

GABBER'S LEMMA

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1. INTRODUCTION

For our study of Faltings' heights on abelian varieties over number fields, it will be convenient if we can compactify the “moduli space” of g -dimensional abelian varieties (over \mathbf{Z}) such that the universal abelian scheme extends to a semi-abelian scheme over the compactification. The reason for this wish is that ideally we'd like to set up a theory for which the Faltings height of an abelian variety over a number field K (viewed as a K -point on the moduli space, which in turn sits in the sought-after compactification) is related to the height of a point on a projective variety in the sense of Weil's original conception of a height function (so we can apply finiteness properties of such “Weil heights” to the study of finiteness problems for abelian varieties.) There are several serious problems that arise in this plan.

First of all, to have a meaningful notion of “moduli space” we need to fix the degree of a polarization, yet as we vary even though an isogeny class of abelian varieties over a fixed number field it is not at all clear what degrees are possible for polarization. In particular, it might not happen that the abelian variety admits a principal polarization, nor is even isogenous to one. Over an algebraically closed field the latter can always be achieved, yet this comes at a cost: understanding the effect of isogenies on the Faltings height will be a serious issue to be confronted later, not something we want to deal with right at the start of the theory.

A second problem is that even if we focus on a fixed polarization degree, say d^2 , the theory only works nicely over $\mathbf{Z}[1/d]$ (not \mathbf{Z} , if $d > 1$) and the “moduli space” is actually a Deligne–Mumford stack and not a scheme. Compactifying a stack (especially over $\text{Spec } \mathbf{Z}[1/d]$) is a highly nontrivial matter. In the case $g = 1$ there is a natural compactification via the Deligne–Rapoport theory of generalized elliptic curves, but beyond dimension 1 matters get very complicated. We'd like to avoid such things. Gabber's result, to be stated a bit later, provides a substitute which will be sufficient for the purpose of developing properties of Faltings' heights to prove the Mordell and Tate and Shafarevich conjectures. (For other purposes, such as integral models of Shimura varieties, one has to get into the work of Faltings–Chai and its generalizations on the compactification of moduli stacks.)

Gabber's Lemma (really a theorem) will provide a higher-dimensional generalization of the semi-stable reduction theorems for curves and abelian varieties. Let's recall the statements of those two fundamental results, which have been discussed in earlier lectures (by Christian and me).

Theorem 1.1 (Semi-stable reduction for curves). *Let R be a discrete valuation ring, K its fraction field, and C a smooth proper and geometrically connected curve over K of genus $g \geq 2$. There exists a finite separable extension K'/K such that for the R -finite integral*

closure R' of R in K' the base change $C_{K'}$ extends to a proper flat map $\mathcal{C} \rightarrow \text{Spec } R$ with semistable closed fibers.

In the preceding theorem, one can be much more precise: such a \mathcal{C} can be arranged to be regular as well as to satisfy a certain “minimality” property, and as such it is the minimal regular proper model of its generic fiber. These refinements have no analogue over a higher-dimensional base, so we do not focus on them (and will not need them).

Theorem 1.2 (Semi-stable reduction for abelian varieties). *Let R be a discrete valuation ring, K its fraction field, and A an abelian variety over K . There exists a finite separable extension K'/K such that for the R -finite integral closure R' of R in K' the base change $A_{K'}$ extends to a semi-abelian scheme $\mathcal{A} \rightarrow \text{Spec } R$.*

Remark 1.3. Recall that the earlier formulation of this result was in terms of the Néron model $N(A_{K'})$ over R' having special fibers with semi-abelian identity component; i.e., the relative identity component $N(A_{K'})^0$ is a semi-abelian scheme. This is really equivalent to Theorem 1.2 because of the general fact (Theorem 4.4 in the notes for my earlier lecture on semistable reduction for abelian varieties) that a semi-abelian scheme with proper generic fiber over a discrete valuation ring is necessarily the relative identity component of the Néron model of its generic fiber.

We have avoided any mention of Néron models in the statement of Theorem 1.2 (even though they are essential in the proof) because the concept of Néron model has no higher-dimensional analogue. Néron models are nonetheless a key ingredient in the proof of the following lemma of Faltings (Lemma 35 in the notes from Christian’s talk) which will be very useful in what follows:

Lemma 1.4 (Faltings’ lemma with proof in 1000 ways). *Let S be a normal noetherian scheme, and $U \subseteq S$ a dense open subscheme. The functor $A \rightarrow A_U$ from the category of semi-abelian schemes over A to the category of semi-abelian schemes over U is fully faithful.*

Remark 1.5. We will need this lemma only in the case that the generic fibers are proper, or equivalently (by the “spreading out” principle) that our semi-abelian schemes are abelian schemes over some dense open in the base.

