BACKGROUND ON R-GROUPS (NOTES BY I. BOREICO, B. CONRAD)

1. Introduction

Let $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ be a congruence subgroup. For the upper half-plane $\mathscr{H}$, the quotient space

$$\Gamma\backslash\mathscr{H} = \Gamma\backslash\mathrm{SL}_2(\mathbf{R})/\mathrm{SO}_2(\mathbf{R})$$

can also be rewritten as

$$\Gamma\backslash\mathrm{GL}_2(\mathbf{R})/(\mathrm{O}_2(\mathbf{R})\cdot\mathbf{R}^\times)$$

(as a real-analytic manifold). In the latter presentation, note that $\mathrm{O}_2(\mathbf{R})$ is a maximal compact of $\mathrm{GL}_2(\mathbf{R})$, and $\mathbf{R}^\times$ is the center of $\mathrm{GL}_2(\mathbf{R})$. Also, $\mathrm{GL}_2(\mathbf{R})$ and $\mathrm{O}_2(\mathbf{R})$ each two connected components, with respective identity components $\mathrm{SL}_2(\mathbf{R})$ and $\mathrm{SO}_2(\mathbf{R})$.

Remark 1.1. We prefer to deal with GL_2 and O_2 instead of SL_2 and SO_2 because the center $\mathbf{G}_m$ of GL_2 is a torus rather than μ_2; tori have better cohomological properties (for vanishing, finiteness, and duality) than finite multiplicative groups.

We want to generalize the construction above with GL_2 replaced by a *wider class* of connected reductive groups G over $\mathbf{Q}$. This "wider class" of groups is given by a set of conditions which we call "Shimura datum axioms". We aim to consider $\Gamma\backslash G(\mathbf{R})/K\cdot Z_G(\mathbf{R})$ for a maximal compact subgroup K of $G(\mathbf{R})$ and suitable subgroups $\Gamma \subset G(\mathbf{Q})$ that are discrete in $G(\mathbf{R})$.

Example 1.2. Let F be a totally real number field, and $G = \mathrm{R}_{F/\mathbf{Q}}(\mathrm{GL}_2)$ the Weil restriction of the F-group GL_2. Then

$$G(\mathbf{R}) = \prod \mathrm{GL}_2(F_v)$$

where v runs through the real places of F (as $F\otimes_{\mathbf{Q}}\mathbf{R} \simeq \prod_v F_v$). This product is identified with $\prod_j \mathrm{GL}_2(\mathbf{R})$ as j varies through the $\mathbf{Q}$-embeddings $j : F \to \mathbf{R}$.

Issues to deal with:

1. What replaces the role of $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$?

Naive guess. Choose a $\mathbf{Q}$-group inclusion $G \hookrightarrow \mathrm{GL}_n$, with the target viewed as the generic fiber of the $\mathbf{Z}$-group GL_n. Define $\mathbf{G}$ to be the Zariski closure of G in the $\mathbf{Z}$-group GL_n. Because $\mathbf{Z}$ is Dedekind, this is a flat closed $\mathbf{Z}$-subgroup of GL_n, so it is a flat affine $\mathbf{Z}$-group of finite type with generic fiber G. (Conversely, it is a good exercise to check that every flat affine $\mathbf{Z}$-group with generic fiber G arises in this way, and likewise with any Dedekind domain in place of $\mathbf{Z}$.) We can try taking $\Gamma \subset \mathbf{G}(\mathbf{Z})$ to be a congruence subgroup (i.e., the kernel of $\mathbf{G}(\mathbf{Z}) \to G(\mathbf{Z}/m\mathbf{Z})$ for some $m > 0$).

Defect. The above definition depends on $\mathbf{G}$ (or more concretely, on $G \hookrightarrow \mathrm{GL}_n$). This is not suitable for our purposes, because if $G_1 \xrightarrow{f} G_2$ is an isogeny of connective reductive $\mathbf{Q}$-groups then such $\mathbf{Z}$-models (chosen "prior" to f) need not interact well with f.

Rectification. We work up to finite index: choose $\Gamma \subset G(\mathbf{Q})$ which is commensurable with $\mathbf{G}(\mathbf{Z})$ for some (equivalently, any) flat affine finite type $\mathbf{Z}$-model $\mathbf{G}$ of G. Such a Γ is called an *congruence group in* G. These are precisely the groups commensurable with the intersection of $G(\mathbf{Q})$ and a compact open subgroup of $G(\mathbf{A}_f)$, thereby giving a description intrinsic to the $\mathbf{Q}$-group G (and thereby well-suited to interaction with $\mathbf{Q}$-isogenies in G). Note that is is NOT the same to ask that Γ contains a congruence subgroup

$$\ker(\mathbf{G}(\mathbf{Z}) \to \mathbf{G}(\mathbf{Z}/m\mathbf{Z}))$$

for some $m > 1$ (since SL_2 over $\mathbf{Q}$ fails to satisfy the "congruence subgroup property").

2. What is the conceptual way to describe the complex structure on $G(\mathbf{R})/K \cdot Z_G(\mathbf{R})$ without mentioning the upper half-plane? In other words, we seek to describe the complex structure on $GL_2(\mathbf{R})/\mathrm{O}_2(\mathbf{R})\mathbf{R}^\times$ in terms of the Lie group $\mathrm{GL}_2(\mathbf{R})$. The answer is given in terms of Cartan involutions of the $\mathbf{R}$-group GL_2, as Brandon indicated in his lectures.

3. How to deal with the maximal compact subgroups of disconnected Lie groups?

For many connected reductive $\mathbf{R}$-groups G, the group $G(\mathbf{R})$ is disconnected. For example, $\mathrm{GL}_n(\mathbf{R})$ is disconnected for all $n > 1$, and if n is even then $\mathrm{PGL}_n(\mathbf{R})$ has 2 connected components. Also, the groups $\mathrm{SO}(p,q)(\mathbf{R})$ can have 1, 2, or 4 connected components.

The theory of Cartan involutions is usually developed for *connected* Lie groups with semisimple Lie algebra and finite center. But we need to allow a finite component group that may be nontrivial, so we will extend the notion of Cartan involution to a class of disconnected groups that includes $G(\mathbf{R})$ for any connected semisimple $\mathbf{R}$-group G (and even any connected reductive $\mathbf{R}$-group G).

Digression on some exact sequences for connected reductive groups

In this subsection, k will be an arbitrary field. Every connected reductive k-group G fits into a short exact sequence of k-groups

$$1 \to Z_G \to G \to G^{\mathrm{ad}} := G/Z_G \to 1$$

where Z_G is the schematic center of G. Since Z_G is of multiplicative type, the reduced identity component $(Z_G)^0_{\mathrm{red}}$ is a torus.

If $\mathfrak{g}$ is the Lie algebra of G then the adjoint representation $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ has image identified with G^{ad} as quotients of G (i.e., $\ker \mathrm{Ad}_G = Z_G$). The connected reductive group G^{ad} is semisimple. That is, its center is finite or equivalently its is equal to its own derived group.

Definition 1.3. A connected semisimple group H over a field is called *adjoint* if the schematic center Z_H is trivial. Equivalently, $\mathrm{Ad}_H : H \to \mathrm{GL}(\mathfrak{h})$ is a closed immersion, where $\mathfrak{h} = \mathrm{Lie}(H)$.

