Faltings' Finiteness Theorems

Michael Lipnowski

Introduction: How Everything Fits Together

This note outlines Faltings' proof of the finiteness theorems for abelian varieties and curves. Let K be a number field and S a finite set of places of K. We will demonstrate the following, in order:

- There are only finitely many isogeny classes of abelian varieties over K of dimension g with a polarization of degree d and with good reduction outside of S.
  - This ultimately relies on the Tate Conjecture and the semisimplicity of the action of G_K on $V_\ell(A)$ for an abelian variety A/K.
- (Shafarevich conjecture)
  There are only finitely many isomorphism classes of abelian varieties over K of a given dimension g, with polarization of degree d, which have good reduction outside S.
  - Faltings' amazing insight was that this could be done by understanding how the Faltings height $h(A)$ varies for A/K an abelian variety within a fixed isogeny class.

    Let $B \to A$ be a K-isogeny. Then $\exp(2[K:\mathbb{Q}](h(B) - h(A))$ (read: "change in height under isogeny") is a rational number.
    * For any individual prime ℓ, the ℓ-adic valuation of $\exp(2[K:\mathbb{Q}](h(B) - h(A))$ can be controlled by the Tate Conjecture.
    * Using results of Raynaud, Faltings shows that for large ℓ, depending only on the field K and the places S where A has good reduction, the ℓ-adic valuation doesn't change at all.

    These together imply that $h(B)$ remains bounded within the isogeny class of A.
    * It is a technically difficult but true fact that h is a Weil height, i.e. that there are only finitely many isomorphism classes of B/K of abelian varieties of fixed dimension g with $h(B) \leq$ any fixed upper bound. This was explained in Akshay's and Peter's talks; it rests on the existence of enough Siegel modular forms, defined over $\mathbb{Z}$, to separate points in the moduli space $A_{g,n}$ of principally polarized abelian varieties of dimension g with full level n structure.
      As Brandon explained in his talk, this fact is also essential in Faltings' proof of the Tate conjectures.

    There are thus only finitely many isomorphism classes of abelian varieties over K, equipped with a polarization of fixed degree, isogenous to A. Combined with finiteness of the number of isogeny classes with good reduction outside S, this proves the Shafarevich Conjecture.

Let's get started!

Preliminaries

Let K be a number field, O_K its ring of integers.
Let $A_1, A_2 \to Spec(O_K)$ be semiabelian schemes of relative dimension g with proper generic fiber with $s : Spec(O_K) \to A_1$ the zero section. Let $\phi : A_1 \to A_2$ be an isogeny between them with kernel $\mathcal{G}/O_K$.

Lemma 5. $h(A_2) = h(A_1) + 1/2 \log(\deg(\phi)) - [K : Q] \log(\#s^*(\Omega^1_{\mathcal{G}/O_K}))$.

We will make good use of this formula.

Proofs of Finiteness Theorems

Theorem 5. *Let S be a finite set of places of number field K. Then there are only finitely many isogeny classes of abelian varieties of a given dimension g with good reduction outside S.*

Proof. Let $A_1, A_2/K$ be an abelian varieties with good reduction outside S.

Let $M \subset End_{\mathbb{Z}_\ell}(T_\ell(A_1)) \times End_{\mathbb{Z}_\ell}(T_\ell(A_2))$ be the $\mathbb{Z}_\ell$ algebra spanned by the image of $\pi = G_K$. M is a $\mathbb{Z}_\ell$ module of rank $\leq 8g^2$ which acts on $End_{\mathbb{Z}_\ell}(T_\ell(A_1))$ and $End_{\mathbb{Z}_\ell}(T_\ell(A_2)$.

