O}_E)$. Then, E is an exceptional divisor if and only if $E \cong \mathbb{P}^1_{k'}$ and $E^2 = -[k' : \kappa(s)]$.*

Proof. We choose relatively ample invertible sheaf $\mathcal{L}$ on $\pi : X \to S$. Then, we set $\mathcal{M} := \mathcal{L}^{\otimes m}(-rE)$, where $m := -E^2$ and $r := \deg_E \mathcal{L}$. Then, it turns out that the a priori only rational map

$$X \dashrightarrow Y := \operatorname{Proj} \bigoplus_{i \geq 0} \pi_*(\mathcal{M}^{\otimes i})$$

is in fact an S-morphism. It contracts E to a point and is an isomorphism outside E. The main difficulty is to check that the so-constructed Y is regular. A proper proof can be found in [Liu, Theorem 9.3.8]. □

Proposition 7. *Let $\pi : X \to S$ be a regular arithmetic surface. Then, there exists a birational morphism $f : X \to Y$ of arithmetic surfaces over S, with Y regular and without exceptional divisors.*

Proof. By Castelnuovo's Criterion, we can contract exceptional curves. Now, there are only finitely many fibers containing exceptional divisors [Liu, Lemma 9.3.17], and also, every fiber has only finitely many components. Thus, successively contracting exceptional curves, we eventually arrive at a regular model without exceptional divisors. A proper proof can be found in [Liu, Proposition 9.3.19]. □

Definition 8. A regular arithmetic surface $\pi : X \to S$ is called *relatively minimal* if it contains no exceptional divisors.

By what we have just seen, given a regular arithmetic surface, we can contract its exceptional divisors and arrive at a regular arithmetic surface that is relatively minimal.

Example 9. Let $\pi : X \to S$ be an arithmetic surface that is smooth over S. Then, X is relatively minimal. In fact, the fibers of π are connected and so smoothness forces them to be irreducible. But an irreducible fiber F satisfies $F^2 = 0$ and thus, cannot be exceptional.

2. Minimal Models

In view of the algorithm in the proof of Proposition 7, one might hope that for a given regular arithmetic surface $\pi : X \to S$ there exists a unique relatively minimal arithmetic surface that is S-birational to X. It turns out that this is "almost" true.

Definition 10. A regular arithmetic surface $\pi : X \to S$ is called *minimal* if every birational map $Y \dashrightarrow X$ of regular arithmetic surfaces over S is a birational morphism.

Given a regular arithmetic surface $Y \to S$, we call $X \to S$ a *minimal model* of Y if $X \to S$ is minimal and if there exists a birational map $Y \dashrightarrow X$ over S (which, of course, then extends to a birational morphism).

Remarks 11. It is not difficult to see (exercise!)

(1) If a regular arithmetic surface admits a minimal model, then this minimal model is unique up to S-isomorphism.
(2) A minimal arithmetic surface $\pi : X \to S$ is relatively minimal.

Here is another nice property of minimal arithmetic surfaces:

Proposition 12. *Let $X \to S$ be a minimal arithmetic surface. Let η be the generic point of S. Then, the canonical map*

$$\mathrm{Aut}_S(X) \to \mathrm{Aut}_{K(S)}(X_\eta)$$

is bijective. In other words, every automorphism of X_η extends in a unique way to an automorphism of X.

Proof. Let $\sigma : X_\eta \to X_\eta$ be an automorphism. This induces a birational map $\sigma' : X \dashrightarrow X$, which is a morphism by hypothesis. Applying the same argument to σ^{-1}, we conclude that σ' is an automorphism. See also [Liu, Proposition 9.3.13]. □

This shows that arithmetic surfaces, whose generic fibers have "many" automorphisms, might not admit minimal models. Indeed, we have the following:

Example 13. Consider $X_1 := \mathbb{P}^1_S$.