Remark 1.4. If $f : H_1 \xrightarrow{f} H_2$ is a surjective homomorphism between connected reductive k-groups and $\mathrm{Ker}(f) \subset Z_{H_1}$ then the natural map $H_1^{\mathrm{ad}} \to H_2^{\mathrm{ad}}$ is an isomorphism. For example, the surjection $\mathbf{G}_m \times \mathrm{SL}_n \twoheadrightarrow \mathrm{GL}_n$ yields the isomorphism $\mathrm{SL}_n/\mu_n \xrightarrow{\sim} \mathrm{PGL}_n$

Also, the derived group $G' := \mathscr{D}G$ of G is semisimple and the injection $G' \hookrightarrow G$ yields an isomorphism $(G')^{\mathrm{ad}} \to G^{\mathrm{ad}}$. For example, since the derived group of GL_n is SL_n, we again obtain the isomorphism $\mathrm{SL}_n/\mu_n \xrightarrow{\sim} \mathrm{PGL}_n$.

• Another short exact sequence is obtained by injecting the derived group G' into G. The quotient is commutative, hence a torus.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & T & \longrightarrow & 1 \\ & & & & \uparrow & \nearrow_{\text{isogeny of tori}} & & & \\ & & & & (Z_G^0)_{\text{red}} & & & & \end{array}$$

A special case is provided by the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{SL}_n & \longrightarrow & \mathrm{GL}_n & \xrightarrow{\det} & \mathbf{G}_m & \longrightarrow & 1 \\ & & & & \uparrow & \nearrow_{t \to t^n} & & & \\ & & & & \mathbf{G}_m & & & & \end{array}$$

Note that the maps

$$(Z_G^0)_{\text{red}} \times G' \to G$$
$$G \to G^{\text{ad}} \times G/G'$$

are both central isogenies.

• Let G be connected semisimple over k. It has a unique "simply connected" central cover $\widetilde{G}$:

$$1 \to \mu \to \widetilde{G} \to G \to 1$$

where μ is central and finite. Here, "simply connected" is taken in the sense of algebraic groups: a connected semisimple k-group $\widetilde{G}$ is simply connected if and only if it has no nontrivial central isogenous cover by a connected semisimple k-group. If $k = \mathbf{C}$, it is equivalent to $\widetilde{G}(\mathbf{C})$ being simply connected in the sense of topology (but this equivalence is not obvious).

Example 1.5. For $n > 1$, if $G = \mathrm{PGL}_n$ then $\widetilde{G} = \mathrm{SL}_n$. Note that $\mathrm{SL}_n(\mathbf{R})$ is not topologically simply connected; e.g., $\mathrm{SL}_2(\mathbf{R}) \simeq S^1 \times \mathbf{R}^2$ as topological spaces.

Example 1.6. If $G = \mathrm{SO}(q)$ for a non-degenerate quadratic space (V, q) then $\mathrm{Spin}(q)$ is its simply connected central cover; the covering map is 2:1. The symplectic group Sp_{2g} is simply connected.

Theorem 1.7 (E. Cartan)**.** *Let G be a connected semisimple $\mathbf{R}$-group which is simply connected. Then $G(\mathbf{R})$ is connected.*

Remark 1.8. If G is split, by using an open cell and the Bruhat decomposition we can reduce the proof of Cartan's theorem to the case of SL_2. In general, however, the proof is harder.

Corollary 1.9. *Let G be a connected semisimple $\mathbf{R}$-group. Then $\pi_0(G(\mathbf{R}))$ is a finite abelian group of exponent 2 (i.e. a finite direct product of copies $\mathbf{Z}/2\mathbf{Z}$).*

Proof. Consider the short exact sequence:

$$1 \to \mu \to \widetilde{G} \to G \to 1$$

This yields to the exact sequence

$$\widetilde{G}(\mathbf{R}) \to G(\mathbf{R}) \xrightarrow{\delta} \mathrm{H}^1(\mathbf{R}, \mu)$$

Note that $\widetilde{G}(\mathbf{R})$ is connected by Cartan's theorem and the map $\widetilde{G}(\mathbf{R}) \to G(\mathbf{R})$ has finite kernel because it induces an isomorphism of Lie algebras. Also, note that $\mathrm{H}^1(\mathbf{R}, \mu)$ is a finite abelian group of exponent 2, so the claim concerning π_0 follows.. (If $S \subset G$ is a maximal split $\mathbf{R}$-torus then by using the Borel–Tits structure theory one can show that $\pi_0(S(\mathbf{R})) \to \pi_0(G(\mathbf{R}))$ is surjective, so since gives another proof since $\pi_0(S(\mathbf{R})) = \prod \pi_0(\mathbf{R}^\times) = \prod(\mathbf{Z}/2\mathbf{Z})$.) □

2. Maximal compact subgroups and the Cartan Involution

Example 2.1. If G is $\mathbf{R}$-anisotropic (i.e., there is no inclusion of $\mathbf{R}$-groups $\mathbf{G}_m \hookrightarrow G$) then $G(\mathbf{R})$ is compact. This is true for non-archimedean local fields too.

Theorem 2.2 (Tannaka-Krein)**.** *The assignment $G \mapsto G(\mathbf{R})$ provides an equivalence of categories between*

- $\mathbf{R}$*-anisotropic reductive* $\mathbf{R}$*-groups G such that every connected component of G has an* $\mathbf{R}$*-point*
- *compact Lie Groups.*

Moreover, such a G is connected for the Zariski topology if and only if $G(\mathbf{R})$ is connected.

Remark 2.3. If G is connected semisimple over $\mathbf{R}$ and is $\mathbf{R}$-anisotropic but is not simply connected then the exact sequence

$$1 \to \mu \to \widetilde{G} \to G \to 1$$

yields

$$\widetilde{G}(\mathbf{R})/\mu(\mathbf{R}) \to G(\mathbf{R}) \xrightarrow{\delta} H^1(\mathbf{R}, \mu) \to H^1(\mathbf{R}, \widetilde{G}).$$

The map $\widetilde{G}(\mathbf{R})/\mu(\mathbf{R}) \to G(\mathbf{R})$ is an isomorphism (since $G(\mathbf{R})$ is connected by Tannaka-Krein!), hence $\delta = 0$ but $\mathrm{H}^1(\mathbf{R}, \mu)$ and $\mathrm{H}^1(\mathbf{R}, \widetilde{G})$ are often non-trivial.

Remark 2.4. If G is connected and semisimple over a field k, and is moreover simply connected then

$$G \simeq \prod \mathrm{R}_{k_i/k}(G_i)$$

where k_i/k is finite separable and each G_i is connected semisimple over k_i with $G_i \otimes_{k_i} \overline{k}$ simple. [Equivalently, the absolute root system of G_i is irreducible.]

In particular, for such G over $k = \mathbf{R}$:

$$G = \prod G_i \times \prod \mathrm{R}_{\mathbf{C}/\mathbf{R}}(H_j)$$

where G_i are absolutely simple and simply connected over $\mathbf{R}$, and each H_j is connected semisimple over $\mathbf{C}$ and is simple and simply connected. Note that in this case, $H_j(\mathbf{C})$ is never compact. We then get $\mathrm{H}^1(\mathbf{R}, G) = \prod \mathrm{H}^1(\mathbf{R}, G_i)$. This is trivial if each G_i has non-compact group of $\mathbf{R}$-points; i.e. all G_i are $\mathbf{R}$-isotropic.

Theorem 2.5. *Let $\mathscr{G}$ be a Lie group with finite component group. Then:*

(1) *Every compact subgroup of $\mathscr{G}$ (e.g., finite subgroups!) lies in a maximal compact subgroup.*

(2) *All maximal compact subgroups $\mathcal{K} \subset \mathscr{G}$ are conjugate and such $\mathcal{K}$ meet every connected component of $\mathscr{G}$, with $\mathcal{K} \bigcap \mathscr{G}^0 = \mathcal{K}^0$ a maximal compact in $\mathscr{G}^0$*

A proof of this theorem is given in Hochschild's book "Structure of Lie groups".