2. A WARM-UP WITH CURVES

As a prelude to the case of abelian varieties, let’s first discuss an extension problem for proper flat families of curves. This is not only an instructive preparation for the case of abelian varieties (which is the actual focus of Gabber’s Lemma), but it is also logically relevant as an ingredient in the proof of the result to be given in the case of abelian varieties. Roughly speaking, our approach to the study of abelian varieties will be parallel to the approach we took in the proof of the semi-stable reduction theorem for abelian varieties: we will ultimately reduce problems for abelian varieties to problems for curves by using the fact that any abelian variety over a field is isogenous to a direct factor of the Jacobian of a smooth proper (and geometrically connected) curve.

Here is the main result for curves, which we will see is a natural higher-dimensional generalization of Theorem 1.1.

Proposition 2.1. *Let S be a noetherian scheme, and $f : X \rightarrow S$ a separated map of finite type. Let $u : C \rightarrow X$ be a family of smooth proper curves of genus $g \geq 2$: a smooth proper map whose fibers are geometrically connected curves of genus $g \geq 2$.*

Then there exists a proper surjection $\pi : X' \rightarrow X$ and an open immersion $j : X' \hookrightarrow \overline{X}'$ into a proper S -scheme such that the pullback family $C_{X'} \rightarrow X'$ extends across \overline{X}' to a proper semistable curve. That is, there is a commutative diagram

$$\begin{array}{ccc}
 C_{X'} & \longrightarrow & \mathcal{C} \\
 u_{X'} \downarrow & & \downarrow \bar{u} \\
 X' & \xrightarrow{j} & \overline{X}' \\
 \pi \downarrow & & \downarrow \bar{f}' \\
 X & \xrightarrow{f} & S
 \end{array}$$

in which \bar{f}' is proper, j is an open immersion, π is a proper surjection, the top square is cartesian, and \bar{u} is a proper flat map whose fibers are geometrically connected semistable curve.

Before proving this result, we make some remarks.

Remark 2.2. A very special case of Proposition 2.1 is a consequence of Theorem 1.1: take $S = \text{Spec } R$ and $X = \text{Spec } K$. Then we take $X' = \text{Spec } K'$ and $\overline{X}' = \text{Spec } R'$ (which is S -proper, even S -finite), and \mathcal{C} to be the minimal regular proper model of $C_{K'}$.

For our actual applications, the base scheme S of most interest will be $\text{Spec } \mathbf{Z}$. In particular, Proposition 2.1 will only be needed by us in the case that S is a \mathbf{Z} -scheme of finite type (though that setting is far too restrictive to be regarded as satisfactory in general).

Remark 2.3. The use of base change through proper surjections to bypass difficulties is a very effective technique in algebraic geometry, and is not as horrible as it may seem (e.g., the fibers of a proper surjection can be awful, especially when one has as little control as we will on the geometric details of the surjections we'll use).

First of all, recall that Chow's Lemma provides a powerful technique for reducing problems for proper maps to the case of projective maps: it says that for any finite type and separated map $X \rightarrow S$ to a noetherian scheme, there is a surjective projective map $\pi : X' \rightarrow X$ (to be precise, X' is closed in some \mathbf{P}_X^m) such that X' is quasi-projective over S (i.e., X is open in a closed subscheme of some \mathbf{P}_S^n). In particular, by using an open immersion $j : X' \hookrightarrow \overline{X}'$ into a projective closure, we can "compactify" X after the base change π . This is a poor man's version of Nagata compactification, but for many theoretical purposes it is entirely sufficient (and certainly a lot easier to prove; the proof of Nagata's theorem requires refinements of Chow's Lemma and a whole lot more anyway).

Secondly, and perhaps more interestingly, proper surjective maps arise in the classical "graph trick" for removing indeterminacies in a rational map. We now recall how this goes (keeping in mind that in geometrically nice settings, one can exert much more control over the construction). Let U be a dense open subscheme of a scheme X that is of finite type over a noetherian scheme S , and suppose there is given an S -morphism $f : U \rightarrow Y$ to a

proper S -scheme Y . We'd like to extend f to X , at least after a proper surjective base change. That is, we seek a proper surjective map $\pi : X' \rightarrow X$ such that the pullback $f \circ \pi : U' = \pi^{-1}(U) \rightarrow Y$ extends to an S -map $X' \rightarrow Y$.

To make X' , consider the graph morphism $\Gamma_f : U \rightarrow U \times_S Y$; this is a closed immersion since it is a base change of $\Delta_{Y/S}$. Thus, the composite map

$$U \xrightarrow{\Gamma_f} U \times_S Y \xrightarrow{j \times 1} X \times_S Y$$

is a locally closed immersion. Hence, the schematic closure Z in $X \times_S Y$ contains U as an open subscheme. More specifically, since Γ_f is a closed immersion, the restriction of $Z \rightarrow X$ over U is an isomorphism $p_1^{-1}(U) \simeq U$; in particular, the map $\pi : Z \rightarrow X$ is proper (since Y is S -proper) and dominant, so it is surjective, and by construction $f \circ \pi : \pi^{-1}(U) \rightarrow Y$ extends to $p_2 : Z \rightarrow Y$. Hence, we can take $X' = Z$.