Let X be the blow-up of X_1 in a closed point $x \in (X_1)_s$ for some $s \in S$. In the fiber X_s, the strict transform E of of $(X_1)_s$ is an exceptional divisor. Let $X \to X_2$ be the contraction of E. Then, see [Liu, Remark 9.3.23]

(1) the models X_1 and X_2 are relatively minimal, but
(2) the birational map $X_1 \dashrightarrow X_2$ induced by the identity on their generic fibers does *not* extend to an isomorphism.

Working a little bit harder, one can explicitly classify all relatively minimal arithmetic surfaces, whose generic fiber is $(X_1)_\eta = \mathbb{P}^1_\eta$. For all of them a similar reasoning as before shows that they are not minimal. Thus,

(3) X_1 does not admit a minimal model,

see [Liu, Exercise 9.3.1].

On the positive side, we have the following result, due to Castelnuovo and Enriques in the setting of complex projective surfaces:

Theorem 14 (Lichtenbaum–Shafarevich)**.** *Let $\pi : X \to S$ be a regular arithmetic surface with generic fiber of arithmetic genus $p_a(X_\eta) \geq 1$. Then, X admits a unique minimal model over S.*

Proof. Uniqueness is clear by Remark 11. We also know that $\pi : X \to S$ admits a relatively minimal model by Proposition 7. It suffices to show that any two relatively minimal models $\pi_i : X_i \to S$, $i = 1, 2$ of $\pi : X \to S$ are isomorphic. If not, we can find an arithmetic surface $Z \to S$ dominating both, X_1 and X_2, as well as exceptional divisors $E_1, E_2 \subset Z$ such that E_1 is contained in the exceptional locus of $Z \to X_1$, and there exists an integer μ with $(E_1 + \mu E_2)^2 \geq 0$, see [Liu, Lemma 9.3.20]. Using the adjunction formula, this implies that $-K_{X/S}$ is ample and thus $p_a(X_\eta) < 1$, a contradiction. For details, we refer to [Liu, Theorem 9.3.21]. □

Corollary 15. *Let $\pi : X \to S$ be an arithmetic surface that is relatively minimal and that satisfies $p_a(X_\eta) \geq 1$. Then, X is minimal.*

3. Canonical Models

For a regular arithmetic surface $\pi : X \to S$ with generic fiber of arithmetic genus $p_a(X_\eta) \geq 1$ we now have existence and uniqueness of minimal models. Such a minimal model arises from an arbitrary regular model by successively contracting all exceptional curves. In case $p_a(X_\eta) \geq 2$, we can contract further: the resulting so-called *canonical model* X_{can} has the property that its relative dualizing sheaf $\omega_{X_{\mathrm{can}}/S}$ is ample. The prize we have to pay for this is that this canonical model is in general no longer regular, but has mild singularities.

We recall from Definition 4 that blow-ups in smooth points correspond to exceptional curves, or, (-1)-curves. These curves have been characterized in Proposition 5 and the most important result is Castelnuovo's Criterion, i.e., Theorem 6. The following results are analogues of these results:

Proposition 16. *Let $\pi : X \to S$ be a regular arithmetic surface with $p_a(X_\eta) \geq 2$. Let $s \in S$ be a closed point, let $\Gamma \subseteq X_s$ be a vertical prime divisor, and $k' = H^0(\Gamma, \mathcal{O}_\Gamma)$. Then, the following are equivalent:*

(1) $K_{X/S} \cdot \Gamma = 0$,
(2) $H^1(\Gamma, \mathcal{O}_\Gamma) = 0$ *and* $\Gamma^2 = -2[k' : \kappa(s)]$, *and*
(3) Γ *is a conic over* k' *and* $\deg_{k'} \mathcal{O}_X(\Gamma)|_\Gamma = -2$.