Example 2.6. 1) $\mathscr{G} = \mathrm{GL}_n(\mathbf{R}), \mathcal{K} = \mathrm{O}(n)$, $\mathscr{G}^0 = \mathrm{GL}_n^+(\mathbf{R}), \mathcal{K} = \mathrm{SO}(n)$.

2) $\mathscr{G} = \mathrm{Sp}_{2n}(\mathbf{R})$. Embedding it into $\mathrm{GL}_{2n}(\mathbf{R})$, which contains $\mathrm{GL}_n(\mathbf{C}) \supset \mathrm{U}(n)$, we obtain $\mathcal{K} = \mathrm{SU}(n)$.

3) If H is connected reductive and $\mathbf{R}$-anisotropic then $\mathscr{G} := H(\mathbf{C})$ as a (connected) Lie group has maximal compact subgroup $\mathcal{K} = H(\mathbf{R})$.

In each of these cases, $\mathcal{K}$ is the fixed point locus on an involution θ of $\mathbf{C}$:
In case 1), θ is $A \to ({}^tA)^{-1}$.
In case 2), θ is $A \to ({}^t\overline{A})^{-1}$
In case 3), θ is $h \to \overline{h}$ on $H(\mathbf{C})$ by using the $\mathbf{R}$-structure on $\mathbf{C}$.

We now generalize the preceding examples by finding an involution whose fixed-point locus is a maximal compact subgroup. Let $\mathscr{G}$ be a Lie group that is semisimple (i.e., $\mathfrak{g} := \mathrm{Lie}(\mathscr{G})$ is semisimple) and has finite component group, and assume $\mathscr{G}^0$ has *finite* center. Let $\mathcal{K} \subset \mathscr{G}$ be a maximal compact subgroup of $\mathscr{G}$. Denote $\mathrm{Lie}(K)$ by $\mathfrak{k} \subset \mathfrak{g}$. Let $\mathfrak{p}$ be the orthogonal complement to $\mathfrak{k}$ under the Killing form of $\mathfrak{g}$ (which is non-degenerate, as $\mathfrak{g}$ is semisimple). In this generality, we have a replacement for the "polar decomposition" in $\mathrm{GL}_n(\mathbf{R})$:

Theorem 2.7 (Cartan)**.** *The map $\mathcal{K} \times \mathfrak{p} \to \mathscr{G}$ defined by $(k, X) \to k\exp_{\mathscr{G}}(X)$ is a real-analytic isomorphism of manifolds.*

The hypothesis of finite center for $\mathscr{G}^0$ cannot be removed, since the universal cover $\mathscr{G}$ of $\mathrm{SL}_2(\mathbf{R})$ (whose discrete center is infinite) has no nontrivial compact subgroups! Note that to prove this theorem it suffices to treat $(\mathscr{G}^0, \mathcal{K}^0 = \mathcal{K} \cap \mathscr{G}^0)$ in place of $(\mathscr{G}, \mathcal{K})$ since $\mathcal{K}$ meets each connected component of $\mathscr{G}$. Likewise, in the connected case it is harmless to pass to the quotient modulo the finite center Z of $\mathscr{G}$ (as the *compact* subgroup Z lies in some maximal compact subgroup, and hence in all of them), so we may assume $\mathscr{G}$ has *trivial* center. By using semisimplicity of the Lie algebra and connectedness of $\mathscr{G}$ and triviality of the center, it can be shown that the adjoint representation $\mathrm{Ad}_{\mathscr{G}} : \mathscr{G} \to \mathrm{GL}(\mathfrak{g})$ is a closed embedding (and in fact is an isomorphism onto the identity component of the group of $\mathbf{R}$-points of a connected semisimple $\mathbf{R}$-group of adjoint type). Such "adjoint semisimple" cases are treated in §2 of Chapter XV of Hochschild's book "Structure of Lie groups", even allowing a finite component group.

Definition 2.8. The *global Cartan involution* $\theta_{\mathcal{K}}$ associated to $(\mathscr{G}, \mathcal{K})$ as above is defined by

$$\theta_{\mathcal{K}} \colon k\exp_{\mathscr{G}}(X) \to k\exp_{\mathscr{G}}(-X) = k\exp_{\mathscr{G}}(X)^{-1}$$

The reason for our interest in this map is that if there is to exist an involution $\theta : \mathscr{G} \simeq \mathscr{G}$ whose fixed-point set is exactly $\mathcal{K}$ then $\mathfrak{k}$ has to be the 1-eigenspace of θ on $\mathfrak{g}$, so the θ-stable $\mathfrak{p}$ must be acted upon by this involution as -1 and hence compatibility with the exponential map implies that the *only* possibility for such a θ is $\theta_{\mathcal{K}}$ as defined above. It is obvious that $\theta_{\mathcal{K}}$ is an involution of the pointed manifold $\mathscr{G}$ with fixed-point set $\mathcal{K}$, but it is not so obvious that $\theta_{\mathcal{K}}$ is a homomorphism. In "algebraic" cases this property is part of:

Example 2.9. Mostow showed that if G is a connected semisimple $\mathbf{R}$-group and $\mathcal{K}$ is a maximal compact subgroup of $G(\mathbf{R})$ then there is an inclusion $G \hookrightarrow \mathrm{GL}_n$ whose image is stable under $g \to (g^t)^{-1}$

such that $\mathcal{K}$ is the fixed-point locus of the involution of $G(\mathbf{R})$ induced by $g \mapsto (g^t)^{-1}$ (equivalently, $\mathcal{K} = G(\mathbf{R}) \cap \mathrm{O}(n)$). A proof is provided by Proposition 9.2 and Lemma 9.3 of Borel's book "Introduction aux groupes arithmétiques", requiring some delicate work to control disconnectedness aspects. (Mostow also proved a much harder and very useful result, concerning compatible Cartan involutions for any finite rising chain of inclusions between connected semisimple $\mathbf{R}$-groups.)

The homomorphism property is valid without algebraicity hypotheses:

Theorem 2.10. *The map $\theta_{\mathcal{K}}$ is a homomorphism. Also, if $\mathscr{G} = G(\mathbf{R})$ for a connected semisimple $\mathbf{R}$-group G then $\theta_{\mathcal{K}}$ arises from a unique involution $\theta_{\mathcal{K}}^{\mathrm{alg}}$ of the $\mathbf{R}$-group G.*

Proof. To prove the homomorphism property, we need to show that for $k, k' \in \mathcal{K}$ and $X, X' \in \mathfrak{p}$,

$$k \exp_{\mathscr{G}}(-X) k' \exp_{\mathscr{G}}(-X') \stackrel{?}{=} \theta_{\mathcal{K}}(k \exp_{\mathscr{G}}(X) k' \exp_{\mathscr{G}}(X')) = k \theta_{\mathcal{K}}(\exp_{\mathscr{G}}(X) k' \exp_{\mathscr{G}}(X')).$$

But $k'^{-1} \exp_{\mathscr{G}}(X) k' = \exp_{\mathscr{G}}(\mathrm{Ad}_{\mathscr{G}}(k'^{-1})(X))$ and likewise for $-X$ in place of X, so we may assume $k = 1$ and replace X with $\mathrm{Ad}_{\mathscr{G}}(k'^{-1})(X)$ to reduce to checking the homomorphism property

$$\theta_{\mathcal{K}}(gg') = \theta_{\mathcal{K}}(g)\theta_{\mathcal{K}}(g')$$

for $g, g' \in \exp_{\mathscr{G}}(\mathfrak{p}) \subset \mathscr{G}^0$. Since $\theta_{\mathcal{K}}|_{\mathscr{G}^0} = \theta_{\mathcal{K}^0}$, for the proof that $\theta_{\mathcal{K}}$ is a homomorphism it now may be assumed that $\mathscr{G}$ is *connected*.