Having made many remarks, we now prove Proposition 2.1.

Proof. By Chow's Lemma, there is a proper surjective map $X' \rightarrow X$ such that X' is quasi-projective over S . By renaming X' as X and replacing C with $C_{X'}$, we may and do assume that X is quasi-projective over S (i.e., it is a closed subscheme of an open subscheme of some \mathbf{P}_S^n). Let $j : X \hookrightarrow \overline{X}$ be an open immersion into a projective S -scheme. We'll try to extend our family of curves over X to one over \overline{X} , at least after some proper surjective base change, by using the principle of resolving indeterminacies of rational maps via proper surjective base change as in the preceding remark. The idea is that the data of $C \rightarrow X$ "corresponds" to an S -morphism

$$u : X \rightarrow M_g := \overline{\mathcal{M}}_g \times_{\mathrm{Spec} \mathbf{Z}} S$$

to the *proper* moduli space of stable genus- g curves over the category of S -schemes (with $C \rightarrow X$ obtained as a pullback of the "universal curve" $\mathcal{C} \rightarrow M_g$), and so we just need to extend u to an S -map $\overline{u} : \overline{X} \rightarrow M_g$ (and then take the pullback $\overline{u}^*(\mathcal{C}) \rightarrow \overline{X}$ as our semistable \overline{X} -curve extending $C \rightarrow X$).

The problem with this idea is that M_g does not actually exist as a (fine moduli) scheme; it is really a (Deligne–Mumford) stack. Nonetheless, we will carry on and eventually use a trick with "Chow's Lemma for stacks" to return to the scheme setting. Much as Chow's Lemma for schemes is the tool *par excellence* to reduce certain kinds of *general* proofs for proper morphisms to the projective case, there is a Chow's Lemma for algebraic spaces which is similarly effective for reducing certain kinds of *general* proofs for proper algebraic spaces to the case of proper schemes, and likewise there is a Chow's Lemma for Deligne–Mumford (even Artin) stacks which reduces problems for such stacks to the case of algebraic spaces (and then even schemes). We'll come to this issue later in the argument. For now, let's prove the result we seek, modulo the problem that our solution will only be a stack (but otherwise have all of the properties we seek!).

Although this is not the time or place to delve into the theory of stacks, we note that the usefulness of stacks is due to their ability to impart "geometric" concepts to the study of moduli problems. More specifically, all of the basic notions of algebraic geometry (irreducible, finite type, proper, flat, smooth, étale, surjective, fiber product, open immersion, etc.) have reasonable definitions for stacks in a manner which is consistent with the theory of schemes.

In effect, working with stacks (or algebraic spaces) defines the “representability by a scheme” question out of existence and replaces it with the burden of proving that a given moduli problem is “nearly representable” in a sense that allows us to do algebraic geometry with the moduli problem. Fortunately, Artin developed powerful criteria to establish that many interesting moduli problems have this “nearly representable” property, making them be stacks in the sense of Deligne–Mumford or Artin. And these verifications have been carried out by Deligne and Mumford in the case of the moduli problem of “stable genus- g curves”, yielding a stack $\overline{\mathcal{M}}_g$ which is *proper* over \mathbf{Z} and admits a “stable genus- g curve” $\mathcal{C} \rightarrow \overline{\mathcal{M}}_g$ with a suitable universal property (the precise details of which we will not delve into here, except to say that it provides essentially everything we could want from a universal object as if the functor of interest were representable).

The upshot is that the family $C \rightarrow X$ corresponds to a morphism $v : X \rightarrow \overline{\mathcal{M}}_g$ (whatever is meant by “morphism”) insofar as $C = v^*(\mathcal{C})$. Promoting v to an S -morphism $u : X \rightarrow M_g = \overline{\mathcal{M}}_{g/S}$ identifies C with $u^*(\mathcal{C}_S)$. Consider the graph

$$\Gamma_u : X \rightarrow X \times_S M_g;$$

this is a base change of $\Delta_{M_g/S}$, so it is a closed immersion since M_g is S -separated. (Warning: the notion of separatedness for morphisms of Artin stacks exhibits some surprises, but for Deligne–Mumford stacks – which are all that we will need – it does correspond to the “closed immersion” property for the diagonal morphism.) By definition, $C = \Gamma_u^*(p_2^*(\mathcal{C}_S))$. Composing u with the open immersion $X \times_S M_g \hookrightarrow \overline{X} \times_S M_g$ yields a locally closed immersion

$$h : X \rightarrow \overline{X} \times_S M_g$$

such that $C \rightarrow X$ is the pullback of the semistable curve over $\overline{X} \times_S M_g$ obtained by p_2 -pullback of the universal curve over M_g . Observe that $\overline{X} \times_S M_g$ is a *proper* Deligne–Mumford stack over S , since \overline{X} is a proper S -scheme and M_g is a proper Deligne–Mumford stack.