By the Tate Conjecture, A_1 and A_2 are isogenous iff $Tr(m|T_\ell(A_1)) = Tr(m|T_\ell(A_2))$ for all $m \in M$, i.e. iff their Tate modules are $\mathbb{Z}_\ell[\pi]$ isomorphic. Thus, it suffices to prove this for a set of $\mathbb{Z}_\ell$-module generators of M, which is the same as a set of $\mathbb{Z}_\ell$-module generators for $M/\ell M$ by Nakayama.
A fortiori, the image of $\rho : \pi \to (M/\ell M)^\times$ generates $M/\ell M$ as a $\mathbb{Z}_\ell$-module. Also note that

- $\#(M/\ell M)^\times \leq l^{8g^2}$.
- ρ is unramified outside of S and ℓ.

Thus, letting K' be the composite, inside $\overline{K}$, of all extensions of K which are unramified outside of S and ℓ AND are of degree $\leq 8g^2$, then K' is finite (Minkowski) and ρ factors through $G_{K'/K}$. Hence, it suffices to show that $Tr(g|T_\ell(A_1)) = Tr(g|T_\ell(A_2))$ for a set of coset representatives for $G_{K'/K} = G_K/G_{K'}$ (or any element in the conjugacy class of such a representative).

By Chebotarev, we can choose these lifts to be finitely many representatives of Frobenius conjugacy classes $Frob_{v_1}, ..., Frob_{v_r}$ for $v_1, ..., v_r \notin S \cup \{\ell\}$.
But $Tr(Frob_v|T_\ell(A_1)) = Tr(Frob_v|T_\ell(A_2))$ whenever the action of $Frob_v$ on $H^1_{et}(A_1)$ and $H^1_{et}(A_2)$ have the same characteristic polynomial. By the Weil conjectures, there are only finitely many possibilities for said characteristic polynomials. These characteristic polynomials correspond precisely to the isogeny classes of A/K of dimension g with good reduction outside S.

□

Theorem 6 ((Weak) Shafarevich Conjecture). *Let S be a finite set of places of $K, d > 0$. There are only finitely many isomorphism classes of abelian varieties over K of a given dimension, with polarization of degree d, which have good reduction outside of S.*

Proof. By Theorem 5, we may assume that all abelian varieties in question are isogenous to a fixed A.
By extending the ground field, we can also assume that all B's extend to semiabelian schemes over $Spec(O_K)$ and that $d = 1$.

Let $\phi : B \to A$ be an isogeny. This induces an isogeny $\phi : \mathcal{B}^0 \to \mathcal{A}^0$ between the connected components of their Neron models, with kernel $\mathcal{G}/O_K$. By the change in height formula,

$$\exp(2[K:\mathbb{Q}](h(B) - h(A))) = \frac{\deg(\phi)^{[K:\mathbb{Q}]}}{(\# s^*\Omega^1_{\mathcal{G}/O_K})^2},$$

We will bound the ℓ-adic valuation of this change in height for each ℓ. Namely,

- For any individual ℓ, we will show that the change in height, among such semiabelian, principally polarized B which are isogenous to A, can be uniformly bounded.
- For large ℓ, large relative to the places S of bad reduction and the ramification of $K/\mathbb{Q}$, we will show that $h(B) = h(A)$.

A useful observation, for what follows, is that height does not change ℓ-adically for $\ell \nmid deg(\phi)$. Indeed, $\mathcal{G}$ is killed by multiplication by $deg(\phi)$ which equals the order of $ker(\phi)$. Since $[n]$ on $\mathcal{G}$ induces multiplication by n on $s^*\Omega^1_{\mathcal{G}/O_K}$, we see that $s^*\Omega^1_{\mathcal{G}/O_K}$ is supported over primes dividing $deg(\phi)$. Staring at the change in height formula, we see that $\exp(2[K:\mathbb{Q}](h(B) - h(A)))$ is prime to ℓ, i.e. there is no ℓ-adic change in height.