Consequently, as for Theorem 2.7, it is harmless to pass to the quotient by the finite center to reduce to the case that $\mathscr{G} = H(\mathbf{R})^0$ for a connected semisimple $\mathbf{R}$-group H of adjoint type. This case is treated in the classical theory of Lie groups, but it can also be understood in more algebraic terms that handle the algebraicity assertion in the theorem: $\mathcal{K}$ is the identity component of a maximal compact subgroup of $H(\mathbf{R})$, so in view of the uniqueness for $\theta_{\mathcal{K}}$ it suffices to prove the existence and uniqueness of an $\mathbf{R}$-group involution $\theta_{\mathcal{K}}^{\mathrm{alg}}$ of H whose fixed-point locus in $H(\mathbf{R})$ is a chosen maximal compact subgroup. The existence of such an involution follows from Example 2.9, and uniqueness follows from the Zariski-density of $H(\mathbf{R})$ in H. □

Corollary 2.11. *If G is a connected semisimple $\mathbf{R}$-group then $K \mapsto K^0 := K \cap G(\mathbf{R})^0$ defines a bijection between the sets of maximal compact subgroups of $G(\mathbf{R})$ and $G(\mathbf{R})^0$. Equivalently, every maximal compact subgroup of $G(\mathbf{R})^0$ lies in exactly one maximal compact subgroup of $G(\mathbf{R})$.*

Proof. If K and K' are maximal compact subgroups of $G(\mathbf{R})$ with the same identity component then θ_K and $\theta_{K'}$ are $\mathbf{R}$-automorphisms of G which coincide on $\mathrm{Lie}(G) = \mathrm{Lie}(G(\mathbf{R})) = \mathrm{Lie}(G(\mathbf{R})^0)$ (as each restricts to the Cartan involution of their common identity component on $G(\mathbf{R})^0$). Thus, Zariski-connectedness of G and algebraicity of these automorphisms forces them to coincide (as we are in characteristic 0). But their respective fixed-point loci are K and K', so $K = K'$. □

Remark 2.12. If $f : G' \to G$ is an isogeny between connected semisimple $\mathbf{R}$-groups then it is generally not surjective on $\mathbf{R}$-points (e.g. $\mathrm{SL}_n \to \mathrm{PGL}_n$ for even n). However, by the preceding corollary it follows that if K' is a maximal compact subgroup of $G'(\mathbf{R})$ then there is a unique a maximal compact subgroup K of $G(\mathbf{R})$ containing $f(K')$, and moreover that $\theta_K \circ f = f \circ \theta_{K'}$.

Let's extend the notion of Cartan involution to $G(\mathbf{R})$ for connected reductive $\mathbf{R}$-groups G (though we will have no use for it, it is extremely convenient in practice). Our construction will initially be expressed in terms of G, rather than just the underlying Lie group $G(\mathbf{R})$. It will be built as an analytic automorphism of $G(\mathbf{R})$ rather than arise from an involution of the $\mathbf{R}$-group G, and will then turn out to be "algebraic": arises from an involution of the $\mathbf{R}$-group G.

Note that $\mathrm{Lie}(G) = \mathrm{Lie}(\mathscr{D}G) \oplus \mathrm{Lie}(Z_G^0)$ and the Killing form is "useful" on $\mathrm{Lie}(\mathscr{D}G)$ but "useless" on $\mathrm{Lie}(Z_G^0)$, so we need to handle the central torus Z_G^0 by more direct means. For any $\mathbf{R}$-torus T, let T_{s} be the maximal split $\mathbf{R}$-subtorus and let T_{a} be the maximal anisotropic $\mathbf{R}$-subtorus, so T_{s} is a power of $\mathbf{G}_m$ and T_{a} is a power of the 1-dimensional anisotropic $\mathbf{R}$-torus

$$\mathbf{S}^1 = \ker(\mathrm{N}_{\mathbf{C}/\mathbf{R}} : \mathrm{R}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m) \to \mathbf{G}_m)$$

satisfying $\mathbf{S}^1(\mathbf{R}) = S^1$. The evident multiplication map $T_{\mathrm{s}} \times T_{\mathrm{a}} \to T$ is an isogeny. For any $\mathbf{R}$-torus T (such as Z_G^0), we define the (algebraic) *Cartan involution* θ_T^{alg} to be inversion on T_{s} and to be the identity on T_{a}. This makes sense since $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-lattice considerations show that $T_{\mathrm{s}} \cap T_{\mathrm{a}}$ must be a power of μ_2, on which inversion is trivial. To "justify" this terminology, let's check:

Lemma 2.13. *The fixed-point locus of θ_T^{alg} on $T(\mathbf{R})$ is its unique maximal compact subgroup K_T.*

Proof. The main point of the proof is not to get ourselves totally confused about 2-torsion issues. To save notation, let $\theta = \theta_T^{\mathrm{alg}}$. The fixed-point locus of θ on $\mathrm{Lie}(T) = \mathrm{Lie}(T(\mathbf{R}))$ is exactly $\mathrm{Lie}(T_{\mathrm{a}})$, so T^θ has T_{a} as its identity component and hence $T^\theta(\mathbf{R})$ is compact. Our problem therefore is to show that K_T is fixed pointwise by θ (so $K_T \subseteq T^\theta(\mathbf{R})$, forcing equality by maximality of K_T).

Consider the $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-lattice $\mathrm{X}(T)$, on which complex conjugation acts as an involution ι. By commutativity of the Galois group, $-\iota$ is a Galois-equivariant automorphism of this lattice, so it corresponds to an involution θ' of the $\mathbf{R}$-torus T. Since T_{s} and T_{a} correspond to the maximal Galois-equivariant torsion-free quotients of $\mathrm{X}(T)$ on which the effect of complex conjugation is respectively trivial and negation, θ' preserves T_{s} and T_{a} with respective effect on these $\mathbf{R}$-tori given by inversion and the identity. In other words, θ' coincides with θ on T_{s} and T_{a}, so $\theta' = \theta$ on the $\mathbf{R}$-torus $T = T_{\mathrm{s}} \cdot T_{\mathrm{a}}$. Hence, our problem can be recast as showing that K_T is fixed pointwise by θ'.