Let Z denote the “schematic closure” of h , which is to say the zero locus in $\overline{X} \times_S M_g$ defined by the (quasi-)coherent ideal sheaf $\mathcal{O}_{\overline{X} \times_S M_g} \rightarrow h_*(\mathcal{O}_X)$. This is a closed substack of $\overline{X} \times_S M_g$, so Z is also S -proper (but alas, a Deligne–Mumford stack rather than a scheme). Also, naturally X is an open substack of Z . By construction, if we form the cartesian diagram

$$\begin{array}{ccc} \overline{C} & \longrightarrow & \mathcal{C}_S \\ \downarrow & & \downarrow \\ Z & \longrightarrow & M_g \end{array}$$

then $\overline{C} \rightarrow Z$ restricts to $C \rightarrow X$ over the open substack $X \subseteq Z$. Thus, if Z were a scheme then we would be done! To get around this problem, we will use the following handy general result for stacks (whose proof requires some serious new ideas beyond the case of schemes, building on the new ideas needed in the case of algebraic spaces):

Lemma 2.4 (Chow’s Lemma for DM stacks). *Let S be a noetherian scheme, M a Deligne–Mumford stack, and $M \rightarrow S$ a separated morphism of finite type. There exists a quasi-projective S -scheme M' and a proper surjective morphism $M' \twoheadrightarrow M$. Moreover, M is S -proper if and only if M' is S -projective.*

Proof. This is proved in the book *Champs Algébriques* by Laumon and Moret-Bailly (look up the index or table or contents). We note that they actually prove a more general version in which M is permitted to be an Artin stack. ■

By Chow's Lemma for DM stacks, there is a proper surjection $Z' \twoheadrightarrow Z$ in which Z' is a quasi-projective S -scheme. The open immersion $X \hookrightarrow Z$ pulls back to an open immersion $X' \hookrightarrow Z'$, so X' is a scheme that is proper surjective over X' and by design

$$C_{X'} = (\overline{C} \times_Z X) \times_X X' = (\overline{C} \times_Z Z') \times_{Z'} X',$$

so for $\overline{C}' := \overline{C} \times_Z Z' \rightarrow Z'$ is a proper flat semistable curve whose restriction over X' is $C_{X'}$. We can therefore name Z' as \overline{X}' to conclude the proof of Proposition 2.1. ■

3. GABBER'S LEMMA

We are finally in position to state Gabber's "Lemma" (which is a lemma towards Faltings theorems, but really is a basic result in its own right). The reader should compare this against the statement of Proposition 2.1, and also note (using Remark 1.3) how in a very special case it is a consequence of Theorem 1.2 much as Proposition 2.1 in a very special case is a consequence of Theorem 1.1 (as explained in Remark 2.2 in the case of curves).

Theorem 3.1 (Gabber). *Let S be a noetherian scheme, and $f : X \rightarrow S$ a separated map of finite type. Let $u : A \rightarrow X$ be an abelian scheme. Then there exists a proper surjection $\pi : X' \twoheadrightarrow X$ and an open immersion $j : X' \hookrightarrow \overline{X}'$ into a proper S -scheme such that the pullback abelian scheme $A_{X'} \rightarrow X'$ extends across \overline{X}' to a semi-abelian scheme.*

That is, there is a commutative diagram

$$\begin{array}{ccc} A_{X'} & \longrightarrow & A' \\ u_{X'} \downarrow & & \downarrow \overline{u} \\ X' & \xrightarrow{j} & \overline{X}' \\ \pi \downarrow & & \downarrow \overline{f}' \\ X & \xrightarrow{f} & S \end{array}$$

in which \overline{f}' is proper, j is an open immersion, π is a proper surjection, the top square is cartesian, and \overline{u} is a semi-abelian scheme.

The basic principle behind the proof of Theorem 3.1 is to deduce it from Proposition 2.1 by eventually reducing to the case when $A \rightarrow X$ is a family of Jacobians. To make this work, we will use Lemma 1.4 several times.

Let's first carry out some preliminary simplifications. By Chow's Lemma we may and do assume that X is quasi-projective over S . Let $\{X_i\}$ be the set of irreducible components of X , each given the reduced structure. Thus, $\coprod X_i \rightarrow X$ is a proper surjection. Renaming this disjoint union as X (as we may) reduces us to the case when each irreducible component of X is integral. Hence, we may and do assume that X is integral.

Consider the normalization map $\widetilde{X} \rightarrow X$. This is a *finite* morphism when X is an excellent scheme. The notion of "excellence" was introduced by Grothendieck as the technical

answer to many prayers in commutative algebra; it is a concept which is satisfied for nearly all schemes that arise in “real life” such as finite type schemes over Dedekind domains of generic characteristic 0 (such as \mathbf{Z}) or complete discrete valuation rings, preserved under operations such as localization and the formation of locally finite type morphisms, and ensures that normalizations are finite and the formation of completions preserves many homological properties of local noetherian rings (such as reducedness and normality). Nonetheless, there are examples of non-excellent 1-dimensional local noetherian rings of characteristic $p > 0$ (with normalization that is *not* finite).

