SHIMURA VARIETIES: THE ADELIC POINT OF VIEW

SAM LICHTENSTEIN

1. Introduction

1.1. **Disclaimer.** Nothing in these notes is original; we follow primarily [5, Ch. 5] and [3].

1.2. **Where have we been and where are we going?** We saw in Martin and Brian's talks the definition of a connected Shimura datum and a connected Shimura variety, which I now recall.

1.2.1. **Definition.** A **connected Shimura datum** is a pair consisting of a connected, semisimple algebraic group $G_{/\mathbf{Q}}$ and a $G^{\mathrm{ad}}(\mathbf{R})^+$-conjugacy class X^+ of homomorphisms

$$h : \mathrm{R}_{\mathbf{C}/\mathbf{R}}\mathbf{G}_{\mathrm{m}} =: \mathbf{S} \to G^{\mathrm{ad}}_{\mathbf{R}}$$

such that the following three axioms hold:

(1) For one (any) $h \in X^+$, the $\mathbf{R}$-Hodge structure on $\mathfrak{g} := \mathrm{Lie}(G(\mathbf{R}))$ defined by composing h with the adjoint representation is required to be of type $\{(-1,1),(0,0),(1,-1)\}$.

$$\mathbf{S} \xrightarrow{h} G^{\mathrm{ad}}_{\mathbf{R}} \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathfrak{g})$$

Concretely, $\mathbf{C}^\times$ acts on $\mathfrak{g}_{\mathbf{C}}$ via the characters $z/\overline{z}, 1, \overline{z}/z$.

(2) $\mathrm{ad}(h(\pm i))$ is a Cartan involution of $G^{\mathrm{ad}}(\mathbf{R})$.

(3) There is no $\mathbf{Q}$-simple factor G_i of G such that $\mathbf{S} \xrightarrow{h} G^{\mathrm{ad}}_{\mathbf{R}} \to G^{\mathrm{ad}}_{i,\mathbf{R}}$ is trivial.

1.2.2. **Remark.** Brian gave a slightly different defintion of a connected Shimura datum, using the continuous map $u : S^1 \to G^{\mathrm{ad}}(\mathbf{R})^+$ obtained by restricting $h(\mathbf{R})$ to the unit circle $S^1 \subset \mathbf{C}^\times = \mathbf{S}(\mathbf{R})$. This turns out to be equivalent, as he explained in §3 of his notes (or see [5, p. 50]).

We now know that these are nice in several ways. First, X^+ is a hermitian symmetric domain with a nice holomorphic action of $G(\mathbf{R})^+$. This was more-or-less explained in the talks by Brandon, Martin, and Brian, but I'll say something about it below to tie together what has been covered so far.

Moreover, by the Baily–Borel theorem from Mike's talk, for any torsion-free arithmetic subgroup $\Gamma \subset G^{\mathrm{ad}}(\mathbf{R})^+$ the locally symmetric quotient $\mathrm{Sh}^\circ_\Gamma(G, X^+) = \Gamma\backslash X^+$ has the structure of a complex algebraic variety. Again, it's worth being precise about exactly how the Baily–Borel theorem is being applied, so I will say something about this below.

In this talk I shall explain another nice way to think about these quotients: when G is simply connected and $\Gamma = G(\mathbf{Q}) \cap K \subset G(\mathbf{A}_f)$ is a *congruence* subgroup (arising from a compact open subgroup $K \subset G(\mathbf{A}_f)$), then they can be described adelically as

$$\mathrm{Sh}^0_K(G, X^+) := \mathrm{Sh}^0_{K\cap G(\mathbf{Q})}(G, X^+) = G(\mathbf{Q})\backslash(X^+ \times G(\mathbf{A}_f))/K.$$

Furthermore, these varieties fit together into a nice inverse system with

$$\mathrm{Sh}^\circ(G, X^+) := \varprojlim_K \mathrm{Sh}^\circ_K(G, X^+) = G(\mathbf{Q})\backslash(X^+ \times G(\mathbf{A}_f)).$$

(All of these adelic descriptions are really *homeomorphisms* with respect to the usual adelic topology.) Call this sort of thing a **connected Shimura variety**.

One aim of this talk is to prove these facts. Another is to explain how they extend to *general* Shimura data. Recall that Brian gave the following modification of the definition of a connected Shimura datum.