Note that T/T_{a} is a split torus (it is an isogenous quotient of T_{s}), so $(T/T_{\mathrm{a}})(\mathbf{R})[2] = (T/T_{\mathrm{a}})[2](\mathbf{R})$ and this is the maximal compact subgroup of $(T/T_{\mathrm{a}})(\mathbf{R})$. Hence, under the quotient map $q : T \to T/T_{\mathrm{a}}$ any compact subgroup of $T(\mathbf{R})$ is carried into $(T/T_{\mathrm{a}})[2](\mathbf{R})$. It follows from the compactness of $T_{\mathrm{a}}(\mathbf{R})$ that K_T is the group of $\mathbf{R}$-points of $q^{-1}((T/T_{\mathrm{a}})[2])$. Hence, $K_T/T_{\mathrm{a}}(\mathbf{R})$ is a finite abelian group killed by 2. But $T_{\mathrm{a}}(\mathbf{R})$ is a power of S^1, on which squaring is surjective, so $K_T[2] \to K_T/T_{\mathrm{a}}(\mathbf{R})$ is surjective by the snake lemma. Since $T[2](\mathbf{R})$ is compact, by maximality of K_T we have $K_T[2] = T[2](\mathbf{R})$, so K_T is generated by $T_{\mathrm{a}}(\mathbf{R})$ and $T[2](\mathbf{R})$. Thus, the involution $\theta = \theta'$ fixes K_T pointwise provided that θ' acts as the identity on $T[2](\mathbf{R})$. By construction, θ' is the negative of complex conjugation on $T[2](\mathbf{C})$, so it is negation on $T[2](\mathbf{R})$, which is visibly the identity map. □

Pick a maximal compact subgroup K of $G(\mathbf{R})$, so $K' := K \cap (\mathscr{D}G)(\mathbf{R})$ is a maximal compact subgroup of $(\mathscr{D}G)(\mathbf{R})$; the formation of maximal compact subgroups of Lie groups with finite component group is always well-behaved in this way with respect to closed normal Lie subgroups with finite component group. Let $\theta_{K'}^{\mathrm{alg}}$ be the unique involution of the $\mathbf{R}$-group $\mathscr{D}G$ that induces the analytic

Cartan involution on $(\mathscr{D}G)(\mathbf{R})$ associated to K'. The group $G(\mathbf{R})^0$ is generated by the commuting subgroups $(\mathscr{D}G)(\mathbf{R})^0$ and $Z_G^0(\mathbf{R})^0$ whose intersection is a finite central subgroup of $(\mathscr{D}G)(\mathbf{R})^0$ and hence lies in the maximal compact subgroup $(K')^0$ as well as in the maximal compact subgroup of $Z_G^0(\mathbf{R})$. Thus, $\theta_{K'}^{\mathrm{alg}}$ on $(\mathscr{D}G)(\mathbf{R})^0$ and $\theta_{Z_G^0}^{\mathrm{alg}}$ on $Z_G^0(\mathbf{R})^0$ agree on their intersection, so they glue to define an involution θ_K^0 of $G(\mathbf{R})^0$ whose fixed-point locus has identity component $K'^0 \cdot (Z_G^0)_{\mathrm{a}}(\mathbf{R})^0$ that must coincide with K^0 for Lie algebra reasons. But θ_K^0 is an involution, so by uniqueness of Cartan involutions for connected semisimple Lie groups with finite center we see that $\theta_K^0 = \theta_{K^0}$ on $G(\mathbf{R})^0$.

Since $K \cdot G(\mathbf{R})^0 = G(\mathbf{R})$ and $K \cap G(\mathbf{R})^0 = K^0$, we get a well-defined real-analytic involution θ_K of the pointed manifold $G(\mathbf{R})$ by defining

$$\theta_K(kg^0) = k\theta_K^0(g^0) = k\theta_{K^0}(g^0)$$

for $k \in K$ and $g^0 \in G(\mathbf{R})^0$. (This is initially a set-theoretic involution, but it is left-invariant under K and agrees with θ_K^0 on $G(\mathbf{R})^0$, so it is real-analytic.) To prove that θ_K is a group homomorphism, we can argue as in the proof of Theorem 2.10 to reduce to checking that θ_{K^0} on $G(\mathbf{R})^0$ is equivariant with respect to the effect of k-conjugation for any $k \in K$. But such conjugation preserves K^0, so the unique characterization of θ_{K^0} as an involution of $G(\mathbf{R})^0$ with fixed-point locus K^0 implies that θ_{K^0} commutes with K-conjugation on $G(\mathbf{R})^0$.

By construction, the fixed-point locus of θ_K on $G(\mathbf{R})$ is $K \cdot (G(\mathbf{R})^0)^{\theta_{K^0}} = K \cdot K^0 = K$, so it is "justified" to call θ_K the *Cartan involution* of $G(\mathbf{R})$ attached to K, though we make no uniqueness claim yet for this Lie group automorphism and its construction depends on the $\mathbf{R}$-group G, not just on the Lie group $G(\mathbf{R})$. It is *not* obvious if θ_K preserves $(\mathscr{D}G)(\mathbf{R})$, due to disconnectedness issues, since we have not yet proved that θ_K arises from a (necessarily unique) involution θ_K^{alg} of the connected $\mathbf{R}$-group G. All works out well:

Proposition 2.14. *There is a unique involution θ_K^{alg} of the $\mathbf{R}$-group G whose fixed-point locus is K, and its effect on $G(\mathbf{R})$ is θ_K. Moreover, the restriction of θ_K^{alg} to $\mathscr{D}G$ recovers $\theta_{K'}^{\mathrm{alg}}$ and its restriction to Z_G^0 recovers $\theta_{Z_G^0}^{\mathrm{alg}}$.*

The description of θ_K^{alg} on Z_G^0 can be strengthend: the effect of θ_K^{alg} on the geometric character group $\mathrm{X}(Z_G)$ is the negative of the effect of complex conjugation. However, the only clean proof I know involves invoking a "connected reductive" generalization of Mostow's generalization of Example 2.9, and that would be too much of a digression to explain here. (I tried to make a bare-hands proof using induction on the dimension of Z_G, but even the base of the induction with semisimple G seems not at all obvious by elementary means.)

Proof. Grant the existence of an automorphism θ_K^{alg} of G that recovers θ_K on $G(\mathbf{R})$ (so it has fixed-point locus on $G(\mathbf{R})$ equal to K, and is an involution since such a property can be checked on the Zariski-dense subset $G(\mathbf{R})$). It must preserve $\mathscr{D}G$ and Z_G (as for any $\mathbf{R}$-automorphism of G). The compatibility of such a θ_K^{alg} with $\theta_{K'}^{\mathrm{alg}}$ on $\mathscr{D}G$ and with $\theta_{Z_G^0}^{\mathrm{alg}}$ on Z_G^0 is reduced to a computation with Lie algebras (since $\mathscr{D}G$ and Z_G^0 are Zariski-connected). Since the Lie algebra of a linear algebraic $\mathbf{R}$-group naturally coincides with that of the identity component of its $\mathbf{R}$-points, the equality $\theta_K|_{G(\mathbf{R})^0} = \theta_{K^0}$ provides what we need for the Lie algebra computation.

To prove that there is at most one involution θ of G with fixed-point locus K, note that any such involution restricts to $\mathscr{D}G$ and Z_G^0 with analogous properties for the maximal compact subgroups $K \cap (\mathscr{D}G)(\mathbf{R})$ and $K \cap (Z_G^0)(\mathbf{R})$ of $(\mathscr{D}G)(\mathbf{R})$ and $(Z_G^0)(\mathbf{R})$ respectively, so for the uniqueness aspect it suffices to treat the connected semisimple and torus cases separately. The connected semisimple case is known from our discussion of Cartan involutions for connected semisimple $\mathbf{R}$-groups, and the case of tori is reduced to the separate cases of anisotropic and split tori (by passing to the maximal anisotropic $\mathbf{R}$-subtorus and the maximal split $\mathbf{R}$-subtorus), both of which are trivial.