In particular, if S is excellent then $\tilde{X} \rightarrow X$ is finite. This applies when S is finite type over \mathbf{Z} , which is the case we will need. There are powerful “limit arguments” in EGA IV₃, §8–§11ff. which allow one to reduce the proof of Theorem 3.1 to the case of S of finite type over \mathbf{Z} anyway, so the reader interested in the general case loses nothing by assuming S is excellent.

The upshot is that we may replace X with \bar{X} to reduce to the case that X is normal and irreducible. In particular, Lemma 1.4 can be applied. The importance of this is that if we can find a proper surjection $X' \twoheadrightarrow X$ and an open immersion $X' \hookrightarrow \bar{X}'$ into a proper S -scheme such that there is a semi-abelian scheme $A' \rightarrow \bar{X}'$ satisfying $A'|_{U'} = A_{U'}$ for a dense open $U' \subseteq X'$ then by replacing \bar{X}' with the normalization of the reduced closure of an irreducible component of X' that dominates the irreducible X (as we may certainly do), *necessarily* this identification extends to an isomorphism $A'|_{X'} \simeq A_{X'}$. That is, for the purposes of checking that a given A' really works, we just need to check this over some dense open of X' . Loosely speaking, our problem now only depends on A' through the specification of the abelian variety generic fiber A_η of A over X (by the principle of “spreading out” from a generic fiber). This suggests that ideas from the theory of abelian varieties over *fields* (such as identifying an abelian variety as an isogeny factor of a Jacobian) may be profitably applied to our problem.

Pick an abelian variety B_η over η such that $A_\eta \times B_\eta$ is isogenous to the Jacobian of a smooth proper and geometrically connected curve C_η over η . By inserting some extra elliptic curve factors before creating the Jacobian we can ensure that $B_\eta \neq 0$, so the Jacobian has dimension ≥ 2 and hence C_η has genus ≥ 2 . Since η is a “limit” of dense opens in X , by the “spreading out” principle there is a dense open $U \subseteq X$ and an abelian scheme B_U with generic fiber B_η and smooth proper curve $C_U \rightarrow U$ (with geometrically connected fibers of genus g) such that there is an isogeny of abelian U -schemes

$$\mathrm{Pic}_{C_U/U}^0 \rightarrow A_U \times B_U.$$

Pick an open immersion $X \hookrightarrow \bar{X}$ into a proper S -scheme. (Recall that we arranged for X to be quasi-projective over S .) The curve C_U corresponds to an S -morphism $U \rightarrow \overline{\mathcal{M}}_{g/S}$ to a proper Deligne–Mumford stack over S . Does this extend to an S -morphism from \bar{X} ? That would imply that C_U extends to a semistable \bar{X} -curve, which may not be possible. But the “graph trick” argument as in the proof of Proposition 2.1 can be applied to our present circumstances to construct a proper surjective map $\pi : X' \twoheadrightarrow X$ and an open immersion $X' \hookrightarrow \bar{X}'$ into a *normal* irreducible proper S -scheme such that for $U' = \pi^{-1}(U)$ the pullback

curve $(C_U)_{U'}$ extends to a proper semistable curve $\overline{C}' \rightarrow \overline{X}'$. Hence, the abelian scheme $(\text{Pic}_{C_U/U}^0)_{U'}$ over U' extends to the semi-abelian scheme $\text{Pic}_{\overline{C}'/\overline{X}'}^0$ over the entirety of \overline{X}' .

Remark 3.2. Why does $\text{Pic}_{\overline{C}'/\overline{X}'}^0$ exist as a scheme? Grothendieck's work on representability of Picard functors required (among other things) the hypothesis of geometric integrality of the fibers. Hence, in the case of a proper flat family of curves (say over a noetherian base), his results do not apply in the presence of reducible geometric fibers. That is a serious restriction for the study of semistable curves!

Fortunately, Artin's techniques with algebraic spaces can be applied under very mild hypotheses that are always satisfied in the case of proper semistable curves (over a scheme). These matters were discussed around Theorem 7.12 in my notes on semistable reduction, the upshot of which is that for a proper semistable curve $C \rightarrow T$ over a noetherian scheme T , there is a finite type algebraic space group $\text{Pic}_{C/T}^0$ over T that is smooth (due to the functorial criterion for smoothness and the 1-dimensionality of curves) and separated (a deep result of Raynaud) with geometric fibers that are semi-abelian (see Proposition 7.14 of my notes on semistable reduction for abelian varieties, applied to fibers over geometric points of T). In other words, $\text{Pic}_{C/T}^0$ exists as a "semi-abelian algebraic space" over T .