<u>For small ℓ</u>:
Suppose $T_\ell(B_1) \xrightarrow{\phi} T_\ell(B_2)$ as $\mathbb{Z}_\ell[\pi]$-modules, $\pi = G_K$, for two abelian varieties $B_1, B_2/K$. Then by the Tate Conjecture, there is some "isogeny"

$$\sum f_i \otimes a_i = \phi$$

for some $f_i \in Hom(B_1, B_2), a_i \in \mathbb{Z}_\ell$. But the set $Isom(T_\ell(B_1), T_\ell(B_2))$ is open $Hom_\pi(T_\ell(B_1), T_\ell(B_2))$ with its ℓ-adic topology. Thus, for good ℓ-adic approximations $\tilde{a_i} \in \mathbb{Z}$ to the $a_i, T_\ell(g)$ is an isomorphism for $g = \sum \tilde{a_i} f_i$.

Since $T_\ell(g)$ is an isomorphism, $det(T_\ell(g)) : det(T_\ell(B_1)) \to det(T_\ell(B_2))$ is an isomorphism as well. But $det(T_\ell(g)) = deg(g)$ (Mumford, p.180), and so g is an isogeny of degree prime to ℓ. Thus, the ℓ-adic change in height

$$\exp(2[K:\mathbb{Q}](h(B_1) - h(B_2)))$$

is zero. Suppose that we can show that there are only finitely many isomorphism classes of π-invariant sublattices of $V_\ell(A)$. By the Tate conjecture, these correspond to the Tate modules of finitely many viable B, say $B_1, ..., B_n$. But then for arbitrary B in the isogeny class of $A, T_\ell(B) \cong T_\ell(B_i)$ for some i, whence

$$\exp(2[K:\mathbb{Q}](h(B) - h(A))) = \exp(2[K:\mathbb{Q}](h(B) - h(B_i))) \exp(2[K:\mathbb{Q}](h(B_i) - h(A)))$$

whose ℓ-adic valuation equals that of $\exp(2[K:\mathbb{Q}](h(B_i) - h(A)))$, independent of B.

Now we'll prove that there are only finitely many isomorphism classes of π-invariant sublattices.

By the Tate Conjecture, $End_{\mathbb{Q}_\ell[\pi]}(V_\ell(A))$ is semisimple. Thus, it decomposes as

First note that if $L_1 \xrightarrow{f} L_2$ is a $\mathbb{Z}_\ell[\pi]$-isomorphism of π-stable sublattices of $V = V_\ell(A)$, then it induces an automorphism of V. By Schur's Lemma, $Aut_{\mathbb{Q}_\ell[\pi]}(V) = End_{\mathbb{Q}_\ell}(D_1) \times .. \times End_{\mathbb{Q}_\ell}(D_n)$ for some division algebras $D_i/\mathbb{Q}_\ell$. Make a finite unramified ground field extension $F/\mathbb{Q}_\ell$ so that each D_i-splits, i.e. $D_i \cong End_F(V_{i,1}) \times ... \times End_F(V_{i,m})$ for subspaces $V_{i,1}, ..., V_{i,m} \subset V_F$. Then the $V_{i,m}$ are absolutely irreducible representations of π and $Aut_\pi(V_{i,m}) = F$. We will show that there are only finitely many F-homothety classes of π-stable lattices in each $U = W_{i,j}$ (and hence finitely many isomorphism classes of π stable lattices in V_F).

As above, the F-span of π is all of $End_F(U)$. Thus, the O_F-span of π contains $\alpha^N End_{O_F}(M)$, where M is some lattice in U and α is the uniformizer of O_F.
Now take any π-stable lattice $L \subset U$. By scaling appropriately, we may find a homothetic lattice, which we also call L, such that

- $L \subset M$ and
- There is a vector $v \in L$ such that $v \in M$ but $v \notin \alpha M$.

But then

$$M \supset L \supset \alpha^N End_{O_F}(M)v = \alpha^N M.$$

There can be only finitely many such L.
Thus, it suffices to show that the base change map $L \mapsto L \otimes_{\mathbb{Z}_\ell} O_F$ from π-isomorphism classes of lattices in W to π-isomorphism classes of lattices in W_F is finite to one.