It remains to make an involution of the $\mathbf{R}$-group G so that the fixed-point locus on $G(\mathbf{R})$ is exactly K. Note that it suffices to treat a single K, as all such are $G(\mathbf{R})$-conjugate. I tried to make an elementary proof using induction from the known semisimple case by pasting on 1-dimensional central tori one at a time, but the case of pasting on a split torus seems to get bogged down in unpleasant technicalities related to disconnectedness. Consequently, the only method that I know is to construct an $\mathbf{R}$-subgroup inclusion $j : G \hookrightarrow \mathrm{GL}_n$ so that (i) $j(G)$ is stable under transpose-inverse (by Zariski-density of $G(\mathbf{R})$ in G, it is equivalent to say that $G(\mathbf{R})$ is stable under transpose-inverse on $\mathrm{GL}_n(\mathbf{R})$) and (ii) $K := G(\mathbf{R}) \cap \mathrm{O}(n)$ is a maximal compact subgroup of $G(\mathbf{R})$. In such cases, the effect on G of transpose-inverse on GL_n is the desired involution θ_K^{alg} for K as in (ii). The existence of such a j was noted (with reference for a proof in Borel's book on arithmetic groups) in Example 2.9 for the case of semisimple G. But in fact, the argument in Borel's book for the semisimple case can be adapted with minor changes so that it works in the reductive case. (The main point is that the initial inner product Q chosen on $\mathbf{R}^n$ at the start of that argument should be chosen to make the distinct weight spaces in $\mathbf{R}^n$ for the action of the maximal split torus of G pairwise orthogonal and then it should be averaged over the maximal compact central torus to be invariant under that torus.) $\square$

We can now extend Corollary 2.11 to the reductive case with essentially the same proof, using the algebraicity of our Cartan involution construction in the reductive case:

Corollary 2.15. *For a connected reductive $\mathbf{R}$-group G, $K \mapsto K^0 = K \cap G(\mathbf{R})^0$ is a bijection between the sets of maximal compact subgroups of $G(\mathbf{R})$ and $G(\mathbf{R})^0$. Equivalently, every maximal compact subgroup of $G(\mathbf{R})^0$ lies in a unique maximal compact subgroup of $G(\mathbf{R})$.*

Remark 2.16. In the literature on Shimura varieties, which is focused more on algebraic groups than on Lie groups, one often sees a rather different definition of Cartan involution in terms of $\mathbf{R}$-anisotropic forms. Let's briefly address the compatibility of these two approaches.

Let G be a connected reductive $\mathbf{R}$-group, so we have an associated *connected* complex Lie group $G(\mathbf{C})$. (This is generally *not* functorial in $G(\mathbf{R})$, only in G. For this reason, it is wrong to refer to $G(\mathbf{C})$ as "the complexification" of the Lie group $G(\mathbf{R})$.) For a Cartan involution θ^{alg} of G, we get an induced involution $\theta_{\mathbf{C}}^{\mathrm{alg}}$ of $G_{\mathbf{C}}$. Letting σ denote complex conjugation on $\mathbf{C}$, the $\mathbf{R}$-structure G on $G_{\mathbf{C}}$ provides a natural isomorphism $\iota : \sigma^*(G_{\mathbf{C}}) \simeq G_{\mathbf{C}}$. The isomorphism $\theta' := \theta_{\mathbf{C}}^{\mathrm{alg}} \circ \iota$ satisfies

$$\theta' \circ \sigma^*(\theta') = \theta_{\mathbf{C}}^{\mathrm{alg}} \circ \iota \circ \sigma^*(\theta_{\mathbf{C}}^{\mathrm{alg}}) \circ \iota^{-1} = \theta_{\mathbf{C}}^{\mathrm{alg}} \circ \theta_{\mathbf{C}}^{\mathrm{alg}} = \mathrm{id}$$

(the second-to-last equality using that $\theta_{\mathbf{C}}^{\mathrm{alg}}$ arises from an $\mathbf{R}$-automorphism of G), so it defines a Galois descent datum on $G_{\mathbf{C}}$ corresponding to an $\mathbf{R}$-form G_θ of G *equipped with* an isomorphism of

their $\mathbf{C}$-fibers. Moreover, $G_\theta(\mathbf{R})$ is identified with the subgroup of $G(\mathbf{C})$ given by

$$\{g \in G(\mathbf{C}) \,|\, \theta^{\mathrm{alg}}(\overline{g}) = g\}.$$

The general result (whose proof we do not give here, as it requires a fair amount of additional work) is that $G_\theta(\mathbf{R})$ is a maximal compact subgroup of $G(\mathbf{C})$ and that $K \mapsto G_{\theta_K^{\mathrm{alg}}}(\mathbf{R})$ defines a bijection from the set of maximal compact subgroups of $G(\mathbf{R})$ to the set of maximal compact subgroups of $G(\mathbf{C})$; its inverse is given by $\mathcal{K} \mapsto \mathcal{K} \cap G(\mathbf{R})$. Moreover, under this correspondence, if $\mathcal{K}$ is a maximal compact subgroup of $G(\mathbf{C})$ and $K := \mathcal{K} \cap G(\mathbf{R})$ then the involution $\theta_{\mathcal{K}}$ of the connected Lie group $G(\mathbf{C})$ restricts to θ_K on $G(\mathbf{R})$ and the $\mathbf{R}$-descent G_{θ_K} of $G_{\mathbf{C}}$ has associated "complex conjugation" on $G_{\theta_K}(\mathbf{C}) = G(\mathbf{C})$ given by $g \mapsto \theta_K(\overline{g})$. This turns out to define a bijection between the set of Cartan involutions of G in the sense that we have defined them and set of Galois descent data on $G_{\mathbf{C}}$ whose associated $\mathbf{R}$-descent is $\mathbf{R}$-anisotropic. It follows that Cartan involutions θ of G are precisely the involutions for which the fixed-point locus of the involution $g \mapsto \theta(\overline{g})$ on $G(\mathbf{C})$ is compact, and in some Shimura variety literature one sees this latter description taken as the *definition* of Cartan involutions.

3. The Shimura Datum

We first consider a connected semisimple $\mathbf{Q}$-groups G of *adjoint type* (e.g., $\mathrm{PGL}_n, \mathrm{SO}_{2n+1}$, etc.), so $G = \prod G_j$ where the G_j are $\mathbf{Q}$-simple of adjoint type. Each G_j has the form $\mathrm{R}_{k_j/\mathbf{Q}}(G_j')$ for an absolutely simple adjoint type G_j' over some number field k_j'.

Definition 3.1. A *connected Shimura datum* is a $G(\mathbf{R})^0$-conjugacy class of continuous homomorphisms $u\colon S^1 \to G(\mathbf{R})^0$ that satisfy three axioms:

1) The S^1-action on $\mathrm{Lie}(G(\mathbf{R}))_{\mathbf{C}} = \mathrm{Lie}(G)_{\mathbf{C}}$ has only weights in $\{z^{-1}, 1, z\} \subset \mathrm{Hom}(S^1, \mathbf{C}^\times)$.

2) The involution of $G(\mathbf{R})^0$ given by $u(-1)$-conjugation is a Cartan involution. [In particular, this can only happen when the Cartan involutions of $G(\mathbf{R})^0$ – which are a single conjugacy class of involutions – are inner.] It is equivalent to say that $u(-1)$-conjugation on $G(\mathbf{R})$ is a Cartan involution, since $G(\mathbf{R})^0$ is Zariski-dense in $G_{\mathbf{R}}$.

Note that since $Z_G = 1$ (due to the adjoint type hypothesis on G) and $\mathrm{Lie}(G(\mathbf{R})^0) = \mathrm{Lie}(G)$, conjugation by $u(-1)$ on $G(\mathbf{R})^0$ is non-trivial if and only if $u(-1)$ is non-trivial. This brings us to:

3) The projection of $u(-1)$ into every G_j is non-trivial. In view of 2), this is equivalent to every $G_j(\mathbf{R})$ being non-compact (and hence non-redundant when considering $G(\mathbf{R})/\mathcal{K} = G(\mathbf{R})^0/\mathcal{K}^0$ for a maximal compact subgroup $\mathcal{K}$ of $G(\mathbf{R})$.