Deligne proved that these particular algebraic spaces (with C a semistable curve) are actually schemes. For our purposes (avoiding algebraic spaces and stacks in the final results we wish to use, even if they're used in the middle of some proofs), we only need a rather weaker assertion: this algebraic space becomes a scheme after a proper surjective base change. That is, it is enough that for some proper surjective $T' \twoheadrightarrow T$ and the curve $C' = C_{T'} \rightarrow T'$, the base change $(\text{Pic}_{C/T}^0)_{T'} = \text{Pic}_{C'/T'}^0$ is a scheme.

Here is a proof of an even stronger result: there is a *finite* surjective map $T' \twoheadrightarrow T$ such that $\text{Pic}_{C/T}^0$ becomes a scheme after base change to T' . The starting point is the special case $T = \text{Spec } R$ for a strictly henselian normal local noetherian ring R . In this special case the "projective scheme" property is directly proved, without needing base change, by an alternative construction via birational group laws in 9.3/7 of "Néron Models" (using the strictly henselian property to get enough sections to perform translations). That construction requires the structural morphism $C \rightarrow T$ to be projective Zariski-locally on T , but this condition is always satisfied over a strictly henselian local ring for proper flat curves with geometrically reduced fibers (by lifting a finite étale ample divisor on the special fiber, and using openness in the base for the ampleness condition on fibers).

To build $T' \twoheadrightarrow T$ in general, we first make a finite surjective base change so that T is normal and integral. Hence, for any $t \in T$ the local ring $\mathcal{O}_{T,t}$ is an integrally closed domain. Fix a choice of t . The strict henselization $\mathcal{O}_{T,t}^{\text{sh}}$ is a directed union of localizations of integral closures of $\mathcal{O}_{T,t}$ in certain finite separable extensions of the function field of T . By considering the (finite) normalization of T in Galois closures of such function field extensions, and using *transitivity* of the Galois action on the fibers of such normalization maps, we get a sufficiently big such finite covering $q : T' \twoheadrightarrow T$ (closely approximating $\mathcal{O}_{T,t}^{\text{sh}}$ by localization at some $t' \in T'_t$) such that after base change of $\text{Pic}_{C/T}^0$ to T' and localization at some $t' \in T'_t$ the scheme property is obtained. But transitivity of the Galois action implies that the scheme property is therefore obtained upon localizing at each of the finitely many $t' \in T'_t$, and by

“spreading out” principles (which work for algebraic spaces as they do for schemes) it follows that the scheme property is obtained over some Zariski-open U' in T' containing the t -fiber. Thus, for the closed set $Z = q(T' - U')$ in T not containing t we see that $U = T - Z$ is an open neighborhood of t for which $q^{-1}(U) \subseteq U'$. Hence, $\text{Pic}_{C/T}^0$ becomes a scheme after base change to $q^{-1}(U)$. The open sets U obtained in this way cover the quasi-compact T (as we vary t), so we obtained finitely many opens U_1, \dots, U_n that cover T and finite surjective maps $q_i : T_i \rightarrow T$ from normal irreducible T_i such that $\text{Pic}_{C/T}^0$ becomes surjective after base change to each $q_i^{-1}(U_i)$. By forming a composite of the function fields of the T_i over that of T , we can construct a finite surjective map $q : T' \rightarrow T$ factoring through every q_i . Hence, $q^{-1}(U_i) \rightarrow T$ factors through $q_i^{-1}(U_i)$ and so $(\text{Pic}_{C/T}^0)_{T'}$ is a scheme over $q^{-1}(U_i)$ for all i . But these preimages cover T' , so $(\text{Pic}_{C/T}^0)_{T'}$ is a scheme.

It now remains to prove the following two results, which will be the topics of consideration in the subsequent sections. The first result transfers the extension of $(\text{Pic}_{C_U/U}^0)_{U'}$ over \bar{X}' to an extension of the isogenous $A_U \times B_U$ over the irreducible normal \bar{X}' . The second result handles passage to a direct factor (keeping in mind Lemma 1.4).

Lemma 3.3. *Let $V \subseteq Y$ be an open subscheme of a noetherian scheme Y , and let $J \rightarrow Y$ be a semi-abelian scheme whose restriction J_V over V is an abelian scheme (i.e., proper). Suppose there is given an isogeny $f : J_V \rightarrow J'$ between abelian schemes over V . Then there exists a proper surjective base change $Y' \rightarrow Y$ such that for the open preimage V' of V in Y' the pullback J'_V , extends to a semi-abelian scheme over Y' .*

Lemma 3.4. *Let $V \subseteq Y$ be an open subscheme of a connected normal noetherian scheme Y and let $J \rightarrow Y$ be a semi-abelian scheme whose restriction J_V over V is an abelian scheme. Assume that $J_V = A_1 \times A_2$ for abelian schemes A_1 and A_2 over V . Then each A_i extends to a semi-abelian scheme over Y .*

The proof of Lemma 3.4 will ultimately reduce to a clever flatness argument that I learned from Serre. The proof of Lemma 3.3 lies much deeper, as it relies in a hard (but marvelous) “flattening by blow-up” theorem of Raynaud and Gruson. Thus, we will discuss the proof of Lemma 3.4 first. (The proofs of the two lemmas are logically independent of each other.)