Proof. This follows from the adjunction formula, standard results on curves of arithmetic genus $p_a = 0$ and straight forward computations with intersection numbers, see [Liu, Proposition 9.4.8]. □

Theorem 17 (Artin)**.** *Let $X \to S$ be a regular arithmetic surface, and Γ_i, $i = 1, ..., r$ be vertical prime divisors, such that $K_{X/S} \cdot \Gamma_i = 0$ and such that the intersection matrix $(\Gamma_i \cdot \Gamma_j)_{i,j}$ is negative definite.*

Then, there exists an S-morphism $f : X \to Y$ contracting all Γ_i's that is an isomorphism outside $\bigcup_i \Gamma_i$.

Proof. The idea is similar to the proof of Theorem 6, see [Liu, Theorem 9.4.2] and [Liu, Corollary 9.4.7]. □

After contracting all these curves, we obtain the following

Proposition 18. *Let $\pi : X \to S$ be a minimal arithmetic surface with $p_a(X_\eta) \geq 2$. Let $\mathcal{E}$ be the set of all vertical prime divisors Γ that satisfy $K_{X/S} \cdot \Gamma = 0$. Then*

(1) *the set $\mathcal{E}$ is finite and there exists a birational morphism $f : X \to Y$ that contracts all $\Gamma \in \mathcal{E}$ (and nothing else), and*
(2) *the dualizing sheaf $\omega_{Y/S}$ is relatively ample.*

Proof. See [Liu, Proposition 9.4.20]. □

Definition 19. The arithmetic surface $Y \to S$ from Proposition 18 is called the *canonical model* of $\pi : X \to S$.

We note that the canonical model is singular as soon as there exists at least one contracted component.

4. (Potential) Semi-Stable Reduction of Curves

Now, we have minimal regular models. In general, it is too much to ask for minimal models that are smooth over S, even after extending the base, see Examples 24 below. However, if we allow base extensions, we can always find models whose singular fibers are not "too bad". More precisely,

Definition 20. A curve C over a field k (i.e., pure 1-dimensional k-scheme of finite type) is called *semi-stable* if it is geometrically reduced, and if the finitely many non-smooth points x $C_{\overline{k}}$ are ordinary double points, i.e., $\mathcal{O}^{\wedge}_{C_{\overline{k}},x} \simeq \overline{k}[[u,v]]/(uv)$ as $\overline{k}$-algebras.

Remark 21. Using Artin approximation, this definition is equivalent to something much stronger: for each of the finitely many non-smooth points $x \in C$, (C,x) and $(\mathrm{Spec}(k[u,v]/(u,v)),(0,0))$ admit a common pointed étale neighborhood. In particular, the finite extension $k(x)/k$ is necessarily separable and for some finite separable extension $k'/k(x)$ and $x' \in C_{k'}$ over x we have $\mathcal{O}^{\wedge}_{C_{k'},x'} \simeq k'[[u,v]]/(uv)$ as k'-algebras. This is the 1-dimensional case case of the structure theorem for ordinary double points proved in [FK].

The following result relates semi-stability to properties of the Picard group. This is an important ingredient in the proof of Theorem 27. For a reduced, connected, projective curve C over an algebraically closed field we define

$$t \;:=\; t(C) \;:=\; 1 - c + \sum_{x \in C(k)} (m_x - 1),$$

where c denotes the number of components of C, and m_x the multiplicities of the ordinary multiple points To be precise, one has to pass to a partial normalization of C that resolves the unibranch singularities, and then take the multiplicities there. We refer to [Liu, Chapter 7.5] for details.

Theorem 22. *Let C be a connected projective curve over an algebraically closed field k, with irreducible components $C_1, ..., C_n$. Let $\pi : C' \to C$ be the normalization morphism and C'_i the normalization of C_i. Then, there exists a short exact sequence of group schemes*

$$1 \;\to\; \ker(R) \;\to\; \mathrm{Pic}^0_{C/k} \;\stackrel{R}{\to}\; \prod_{i=1}^{n} \mathrm{Pic}^0_{C'_i/k} \;\to\; 1.$$