Remark 3.2. In the preceding definition, the centralizer of u in $G(\mathbf{R})$ is equal to $\mathcal{K}^0$. Indeed, if $\mathbf{S}^1 := \ker(\mathrm{N}_{\mathbf{C}/\mathbf{R}} : \mathrm{R}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m) \to \mathbf{G}_m)$ denotes the rank-1 anisotropic $\mathbf{R}$-torus corresponding to S^1 then u necessarily arises from an $\mathbf{R}$-homomorphism $u^{\mathrm{alg}} : \mathbf{S}^1 \to G$ (due to compactness and connectedness of S^1) and so $Z_{G(\mathbf{R})}(u) = Z_G(u^{\mathrm{alg}})(\mathbf{R})$ with $Z_G(u^{\mathrm{alg}})$ a connected $\mathbf{R}$-subgroup of G whose Lie algebra is the same as that of $\mathcal{K}$. This equality of Lie algebras forces $Z_G(u^{\mathrm{alg}})(\mathbf{R})^0 = \mathcal{K}^0$, so by compactness of $\mathcal{K}^0$ the connected reductive $Z_G(u^{\mathrm{alg}})$ is $\mathbf{R}$-anisotropic. Consequently, the group $Z_G(u^{\mathrm{alg}})(\mathbf{R})$ is also connected, whence it coincides with $\mathcal{K}^0$ as claimed. It follows that $G(\mathbf{R})/\mathcal{K}^0$ parameterizes the $G(\mathbf{R})$-conjugacy class of u, whereas $G(\mathbf{R})^0/\mathcal{K}^0$ parameterizes the $G(\mathbf{R})^0$-conjugacy class of u.

Now we pass to the more general situation of reductive groups.

Definition 3.3. A *Shimura datum* is a pair (G, X) where G is a connected reductive group over $\mathbf{Q}$ and X is a $G(\mathbf{R})$-conjugacy class of maps $h\colon \mathbf{C}^\times \to G(\mathbf{R})$ that are algebraic (i.e., descend from $\mathrm{R}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m) \to G_{\mathbf{R}}$) and satisfy:

1) the induced $\mathbf{C}^\times$-action on $\mathrm{Lie}(G(\mathbf{R})) = \mathrm{Lie}(G)_{\mathbf{C}}$ defined by $\mathrm{Ad}_G \circ h$ has only weights in the set $\{1, z/\overline{z}, \overline{z}/z\}$ (this forces $h(\mathbf{R}^\times) \subset Z_G(\mathbf{R})$, so $h(-1)$ is central); equivalently, as a Hodge structure the only types that may occur are $(-1,1), (0,0), (1,-1)$.

2) the involution of $(\mathscr{D}G)(\mathbf{R})$ given by $h(\pm i)$-conjugation on $G(\mathbf{R})$ is a Cartan involution. [By Remark 2.12, it is equivalent to say that $\mathrm{Ad}_G(h(\pm i))$-conjugation on $G^{\mathrm{ad}}(\mathbf{R})$ is a Cartan involution, so this can only happen when the Cartan involutions of $G^{\mathrm{ad}}(\mathbf{R})$ are inner. The formulation using $\mathscr{D}G$ is used in Deligne's original Bourbaki report, and the formulation using G^{ad} seems to be more common in recent literature.]

3) the map h does not project trivially into any $\mathbf{Q}$-simple factor of G^{ad}.

To explain the link between Shimura data and connected Shimura data, we assume G is of adjoint type: $Z_G = 1$. Thus, the subgroup $h(\mathbf{R}^\times) \subset Z_G(\mathbf{R})$ is trivial, so h factors through a map $u : \mathbf{C}^\times/\mathbf{R}^\times \to G(\mathbf{R})$. Under the product decomposition $\mathbf{C}^\times = S^1 \times \mathbf{R}^\times_{>0}$ we see that $\mathbf{C}^\times/\mathbf{R}^\times$ is identified with $S^1/\langle -1\rangle = S^1$ via the squaring map, so u is a homomorphism $S^1 \to G(\mathbf{R})$ satisfying $u(t^2) = h(t)$ for $t \in S^1 \subset \mathbf{C}^\times$ (e.g., $u(-1) = h(\pm i)$). Thus, for adjoint semisimple G, axioms (1), (2), and (3) for h are respectively equivalent to axioms (1), (2), and (3) for u. However, u is considered with its $G(\mathbf{R})^0$-conjugacy class whereas h is considered with its $G(\mathbf{R})$-conjugacy class.

Example 3.4. Let (V, Ψ) be a $\mathbf{Q}$-vector space of finite dimension $n > 0$ endowed with a symplectic form. Let $G = \mathrm{GSp}(\Psi) = \{g \in \mathrm{GL}(V) \mid \Psi(gu, gv) = \mu(g)\Psi(u,v)\}$, so $G \simeq \mathrm{GSp}_{2n}$.

Consider Lie group homomorphisms $h\colon \mathbf{C}^\times \to \mathrm{GL}(V_{\mathbf{R}})$ algebraic on $\mathbf{R}^\times$ (or equivalently, $\mathbf{R}$-group homomorphisms $\mathrm{R}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m) \to \mathrm{GL}(V)_{\mathbf{R}}$) for which the associated Hodge structure on $V_{\mathbf{R}}$ has type $\subset \{(-1,0),(0,-1)\}$. Such h have a rather concrete meaning, as we now review.

The condition that the Hodge type consists of pairs of the form $(-p, p-1)$ ($p \in \mathbf{Z}$) says that $h|_{\mathbf{R}^\times}$ is the canonical inclusion, so $h(\pm i)^2 = h(-1) = -1$ and hence $h(\pm i)$ define a complex structure on $V_{\mathbf{R}}$ (namely, $(\pm i).v = h(\pm i)(v)$ for all $v \in V_{\mathbf{R}}$, which is unambiguous since $h(-i) = -h(i)$). Also, $h|_{S^1}$ decomposes as a direct sum of copies of the underlying $\mathbf{R}$-linear representations $t^{1-2p} : S^1 \to \mathbf{C}^\times$. Only $p = 0, 1$ occur, so when we equip $V_{\mathbf{R}}$ with the common $\mathbf{C}$-linear structure W defined by $h(\pm i)$ then h extends to an $\mathbf{R}$-algebra homomorphism $\mathbf{C} \to \mathrm{GL}(W)$. But the $\mathbf{C}$-linear structure on W is designed to make $h(\pm i)$ acts as the linear effect of $\pm i \in \mathbf{C}$, so the h we are considering is precisely the restriction of a $\mathbf{C}$-linear structure V_h on $V_{\mathbf{R}}$.