For this, it suffices to show that $H^1(G_{F/\mathbb{Q}_\ell}, Aut_{O_F[\pi]}(L_F))$ is finite. Since there are no homomorphisms between non-isomorphic representations among the $V_{i,j} \subset V_F$, it suffices to restrict our attention to the lattice $L' = L_F \cap W$, where W is the isotypic component of some absolutely irreducible F-representation of π. Supose $W \cong W' \oplus ... \oplus W'$ for some absolutely irreducible W'. Then

$$Aut_{O_F[\pi]}(L') \cong (O_F^\times)^n \rtimes S_n$$

as $G_{F/\mathbb{Q}_\ell}$-groups. Corresponding to the exact sequence $1 \to (O_F^\times)^n \to (O_F^\times)^n \rtimes S_n \to S_n \to 1$, there is an exact sequence of pointed sets

$$H^1(G_{F/\mathbb{Q}_\ell}, (O_F^\times)^n) \to H^1(G_{F/\mathbb{Q}_\ell}, (O_F^\times)^n \rtimes S_n) \to H^1(G_{F/\mathbb{Q}_\ell}, S_n)$$

(Serre, Ch, 1, § 5). Since the flanking terms are finite, the middle term must be finite as well. Hence, $H^1(G_{F/\mathbb{Q}_\ell}, Aut_{O_F[\pi]}(L_F))$ is finite after all.

<u>For large ℓ:</u>
This explanation is from Rebecca's notes, but is assembled here for convenience.

We may assume that all B in the isogeny class of A have semiabelian reduction over O_K. Since Faltings height is invariant under base change, this assumption is harmless.
Choose ℓ large enough so that A has good reduction for all places $v|\ell$ of K and so that ℓ is unramified in O_K.

Let $\phi : B_1 \to B_2$ be an isogeny of degree ℓ^h between abelian varieties over K isogenous to A. By filtering the kernel, we may assume that multiplication by ℓ kills $G = \ker \phi$. We want to show that $h(B_1) = h(B_2)$.
Note that ϕ extends to a map

$$\widetilde{\phi} : \mathcal{B}_1^0 \to \mathcal{B}_2^0$$

between the connected components of the Neron models of B_1 and B_2 over $O_{K,(\ell)}$. Since good reduction is a property of isogeny classes, B_1, B_2 also have good reduction over ℓ and so $\widetilde{\phi}$ is an isogeny of abelian schemes. Thus, the kernel $\mathcal{G}$ is a finite flat group scheme over $O_{K,(\ell)}$ with generic fiber $\mathcal{G}_K = G$.
Our immediate goal is to use Raynaud's results to relate $\#s^*(\Omega^1_{\mathcal{G}/O_{K,(\ell)}})$, a mysterious constituent in the change of Faltings height formula, to the Galois action on $G(\overline{K})$.

Let $\pi = Gal(\overline{\mathbb{Q}}/\mathbb{Q}), \widetilde{\pi} = Gal(\overline{K}/K)$. Consider the following representations over $\mathbb{Z}/\ell\mathbb{Z}$:

- $V_\ell = T_\ell(B_1)/\ell T_\ell(B_1) = B_1[\ell](\overline{K})$
- $\widetilde{V}_\ell = Ind_{\pi}^{\tilde{\pi}}(V_\ell)$
- $W_\ell = G(\overline{K}) \subset V_\ell$
- $\widetilde{W}_\ell = Ind_{\pi}^{\tilde{\pi}}(W_\ell)$