4. PROOF OF LEMMA 3.4

Let $e \in \text{End}_V(J_V)$ denote the idempotent corresponding to the factor scheme A_1 . By Lemma 1.4, we may uniquely extend e to an endomorphism \bar{e} of J over Y , and $\bar{e}^2 = \bar{e}$ since $e^2 = e$. Thus, the scheme-theoretic kernels

$$\bar{A}_1 = \ker(\bar{e}), \quad \bar{A}_2 = \ker(1 - \bar{e})$$

are closed Y -subgroup schemes of J such that

$$J = \bar{A}_1 \times_Y \bar{A}_2$$

as Y -groups. The geometric fibers of each \bar{A}_j are semi-abelian varieties. Indeed, the product of such fibers over a geometric point \bar{y} of Y is the semi-abelian $J_{\bar{y}}$, and we have:

Lemma 4.1. *Let W and W' be non-empty schemes of finite type over an algebraically closed field k . If $W \times W'$ is smooth then the same holds for W and W' , and likewise for connectedness.*

Proof. The connected case is obvious (argue by contradiction). To prove smoothness, we can argue in many different ways. Here is one via the functorial criterion. When working with finite type schemes over $k = \bar{k}$, it suffices to test against artin local points valued in finite k -algebras. That is, it suffices to prove that if R is a finite local k -algebra and I is a square-zero ideal in R then the natural map $(W \times_k W')(R) \rightarrow (W \times_k W')(R/I)$ is surjective. But this is the direct product of the maps $W(R) \rightarrow W(R/I)$ and $W'(R) \rightarrow W'(R/I)$. Hence, each of this individual maps is surjective provided that $W(R/I)$ and $W'(R/I)$ are non-empty. Since the map

$$W(R/I) \times W'(R/I) = (W \times W')(R/I) \rightarrow (W \times W')(k)$$

is surjective (by the smoothness of $W \times W'$) with non-empty target (since $W, W' \neq \emptyset$ and $k = \bar{k}$), we are done. \blacksquare

We conclude that the commutative separated finite type Y -groups A_1 and A_2 have geometric fibers that are smooth and connected. These fibers are also semi-abelian, by Corollary 3.2 in my notes on semistable reduction. Hence, the only remaining issue is to verify the Y -smoothness, which is to say that A_1 and A_2 are *flat*. For this, we may apply the following nifty observation of Serre.

Lemma 4.2. *Let $Z, Z' \rightrightarrows S$ be morphisms of schemes that that $Z'(S) \neq \emptyset$. If $Z \times_S Z'$ is S -flat then so is Z . In particular, if $Z(S), Z'(S) \neq \emptyset$ and $Z \times_S Z'$ is S -flat then Z and Z' are both S -flat.*

Proof. Choose $\xi \in Z$ at which we wish to prove flatness, and let s be its image in S . It is harmless to make the localization obtained via base change along $\text{Spec } \mathcal{O}_{S,s} \rightarrow S$, we can assume that S is local. Pick a section $z' \in Z'(S)$, so the image of z' lies in an open affine around the image of z' at the closed point of the local S (since the z' -preimage of such an open affine is an open subset of the local S that contains the closed point and hence is the entire space). Then we replace Z' by an open affine around the image of z' at the closed point of S to get to the case when Z' is also affine. Finally, we can replace Z with an open affine around ξ to get to the case when Z is also affine.

Now we have $S = \text{Spec } R$ and $Z = \text{Spec } A$ and $Z' = \text{Spec } A'$ for algebras A and A' over a (local) ring R . But the existence of a section to $Z' \rightarrow S$ gives an R -algebra map $A' \rightarrow R$ whose kernel J provides a direct sum decomposition $R \oplus J = A'$ as R -modules. Since R is thereby a direct factor of A' as an R -module, it follows that $A = A \otimes_R R$ is a direct factor of $A \otimes_R A'$ as R -modules. But $A \otimes_R A'$ is R -flat by hypothesis and a direct factor of a flat module is flat, so we are done. \blacksquare

5. PROOF OF LEMMA 3.3

As usual, by making a preliminary proper (even finite) surjective base change we can assume that Y is normal and irreducible.

The map f is quasi-finite and proper, hence finite, and it is flat via the fibral flatness criterion. Let $H = \ker f$, a finite flat Y -group scheme. By descent theory formalism, we

have $J' = J_V/H$. Suppose we can extend H to a quasi-finite flat closed Y -subgroup H' in J . (This will likely be impossible in general, but will be bypassed by using a proper surjective base change on Y .) Then we can consider the “quotient” J/H' , and by Artin’s work on algebraic spaces such a quotient exists as a smooth separated algebraic space group of finite type over Y . (Its separatedness rests on the fact that H' is closed in J .) The restriction of this quotient over V is J' , and the geometric fibers of J/H' over Y are semiabelian varieties since the same holds for the geometric fibers of J over Y by hypothesis.