Now we restrict our attention a bit further: consider only those h as above for which the $\mathbf{R}$-bilinear $\Psi_{\mathbf{R}}(v, h(i)w)$ is symmetric (a condition unaffected by replacing i with $-i$). Since h extends to an $\mathbf{R}$-algebra map $\mathbf{C} \to \mathrm{GL}(V_{\mathbf{R}})$, so it is equivalent to say that the $\mathbf{R}$-bilinear form $(v,w) \mapsto \Psi_{\mathbf{R}}(v, h(z)w)$ on $V_{\mathbf{R}}$ is symmetric for all $z \in \mathbf{C}^\times$. Since Ψ is alternating we have

$$\Psi_{\mathbf{R}}(v, h(i)w) = -\Psi_{\mathbf{R}}(h(i)w, v) = \Psi_{\mathbf{R}}(h(-i)w, v),$$

whereas the symmetry condition says exactly that $\Psi_{\mathbf{R}}(v, h(i)w) = \Psi_{\mathbf{R}}(w, h(i)v)$, or in other words (via $\mathbf{R}$-bilinearity) $\Psi_{\mathbf{R}}(w, h(z)v) = \Psi_{\mathbf{R}}(h(\overline{z})w, v)$ for all $z \in \mathbf{C}^\times$. Upon replacing w with $h(z)w$, the condition that $\Psi_{\mathbf{R}}(v, h(i)w)$ is symmetric says precisely that $\Psi_{\mathbf{R}}(h(z)v, h(z)w) = |z|^2\Psi_{\mathbf{R}}(v, w)$ for all $z \in \mathbf{C}^\times$, or in other words that h is valued in the $\mathbf{R}$-subgroup $\mathrm{GSp}(\Psi_{\mathbf{R}}) \subset \mathrm{GL}(V_{\mathbf{R}})$.

Finally, we come to the main construction: let $X^{\pm}$ denote the set of continuous homomorphisms $h : \mathbf{C}^\times \to G(\mathbf{R}) = \mathrm{GSp}(\Psi_{\mathbf{R}})$ algebraic on $\mathbf{R}^\times$ such that the Hodge type is contained in $\{(-1,0),(0,-1)\}$ and the necessarily symmetric $\Psi_{\mathbf{R}}(v, h(i)w)$ is *definite* (i.e., $\Psi_{\mathbf{R}}(v, h(i)v)$ is nonzero with a fixed sign for all nonzero $v \in V_{\mathbf{R}}$). In terms of the complex structure V_h on $V_{\mathbf{R}}$ defined by $h(\pm i)$, the condition $h(i) \in G(\mathbf{R})$ says exactly that

$$H_{h,i}(v, w) = \Psi_{\mathbf{R}}(v, h(i)w) + i\Psi_{\mathbf{R}}(v, w)$$

is Hermitian on V_h, and the definiteness hypothesis expresses exactly the property that $H_{h,i}$ is definite. (Since $H_{h,-i} = -H_{h,i}$, the condition on h that $H_{h,i}$ be definite is independent of the choice of i and hence is intrinsic.)

Since such h are the restriction of a complex structurre on $V_{\mathbf{R}}$, the resulting $\mathbf{C}^\times$-action on $\mathrm{Lie}(G)_{\mathbf{C}} \subset \mathfrak{gl}(V_{\mathbf{C}})$ has weights $\{z/\overline{z}, \overline{z}/z\}$ (with each weight space of the same dimension as the other), which is to say the Hodge type is $\{(1,-1),(-1,1)\}$. Using the complex structure V_h on $V_{\mathbf{R}}$ defined by $h(\pm i)$, the resulting connected unitary group $\mathrm{U}(H_{h,i}) = \mathrm{U}(-H_{h,i}) = \mathrm{U}(H_{h,-i})$ is a subgroup of the connected Lie group $G(\mathbf{R})$ (as $\Psi_{\mathbf{R}}$ is the imaginary part of $H_{h,i}$) and it is compact (due to the definiteness of $H_{h,i}$). This meets the connected $(\mathscr{D}G)(\mathbf{R}) = \mathrm{Sp}(\Psi_{\mathbf{R}})$ in $\mathrm{SU}(H_{h,i})$, which is a maximal compact subgroup by dimension reasons. The elements of $\mathrm{U}(H_{h,i})$ are precisely the $T \in \mathrm{GL}(V_{\mathbf{R}})$ that centralize $h(i)$ (or equivalently, centralize h) and satisfy the pair of conditions

$$\Psi_{\mathbf{R}}(Tv, h(i)Tw) = \Psi_{\mathbf{R}}(v, h(i)w), \;\; \Psi_{\mathbf{R}}(Tv, Tw) = \Psi_{\mathbf{R}}(v, w)$$

for all $v, w \in V_{\mathbf{R}}$. The second of these conditions just says $T \in G(\mathbf{R})$, and by using this second condition we can rewrite the first as:

$$\Psi_{\mathbf{R}}(Tv, h(i)Tw) = \Psi_{\mathbf{R}}(Tv, T(h(i)w)).$$

But $\Psi_{\mathbf{R}}$ is non-degenerate, so this final condition expresses exactly that T centralizes $h(i)$. Thus, $h(i)$-conjugation on $(\mathscr{D}G)(\mathbf{R})$ is an involution whose fixed-point set is the maximal compact subgroup $\mathrm{SU}(H_{h,i})$, so $h(i)$-conjugation is a Cartan involution of $(\mathscr{D}G)(\mathbf{R})$.

We conclude that every $G(\mathbf{R})$-conjugacy class in $X^{\pm}$ is a Shimura datum, and it remains to show that $X^{\pm}$ is a single $G(\mathbf{R})$-conjugacy class. The preceding calculations show that an element $h \in X^{\pm}$ is nothing more or less than a complex structure on $V_{\mathbf{R}}$ for which the associated Hermitian form $H_{h,i}$ is definite. If h' is a second such complex structure, then there is a unique $\pm i$ for which the definite $H_{h',\pm i}$ has the same sign as the definite $H_{h,i}$, so there is a $\mathbf{C}$-linear isomorphism $T : V_h \simeq V_{h'}$ carrying $H_{h,i}$ to $H_{h',\pm i}$ since any two n-dimensional Hermitian spaces with the same signature are isomorphic. Viewing T as an element of $\mathrm{GL}(V_{\mathbf{R}})$, we have $T \circ h(i) = h'(i) \circ T$, so T conjugates h into h'. The preservation of the *common* real part $\Psi_{\mathbf{R}}$ of the Hermitian forms $H_{h,i}$ and $H_{h',\pm i}$ implies that $T \in \mathrm{Sp}(V_{\mathbf{R}}, \Psi_{\mathbf{R}}) = (\mathscr{D}G)(\mathbf{R})$, so indeed h' and h are $G(\mathbf{R})$-conjugate.

Proposition 3.5. *For $n > 2$, the $\mathbf{Q}$-groups PGL_n and SL_n do not admit Shimura data.*

Proof. It suffices to check that Cartan involutions of $\mathrm{PGL}_n(\mathbf{R})$ are not inner. These involutions form a single conjugacy class, so it suffices to focus on a single such involution. Since $\mathrm{SL}_n(\mathbf{R})$ is a finite covering of $\mathrm{PGL}_n(\mathbf{R})^0$, it suffices to show that a single Cartan involution of $\mathrm{SL}_n(\mathbf{R})$ is not induced by an inner automorphism of $\mathrm{GL}_n(\mathbf{R})$. We will focus on the transpose-inverse involution, and it is equivalent to work with the $\mathbf{R}$-group SL_n rather than with the Lie group $\mathrm{SL}_n(\mathbf{R})$ since the latter is Zariski-dense in the former.

There are several ways to see that transpose-inverse on SL_n is not induced by a $\mathrm{GL}_n(\mathbf{R})$-conjugation. One way is to observe that such automorphisms of the $\mathbf{R}$-group SL_n act trivially on the center μ_n, yet transpose-inverse on this subgroup is visibly inversion, which is nontrivial since $n > 2$. Alternatively, on diagonal matrices transpose-inverse acts by inverting all the diagonal entries, which visibly does not preserve the trace on such matrices in SL_n (since $n > 2$). $\square$