Let $[K : \mathbb{Q}] = m$. Note that W_ℓ is h-dimensional and $\widetilde{W}_\ell$ is mh-dimensional.
Consider the character $\chi = det(\widetilde{W}_\ell) : \pi \to (\mathbb{Z}/\ell\mathbb{Z})^\times$. Because $\mathcal{B}_1^0$ is semiabelian, the action of inertia $I_v, v \nmid \ell$, on $W_\ell \subset V_\ell$ is unipotent. Hence, $det(W_\ell)$ is trivial away from ℓ. Hence, for any $g \in I_p$, the action of $det(g)$ on $det(\widetilde{W}_\ell)$ is simply by $\epsilon(g)$, where $\epsilon : \pi \to \{\pm 1\}$ is the sign representation of π acting on the cosets $\pi/\widetilde{\pi}$. Thus, $\chi_0 = \chi\epsilon$ is unramified outside of ℓ.

By class field theory, any continuous character $\pi \to (\mathbb{Z}/\ell\mathbb{Z})^\times$ unramified outside of ℓ corresponds to a continuous character

$$\chi_0 : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times / \prod_{p \neq \ell} \mathbb{Z}_p^\times \to (\mathbb{Z}/\ell\mathbb{Z})^\times.$$

Also, any continuous character $\mathbb{Z}_\ell^\times = (1 + \ell\mathbb{Z}_\ell) \times (\mathbb{Z}/\ell\mathbb{Z})^\times \to (\mathbb{Z}/\ell\mathbb{Z})^\times$ must kill the 1-units $1 + \ell\mathbb{Z}_\ell$ (since they form a pro-ℓ group) and so factors through some dth power map $(\mathbb{Z}/\ell\mathbb{Z})^\times$. Then χ_0 must be the dth power of the cyclotomic character (for some $0 \le d \le \ell - 1$):

$$\chi\epsilon = \chi_0 = \chi_\ell^d.$$

Claim 1. *If $\#s^*(\Omega^1_{\mathcal{G}/O_{K,(\ell)}}) = \ell^{d'}$, then $d = d'$.*

Proof. Raynaud's results say that if H is a finite flat group scheme killed by ℓ over a strictly henselian local ring R of mixed characteristic $(0, \ell)$ with low ramification, then the Galois action on $det(H)$ is $\tau_\ell^{v(\mathfrak{d}_{H/R})}$, where τ_ℓ is the canonical tame character $\tau_\ell : I_t \to F_\ell^\times$.

Note that $\chi_\ell|_{I_\ell} = \tau_\ell$. Thus, we are really interested in computing $\chi\epsilon|_{I_\ell}$ in terms of $\mathcal{G}$. But

$$\text{Weil restriction } O_{K,(\ell)}/\mathbb{Z}_{(\ell)} \leftrightarrow Ind_{\pi}^{\tilde{\pi}}$$

and

$$\text{base change to } \mathbb{Z}_{\ell}^{un} (= R) \leftrightarrow Res_{\pi/I_{\ell}}.$$

Thus, we are actually interested in the Galois action on the generic fiber of

$$\mathcal{G}' := Res_{O_{K,(\ell)}/\mathbb{Z}_{(\ell)}}(\mathcal{G}) \otimes R = \mathcal{G}_{i_1} \times_R \ldots \times_R \mathcal{G}_{i_n},$$

where there is one copy of $\mathcal{G}$ for each embedding $i_j : O_{K,(\ell)} \hookrightarrow R$.
Applying Raynaud, we see that inertia acts by the character

$$\tau_{\ell}^{v(\mathfrak{d}_{\mathcal{G}'/R})/\#\mathcal{G}'} = \tau_{\ell}^{v(\mathfrak{d}_{\mathcal{G}'/R})/\ell^{mh}}.$$

It follows that

$$d = v(\mathfrak{d}_{\mathcal{G}'/R})/\#\mathcal{G}' = \sum_i v(\mathfrak{d}_{\mathcal{G}_i/R})/\#\mathcal{G}_i = \sum_i v(\mathfrak{d}_{\mathcal{G}_i/R})/\ell^h.$$