The kernel $\ker(R)$ *is an extension of a torus T by a smooth connected unipotent group U, and*

$$\begin{array}{lcl} \dim T &=& t \\ \dim U &=& \dim_k H^1(\mathcal{O}_C) - \sum_{i=1}^n g(C'_i) - t\,. \end{array}$$

PROOF. See [Liu, Theorem 7.5.19] for a version (sufficient for what follows) which uses just ordinary Picard groups rather than Picard schemes. The k-group $\mathrm{Pic}^0_{C'_i/k}$ is the Jacobian of the smooth projective curve C'_i, whereas the k-group $\mathrm{Pic}^0_{C/k}$ is the open and closed k-subgroup of $\mathrm{Pic}_{C/k}$ whose k-points have degree-0 pullback to each C'_i (and it is also the identity component of $\mathrm{Pic}_{C/k}$. (Next week's talk will discuss Picard schemes in some detail.) □

Definition 23. Let S be a Dedekind scheme with function field $K = K(S)$. Let C be a normal and geometrically connected projective curve over K and let $s \in S$ be a closed point. The curve has *good reduction* (resp. *semi-stable* *reduction*) at s if there exists a model $\pi : \mathcal{C} \to S$ of C such that $\mathcal{C}_s$ is a smooth (resp. semi-stable) curve over $\kappa(s)$.

We illustrate these notions for elliptic curves. In particular, a given curve may have neither good nor semi-stable reduction at a given prime.

Examples 24. Let $p \in \mathbb{Z}$ be a prime $p \geq 5$.

(1) The arithmetic surface $y^2 = x^3 + x^2 + p$ over $\mathbb{Z}$ has semi-stable reduction at p. The minimal discriminant has valuation $\nu = 1$, which implies that its generic fiber cannot have good reduction at p.
(2) The arithmetic surface $y^2 = x^3 + p$ over $\mathbb{Z}$ has neither good, nor semi-stable reduction at p. In fact, the minimal discriminant has valuation $\nu = 2$, so its generic fiber cannot have good or semi-stable reduction at p.
(3) Quite generally, for every elliptic curve $E/\mathbb{Q}$ there exists a prime, where the reduction is not good.

These statements follow, for example, from results on Néron models, the Tate algorithm and minimal discriminants, see [Si, Chapter IV].

Proposition 25. *Let S be a Dedekind scheme, and C be a smooth projective and geometrically connected curve over $K = K(S)$ of genus $g \geq 1$.*

(1) *The curve C has good reduction at $s \in S$ except perhaps for a finite set of closed points of S.*
(2) *The curve C has good reduction over S if and only if the minimal regular model $\mathcal{C} \to S$ of C over S is smooth. In this case, $\mathcal{C} \to S$ is the unique smooth model of C over S.*

Proof. The first assertion follows from taking an arbitrary model and generic smoothness. As for the second assertion: if the minimal regular model is smooth, then C has good reduction over S, and the converse follows from Example 9 and Corollary 15. We refer to [Liu, Proposition 10.1.21] for a proper proof. $\square$

The next result shows that semi-stability is preserved under pull-back. Moreover, the third assertion is the key to showing that it often suffices to construct semi-stable reductions étale locally, or, over completions.

Proposition 26. *Let S be a Dedekind scheme with function field K. Let C be a smooth projective and geometrically connected curve over K of genus $g \geq 1$.*

(1) *If C has semi-stable reduction over S then the minimal regular model of C has semi-stable reduction over S.*

Next, let S' be a Dedekind scheme with function field K'. Suppose that S' dominates S.

(2) *If C has semi-stable reduction over S, then $C_{K'}$ has semi-stable reduction over S'.*

(3) *Assume $S' \to S$ to be surjective and étale or that $S = \operatorname{Spec} \mathcal{O}_K$ is local and $S' = \operatorname{Spec} \widehat{\mathcal{O}}_K$. If $C_{K'}$ has semi-stable reduction over S', then C has semi-stable reduction over S.*

Proof. For the first assertion, one chooses a model over S with semi-stable reduction. An explicit computation shows that also the minimal desingularization has semi-stable reduction. This regular model possesses a birational morphism onto the minimal regular model, which is a successive contraction of exceptional divisors. ¿From this, it is not difficult to see that the minimal regular model has semi-stable reduction. We refer to [Liu, Theorem 10.3.34] for a complete proof.