We will now prove that J/H' is necessarily a scheme, so we would be done *conditional* on the (usually false) hypothesis that H extends to a quasi-finite flat closed Y -subgroup of J' . The usefulness of this conditional conclusion is that it reduces our task to proving that such a quasi-finite flat extension over Y exists after making a proper surjective base change on Y .

Lemma 5.1. *Let A be a semiabelian scheme over a noetherian scheme S , and let G be a quasi-finite flat closed subgroup scheme of A . Then the semiabelian algebraic space A/G over S is a scheme.*

Proof. The fibral rank of $G \rightarrow S$ is bounded (by constructibility considerations, or in a thousand other ways), so we can pick an integer n that is a multiple of the rank of every geometric fiber of $G \rightarrow S$. Since a quasi-finite scheme over an artin local ring is finite (why?), we conclude that the infinitesimal fibers of $G \rightarrow S$ (i.e., fibers over infinitesimal points of S) are *finite* flat group schemes of order dividing n . By Deligne’s theorem, these fibers are then all killed by n , so the map $[n] : G \rightarrow G$ over S vanishes on all infinitesimal fibers. By the Krull Intersection Theorem, it follows that $[n] = 0$, so $G \subseteq A[n]$.

We conclude that there is a natural S -group map $A/G \rightarrow A/A[n]$ that is quasi-finite and separated (as A/G is separated), yet $A/A[n]$ is a scheme since

$$0 \rightarrow A[n] \rightarrow A \xrightarrow{[n]} A \rightarrow 0$$

is a short exact sequence of fppf abelian sheaves due to the semi-abelian hypothesis on A . It is a general fact that an algebraic space that is quasi-finite and separated over a scheme is a scheme (this is almost the “only” method for proving that abstract algebraic spaces are schemes), and we have just seen that A/G is quasi-finite and separated over a scheme. ■

Finally, it remains to finite a proper surjective map $\pi : Y' \twoheadrightarrow Y$ such that for $V' = \pi^{-1}(V)$, $H_{V'}$ extends to a quasi-finite flat *closed* Y' -subgroup scheme of $J_{Y'}$. Recall that we arranged for Y to be normal and irreducible. We may assume that V is non-empty, so it is a dense open in Y .

Let $G \subset J$ denote the Zariski-closure of H . This is a closed subscheme of J such that $G_V = H$, and since Y is integral and J is Y -separated it follows that the closed immersion identity section $e : Y \hookrightarrow J$ factors through the closed subscheme G (as it does so over the dense open V in the integral Y). The inversion automorphism of J carries G into itself over Y since it does so over V (as $G_V = H$). The same idea does not work for multiplication since without knowing Y -flatness for G we cannot conclude that $G \times_Y G$ inside of $J \times_Y J$ is the Zariski closure of $H \times_S H$.

But if G were *flat* over Y then we would win. Indeed, under such a flatness hypothesis, $G \times_Y G$ would be flat over the integral Y and hence its V -restriction $H \times_V H$ would admit

$G \times_Y G$ as its Zariski-closure in $J \times_V J$. Thus, G would be a closed Y -subgroup scheme of J extending H . Moreover, by Y -flatness of G and the quasi-finiteness of G over the dense open V in Y it follows that G is necessarily quasi-finite over the entirety of Y . (This amounts to the general fact that fiber dimension does not vary under flat maps of finite type between noetherian schemes.)

So how do we handle this flatness problem? The key is the following remarkable theorem, which again supports the principle that one can often solve “extension” problems after a proper surjective base change.

Theorem 5.2 (Raynaud–Gruson). *Let S be a noetherian scheme, $X \rightarrow S$ a map of finite type, and V an open subscheme of S . Suppose that there is given a V -flat closed subscheme Z in X_V . There is a proper surjective map $\pi : S' \rightarrow S$ that is an isomorphism over V such that $X_{S'}$ contains an S' -flat closed subscheme Z' whose restriction over $V' = \pi^{-1}(V)$ is $Z_{V'}$.*

Proof. This is a very special case of the deep Theorem 5.2.2 in the amazing unique paper co-authored by Raynaud and L. Gruson. It might be possible to make a more direct argument when X is quasi-projective by using Hilbert schemes, but I am not sure. ■

This result of Raynaud and Gruson implies that after making a proper surjective base change, and then passing once again to a normalization, we can assume that H is the V -restriction of a flat closed subscheme of J . But it follows that H must have this flat closed subscheme as its Zariski closure, since any flat Y -scheme is the Zariski closure of its own V -restriction (due to the integrality of Y).