On the other hand, by computations from Brandon's notes,

$$\#s^*\Omega^1_{\mathcal{G}/O_K} = \prod_{v|\ell} \#s_v^*\Omega^1_{\mathcal{G}_v/O_{K,v}}$$

(essentially the Chinese remainder theorem) and

$$\#s_v^*\Omega^1_{\mathcal{G}_v/O_{K,v}} = \#(O_{K,v}/\mathfrak{d}_{G_v/O_{K,v}})^{1/\#G_v}.$$

Combining, we see that if $\ell^{d'} = \#s^*\Omega^1_{\mathcal{G}/O_K}$, then

$$d' = \sum_{v|\ell} f_v v(\mathfrak{d}_{\mathcal{G}_v/O_{K_v}})/\#\mathcal{G}_v.$$

Making an unramified base change up to R preserves the valuation of the discriminant. Also, each of the f_v embeddings of $k_v \hookrightarrow \overline{\mathbb{F}_{\ell}}$ gives an embedding $O_{K,(\ell)} \hookrightarrow O_{K,v} \hookrightarrow R$. Running over all $v|\ell$ accounts for all possible embeddings $O_{K,(\ell)} \hookrightarrow R$. Thus,

$$\begin{aligned} d' &= \sum_{v|\ell} \sum_{O_{K,v} \hookrightarrow R} v(\mathfrak{d}_{\mathcal{G}_R/R})/\#\mathcal{G}_R \\ &= \sum_{i:O_{K,(\ell)} \hookrightarrow R} v(i^*\mathcal{G})/\#i^*\mathcal{G} = d. \end{aligned}$$

□

This is great news: we can now use global information (the Weil conjectures) to control the change in height.

Define $P_i(T) = det(T - Frob_p | \wedge^i Ind_{\pi}^{\tilde{\pi}} T_{\ell}(A))$.
Then $\chi(Frob_p) = \pm\chi_{\ell}^d(Frob_p) = \pm p^d$ is a zero of $P_{mh}(T)$ modulo ℓ.
But by the Weil conjectures, the zeros of $P_{mh}(T)$ are algebraic with absolute value $p^{mh/2}$ under any complex embedding. We have the a priori crude bound $d \leq gm$.

- Indeed, since G is killed by ℓ, $s^*\Omega^1_{\mathcal{G}/O_{K,(\ell)}}$ is a quotient of $s^*\Omega^1_{\mathcal{B}_1^0[\ell]/O_{K,(\ell)}}$ which has order ℓ^{mg}.

Thus, as long as we choose ℓ large enough not to divide any of the values $P_i(\pm p^j)$ for

$$0 \leq i \leq 2gm$$
$$0 \leq j \leq gm$$
$$j \neq h/2$$

this will force $d = mh/2$.

By the change in height formula,

$$\begin{aligned} h(B_2) - h(B_1) &= \frac{1}{2}\log(deg(\phi)) - \frac{1}{[K:\mathbb{Q}]}\log(\# s^*\Omega^1_{\mathcal{G}/O_K}) \\ &= \frac{1}{2}h\log(\ell) - \frac{1}{m}d\log(\ell) \\ &= 0 \end{aligned}$$

Thus, the height doesn't change under ℓ power isogenies for large ℓ! Combining with the earlier argument for small ℓ, we see that the height remains bounded within isogeny classes. Thus, there can be only finitely many isomorphism classes of abelian varieties over K, equipped with a polarization of bounded degree, within the isogeny class of A. Combined with our earlier result (Theorem 5) on finiteness of the number of good-outside-S isogeny classes, this proves what we've called the Shafarevich conjecture. □

Theorem ((Strong) Shafarevich Conjecture). *Let S be a finite set of places of K. There are only finitely many isomorphism classes of abelian varieites over K of dimension g with good reduction outside S.*

Proof. See Corollary 6.6 in Brian's lecture 13 notes. This includes the statement that weak Shafarevich implies strong Shafarevich. □

References

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