The second assertion follows from a local computation and the fact that flat (resp. proper) morphisms remain flat (resp. proper) after base change, see also [Liu, Proposition 10.3.15].

For the third assertion, we know from (1) that the minimal regular model $\mathcal{C}' \to S'$ of C' has semi-stable reduction. By our assumptions on $S' \to S$, the minimal regular model of C' is $\mathcal{C}' = \mathcal{C} \times_S S' \to S'$, where $\mathcal{C} \to S$ is the minimal regular model of C. Thus, if $\mathcal{C}' \to S'$ is semi-stable, then so is $\mathcal{C} \to S$. We refer to [Liu, Corollary 10.3.36] for details. □

The following result is the *potential semi-stable reduction theorem* for curves:

Theorem 27 (Artin–Winters). *Let S be a Dedekind scheme. Let C be a smooth, projective, and geometrically connected curve of genus $g \geq 1$ over $K(S)$. Then, there exists a Dedekind scheme S' that is finite flat and generically étale over S such that $C_{K(S')}$ has semi-stable reduction over S'.*

Proof. First, one reduces to the case where $S = \operatorname{Spec} R$, where R is a DVR with algebraically closed residue field [Liu, Lemma 10.4.5]. Let K be the field of fractions of R. We fix a prime ℓ different from the residue characteristic of R.

Passing to a finite extension K'/K, we may assume that $C(K') \neq \emptyset$. Possibly extending K' further, we may assume that the ℓ-torsion $\operatorname{Pic}(C)[\ell] = \operatorname{Jac}(C)[\ell]$ is K'-rational. We set $C' := C_{K'}$ and then

$$2g \;=\; \dim_{\mathbb{F}_\ell} \operatorname{Pic}(C')[\ell].$$

Let R' be a DVR with quotient field K' dominating R and set $S' := \operatorname{Spec} R'$. Let $\pi : X' \to S'$ be the minimal regular model of C' over S'. We arranged that $C'(K')$ is non-empty, and this is equal to $X'(S')$ by the valuative criterion for properness. Such a section must pass through the relative smooth locus, so $X'_{s'}$ has non-empty smooth locus (and hence an irreducible component that is generically reduced).

Let a, t, u be the abelian, toric, and unipotent ranks of $\operatorname{Pic}^0_{X'_{s'}/k}$ as in Theorem 22. (By definition, a is the dimension of the right side of the exact sequence there, and $t = \dim T$ and $u = \dim U$, so by definition $a + t + u =$

$h^1(X'_{s'}, \mathcal{O})$.) Since $X'_{s'}$ has a reduced component, cohomological flatness [Liu, Corollary 9.1.24] implies

$$g = h^1(C', \mathcal{O}) = h^1(X'_{s'}, \mathcal{O}) = a + t + u.$$

Then, the Picard groups of special and generic fiber are related by an exact sequence [Liu, Proposition 10.4.17]

$$0 \to \mathrm{Pic}(X'_{s'})[\ell] \to \mathrm{Pic}(C')[\ell] \to \Phi(X')[\ell].$$

Here, $\Phi(X')$ is a finite Abelian torsion group associated to the components of the special fiber (it is the component group of the Néron model of the the Jacobian of C'). From this exact sequence and Theorem 22, we deduce

$$2g = \dim_{\mathbb{F}_\ell} \mathrm{Pic}(C')[\ell] \le \dim_{\mathbb{F}_\ell} \Phi(X')[\ell] + \dim_{\mathbb{F}_\ell} \mathrm{Pic}(X'_{s'})[\ell].$$

In particular, we compute $2g \le t + (2a + t) = 2g - 2u$ and conclude $u = 0$. Thus, $X'_{s'}$ has only ordinary multiple points.