1.2.3. **Definition.** A **Shimura datum** is a pair consisting of connected *reductive* algebraic group $G_{/\mathbf{Q}}$ and a $G(\mathbf{R})$-conjugacy class[1] X of morphisms $h : \mathbf{S} \to G_{\mathbf{R}}$, satisfying the same three axioms in the definition of "connected Shimura datum".

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[1] not just a $G^{\mathrm{ad}}(\mathbf{R})^+$-conjugacy class!

Much as in the case of connected Shimura data, we can attach to this data a **Shimura variety** for any congruence subgroup of $G(\mathbf{Q})$. This will be an inverse system of disjoint unions of connected Shimura varieties, with a similar adelic description.

Following Milne, here are two reasons why you might care about the more general notion.

- Reductive groups are a more natural class to consider. Brian has stressed repeatedly in this seminar in his classes how phrasing proofs in terms of reductive rather than semisimple groups makes inductive arguments cleaner. Moreover we might wonder how to relate the theory on a particular semisimple group we care about to the situation of, say, a nonsemisimple reductive subgroup.
- One of our main goals is to at least *state* some theorems about the existence of canonical models for Shimura varieties over number fields. To get this off the ground, one can first try to study models for connected Shimura varieties. However, there's a problem. The "right" field of definition for $\mathrm{Sh}^\circ_K(G, X^+)$ will tend to grow as K shrinks. Indeed, consider the case of modular curves. As we know, many modular curves X_Γ are geometrically disconnected. For example, the geometrically connected components of $X(N)$ correspond to primitive Nth roots of unity (by considering the Weil pairing). As such, a chosen component $X(N)^{\zeta_N}$ is defined only over $\mathbf{Q}(\mu_N)$. If we blithely continue working with a single connected component, we can *still* descend it to $\mathbf{Q}$ if we want. (Pete Clark has a nice survey [1] explaining many different ways of defining various modular curves over $\mathbf{Q}$.) But we would not end up with the *canonical* model in the sense of Shimura and Deligne. That we have done something a bit bad is manifested by the fact that we lose the natural moduli interpretation; it is probably better (depending on one's aims) to work with the whole geometrically disconnected curve. In particular, whatever tricks you come up with to make all your favorite (connected) modular curves live over $\mathbf{Q}$ are rather unlikely to work for general connected Shimura varieties.

  The (slightly unconvincing) analogy Milne gives is to the $\mathbf{Q}(i)$ point $x = i$ in the affine line $\mathbf{Aff}^1_{\mathbf{Q}}$. To get something defined over $\mathbf{Q}$, if we are willing to sacrifice geometric connectedness, it is reasonable to take its disjoint union with its Galois conjugate $x = -i$ to get the $\mathbf{Q}$-subscheme $\{x^2+1 = 0\} \subset \mathbf{Aff}^1_{\mathbf{Q}}$. By using (disconnected) Shimura varieties, we will be able to find a systematic way of working with disjoint unions of connected Shimura varieties which have models over number fields with good properties (albeit probably lacking moduli descriptions like in the nice case of elliptic modular curves).

  For elaboration of these remarks as regards modular curves, I recommend §3 of Milne's notes [4].

So, in preparation for subsequent lectures on special points and canonical models, we need to describe how to associate adelic double coset spaces to general Shimura data (G, X), and see how they may be related to connected Shimura varieties cooked up out of (G, X) by passing to the derived group of G and a connected component of X. These are the main goals of the lecture. Along the way, we'll also learn a bit about morphisms of Shimura data and the induced morphisms of Shimura varieties, as well as give a bunch of examples of everything above.

2. Connected Shimura data, Hermitian Symmetric Domains, Baily–Borel

2.1. **HSD structure from connected Shimura datum.** This section fills in a loose end from Brandon's talk, at which time we may or may not have actually had enough background on $\mathbf{R}$-groups to prove the desired theorem.

2.1.1. **Proposition.** Let (G, X^+) be a connected Shimura datum. Then X^+ has a natural structure of a Hermitian symmetric domain for which there is a map $G(\mathbf{R})^+ \twoheadrightarrow \prod \mathrm{Hol}(X_i)^+$ with compact kernel, where X_i are the irredicble factors of X^+ (in the sense of Hermitian symmetric domains).

To discuss the proof we need a definition and a lemma, which will also come up again later in the talk.

2.1.2. **Definition.** We say G is *of noncompact type* if for each $\mathbf{Q}$-simple factor G_i, the Lie group $G_i(\mathbf{R})$ is noncompact.