Now, $\pi : X' \to S'$ is regular, i.e., the Zariski-tangent spaces to points of the special fiber satisfy

$$\dim_{\kappa(x)} T_{X'_{s'},x} \le \dim_{\kappa(x)} T_{X',x} = 2.$$

In particular, all ordinary multiple points are ordinary double points. Thus, C' has semi-stable reduction. (There is an issue with non-reduced fiber components, see [Liu, Proposition 10.3.42]).

The complete proof can be found in [Liu, Theorem 10.4.3]. $\square$

Remark 28. This proof uses of the theory of the Picard scheme, though the exposition in [Liu] is designed to avoid logical dependence on those topics (at the cost of masking where some of the ideas really come from). An explicit and geometric proof in case all residue characteristics are zero, is presented in [Liu, Section 10.4.1].

5. (Potential) Stable Reduction of Curves

Of course, we want to do better! It turns out that we can find models of curves, whose singular fibers are semi-stable and have only finite automorphism groups, i.e., are *stable*. However, the price we have to pay is that such models will usually have mild singularities.

Definition 29. A curve C over an algebraically closed field k is called *stable*, if it is semi-stable, and moreover,

(1) C is connected, projective, and of arithmetic genus $p_a(C) \ge 2$,
(2) if $\Gamma \subset C$ is an irreducible component that is isomorphic to $\mathbb{P}^1_k$ then Γ intersects the other irreducible components of C in at least 3 points.

A curve C over an arbitrary field k is called semi-stable if and only if $C_{\overline{k}}$ is semi-stable.

Theorem 30. *Let S be a Dedekind scheme. Let C be a smooth projective curve of genus $g \ge 1$ over $K = K(S)$. Assume that C has semi-stable reduction over S.*

(1) *The minimal model* $\pi : X \to S$ *of* C *is semi-stable over* S.
(2) *If* $g \geq 2$ *and* C *is geometrically connected over* K*, then the canonical model is stable over* S.

Proof. We have seen the first part already in Proposition 26.

For the second assertion, let $\rho : X_{\min} \to X_{\mathrm{can}}$ be the contraction from the minimal to the canonical model. As in the proof of Proposition 26, one concludes that the canonical model has semi-stable reduction. Since the fibers of the canonical model contain neither (-1)-curves nor (-2)-curves, this implies that the fibers are in fact stable. A proper proof can be found in [Liu, Theorem 10.3.34]. □

Theorem 31 (Deligne–Mumford)**.** *Let* S *be a Dedekind scheme. Let* C *be a smooth, projective, and geometrically connected curve of genus* $g \geq 2$ *over* $K(S)$*. Then, there exists a Dedekind scheme* S' *that is finite flat over* S *such that* $C_{K(S')}$ *has a unique stable model over* S'.

Proof. First, one reduces to the case where $S = \mathrm{Spec} R$, where R is a DVR with algebraically closed residue field [Liu, Lemma 10.4.5]. By Theorem 27, there exists a finite flat extension $R \to R'$ over which C acquires semi-stable reduction. By Theorem 30, C also acquires stable reduction over this extension. For a proper proof we refer to [Liu, Theorem 10.4.3]. □

6. Semi-Abelian Schemes

Definition 32. Let S be a scheme. A *semi-Abelian scheme* of relative dimension g over S is a smooth, separated and commutative group scheme $p : G \to S$, whose geometric fibers are connected of dimension g, and are extensions of an Abelian variety by a torus.

Examples 33. For example,

(1) Abelian schemes are semi-Abelian schemes, and
(2) tori are semi-Abelian schemes.
(3) Let E be an elliptic curve over a number field K. Let $\mathcal{O}_K$ be the ring of integers of K and $\mathcal{E} \to \mathrm{Spec}\, \mathcal{O}_K$ be the Néron model of E. For every prime $\mathfrak{p}$ of $\mathcal{O}_K$, the geometric fiber is isomorphic (as a group scheme) to
$$\mathcal{E} \otimes_{\mathcal{O}_K} \overline{\kappa(\mathfrak{p})} \cong \Phi_{\mathfrak{p}} \times F_{\mathfrak{p}}$$
where $\Phi_{\mathfrak{p}}$ is a finite Abelian group, and where $F_{\mathfrak{p}}$ is an elliptic curve, the multiplicative group $\mathbb{G}_m$ (a torus), or the additive group $\mathbb{G}_a$.

If, for all primes $\mathfrak{p}$ of $\mathcal{O}_K$, the group $\Phi_{\mathfrak{p}}$ is always trivial and $F_{\mathfrak{p}}$ is never $\mathbb{G}_a$, then the Néron model of E is a semi-Abelian scheme.
(4) Let $q : C \to S$ be a stable curve of genus g. Then
$$J := \mathrm{Pic}^{\tau}_{C/S} \to S$$
is a semi-Abelian scheme of relative dimension g, see next week's talk by Brian.

(5) The moduli space $\mathcal{A}_g$ of principally polarized Abelian varieties of dimension g can be compactified using semi-Abelian schemes [CF].

One warning, before proceeding:

Caveat 34. In general, semi-Abelian schemes do *not* admit Néron models in the classical sense, since tori do not. However, semi-Abelian schemes admit so-called *lft Néron models*, which are only locally of finite type. We refer to Example 5 of [BLR, Chapter 10.1], where the lft Néron model for $\mathbb{G}_m$ is worked out.

Existence of lft Néron models implies the following result, which is a Néron-type mapping property for semi-Abelian schemes:

Lemma 35. *Let S be locally Noetherian and normal, $U \subseteq S$ open and dense, $p_1 : A_1 \to S$ and $p_2 : A_2 \to S$ be two semi-Abelian schemes, $\phi : A_1|_U \to A_2|_U$ a homomorphism of algebraic groups defined over U. Then, ϕ extends uniquely to S.*

Proof. ([Fa, Lemma §2.1], details provided by Brian.)

Let us first assume that $S = \operatorname{Spec} R$, where R is a complete DVR with generic point η, special point s, and algebraically closed residue field $\kappa(s)$. Then, there exists a lft Néron model $\mathcal{N}_2$ of $(A_2)_\eta$, see Theorem 2 of [BLR, Chapter 10.2]. The identity morphism of $(A_2)_\eta$ induces a morphism (in fact, homomorphism of group schemes) from A_2 to $\mathcal{N}_2$, and identifies the identity component of the special fiber $(\mathcal{N}_2)_s^0$ with the special fiber $(A_2)_s$. Next, $\phi|_\eta : (A_1)_\eta \to (A_2)_\eta$ induces a homomorphism $A_1 \to \mathcal{N}_2$. Since the identity component $(A_1)_s^0$ is connected, this homomorphism factors as $A_1 \to A_2 \to \mathcal{N}_2$. For details, we refer to the margin of page 292 of Brian's copy of [BLR].

For the general case, we let

$$X \subseteq A_1 \times_S A_2$$

be the closure of the graph of ϕ.

We consider the projection $\mathrm{pr}_1 : X \to A_1$ and want to show that it is an isomorphism. For a closed point $s \in A_1$, we choose a complete DVR T with algebraically closed residue field, and a morphism such that the generic point η goes to the generic point of U and the special point maps to the closed point we just started with. Base-changing to T, and using the already established result above, we see that $\mathrm{pr}_1 \times_U T$ is surjective and quasi-finite (in fact, bijective). A descent argument shows that the extension of ϕ already exists over $\kappa(s)$. Thus, ϕ extends.

Now, since both group schemes are semi-Abelian, torsion sections lie dense. Thus, if $\phi|_U$ extends, then this extension is unique.

For details, we refer to the margin of page 10 of Brian's copy of [Fa2]. □

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