2.1.3. **Lemma.** Let (G, X^+) be a connected Shimura datum. Then G is of noncompact type.

Proof. Brian proved this in passing (cf. §3 of the notes from his talk on real groups) using a slightly different (but equivalent) definition of "connected Shimura datum". For convenience we reproduce the proof.

The nondegeneracy axiom (SV3) in the definition of Shimura datum says that the homomorphism $\mathbf{S} \to G^{\mathrm{ad}}_{\mathbf{R}}$ is nontrivial when projected to (the $\mathbf{R}$-group associated to the adjoint form of) any $\mathbf{Q}$-simple factor G_α of G. Call this projection h_α. By the second Shimura datum axiom (SV2), $\mathrm{ad}h(\pm i)$ is a Cartan involution. As explained in Brian's lectures, the identity is a Cartan involution of a real Lie group if and only if the group is compact. So $G^{\mathrm{ad}}_\alpha(\mathbf{R})$ is compact if and only if $\mathrm{ad}h_\alpha(\pm i) = 1$.

In fact, this forces h_α to be the trivial homomorphism. The reason is that by (SV1), the Hodge structure on $\mathfrak{g}_{\alpha,\mathbf{C}}$ induced by $\mathrm{ad}h_\alpha$ has type $\{(-1,1),(0,0),(1,-1)\}$. If X belongs to the $(-1,1)$-graded piece, then $-\frac{z}{\bar{z}}X = \frac{iz}{-i\bar{z}} = \frac{iz}{\bar{iz}}X = \mathrm{ad}h_\alpha(iz)X = \mathrm{ad}h_\alpha(i)\mathrm{ad}h_\alpha(z)X = \mathrm{ad}h_\alpha(z)X = \frac{z}{\bar{z}}X$ and hence $X = 0$; likewise for the $(1,-1)$-graded piece, and so $\mathrm{ad}h_\alpha$ is trivial.

It follows that if $G^{\mathrm{ad}}_\alpha(\mathbf{R})$ is compact, then conjugation by h_α is the trivial map $\mathbf{S} \to \mathrm{Aut}(G^{\mathrm{ad}}_{\alpha,\mathbf{R}})$. Since G^{ad}_α is centerless, saying that conjugation by h_α is the trivial automorphism of $G^{\mathrm{ad}}_{\alpha,\mathbf{R}}$ is the same as saying that h_α itself is the trivial map to $G^{\mathrm{ad}}_{\alpha,\mathbf{R}}$.

Thus, the existence of a compact simple factor of G would contradict (SV3). □

Sketch of proof of the proposition. I think Brandon basically explained this, but I'll recall the idea.

We'll work with the equivalent formulation of connected Shimura data in terms of a $G^{\mathrm{ad}}(\mathbf{R})^+$-conjugacy class D of homomorphism $u : S^1 \to G^{\mathrm{ad}}(\mathbf{R})$. Decompose $G^{\mathrm{ad}}_{\mathbf{R}}$ (*not* $G^{\mathrm{ad}}_{\mathbf{Q}}$!) as a direct product $\mathbf{R}$-simple factors G^{ad}_i. (I believe this will literally be a direct product, not an almost direct product, since we're in the adjoint case.) Write u_i for the projection of u onto $G^{\mathrm{ad}}_i(\mathbf{R})$. Then the $G^{\mathrm{ad}}(\mathbf{R})^+$ conjugacy class of u is just the product of the $G^{\mathrm{ad}}_i(\mathbf{R})^+$-conjugacy classes D_i of the u_is. You can check that that the u_is satisfy (the u-variant forms of) the axioms (SV1) and (SV2). But it is entirely possible that some u_i is trivial; this stems from the distinction between (the $\mathbf{R}$-points of) the $\mathbf{Q}$-simple factors of G and the $\mathbf{R}$-simple factors of $G_{\mathbf{R}}$, and will be illustrated below with an example. If this occurs, then as in the proof of Lemma 2.1.3, the corresponding $G_i(\mathbf{R})$ is compact. This might seem to be a problem, since we would worry that D_i is then some Hermitian symmetric space of compact type, and not a Hermitian symmetric domain, and indeed this is true! But since D_i is the $G^{\mathrm{ad}}_i(\mathbf{R})^+$ conjugacy class of $u_i = 1$, all is well: the symmetric space is just a point, and we are free to ignore it. So $D = \prod D_i$ is a product of (points and) $G^{\mathrm{ad}}_i(\mathbf{R})^+$-conjugacy classes of homomorphisms u_i satisfying (SV1) and (SV2) with $G_i(\mathbf{R})^+$ noncompact. This reduces the problem to showing that each D_i is a Hermitian symmetric space such that the map $G^{\mathrm{ad}}_i(\mathbf{R})^+ \to \mathrm{Hol}(D_i)^+$ is an isomorphism, since such a space (having noncompact isometry group) is necessary a Hermitian symmetric domain.[2] Having made this reduction, we fix some noncompact $\mathbf{R}$-factor G_i of $G_{\mathbf{R}}$ and rename it as G, and rename u_i as u, which as mentioned above satisfies (SV1,2).

Finally we come to what Brandon actually said something about. Since $u(-1)$ is a Cartan involution (axiom (SV2)), the centralizer K of u in $G^{\mathrm{ad}}(\mathbf{R})$ is the identity component of a maximal compact subgroup. Identifying $D = G^{\mathrm{ad}}(\mathbf{R})^+/K$ gives a real-analytic manifold structure on D. The axiom (SV1) endows $T_uD = \mathfrak{g}/\mathfrak{k}$ with a $\mathbf{C}$-structure where multiplication by i is given by conjugation by $u(e^{2\pi i/8})$, and thus (by homogeneity) endows D with an almost-complex structure which turns out (!) to be integrable. Since K_u is compact and acts on T_uD, by averaging one can make a K_u-invariant positive definition Hermitian form. Translating this gives D the structure of a Hermitian symmetric space for which $G(\mathbf{R})^+ = \mathrm{Hol}(X^+)^+$. □

2.1.4. **Remark.** In general it will *not* be the case the $G^{\mathrm{ad}}(\mathbf{R}) = \mathrm{Hol}(D)^+$. Here is an example.

Fix a quaternion algebra B over a totally real field F. Take $G = G^{\mathrm{ad}}$ to be the group

$$G = \underline{B}^\times/\underline{F}^\times.$$

Then

$$G(\mathbf{R})\text{"} = \text{"}(B \otimes_{\mathbf{Q}} \mathbf{R})^\times/(F \otimes_{\mathbf{Q}} \mathbf{R})^\times = \prod_{v:F\hookrightarrow\mathbf{R}} (B \otimes_{F,v} \mathbf{R})^\times/\mathbf{R}^\times$$
$$= (\mathbf{H}^\times/\mathbf{R}^\times) \times \cdots \times (\mathbf{H}^\times/\mathbf{R}^\times) \times \mathrm{PGL}_2(\mathbf{R}) \times \cdots \times \mathrm{PGL}_2(\mathbf{R}).$$

Note that each $\mathbf{H}^\times/\mathbf{R}^\times$ is a copy of $S^3/\pm 1 = \mathrm{SO}(3) = \mathrm{PU}(2)$, which is compact. However, as long as B splits at least one archimidean place of F, $G(\mathbf{R})$ is noncompact. But G is $\mathbf{Q}$-simple (proof?) so if we can construct a connected Shimura datum for G, it will satisfy SV3. This we can clearly do, by taking D to be a product of copies of $\mathfrak{h}$, one for each split archimedean place of B.

[2] For if we have proved this, then it is clearly the case that $G^{\mathrm{ad}}(\mathbf{R})^+ \twoheadrightarrow \prod \mathrm{Hol}(D_i)^+$ has compact kernel equal to $\prod_{i:u_i=1} G^{\mathrm{ad}}_i(\mathbf{R})^+$.

2.2. **Application of Baily–Borel.** The next thing I want to mention is how to apply the Baily–Borel theorem to deduce the algebraicity of $\Gamma\backslash X^+$ for (G, X^+) a connected Shimura datum and $\Gamma \subset G(\mathbf{Q})$ a suitable arithmetic subgroup. The Γs we care about are congruence subgroups of the form $K \cap G(\mathbf{Q})$ for a sufficiently small compact open subgroup $K \subset G(\mathbf{A}_f)$; more about these later. Recall the Baily–Borel theorem as Mike discussed it:

2.2.1. **Theorem.** Let $\mathcal{G}$ be a semisimple $\mathbf{Q}$-group and $\mathcal{K} \subset \mathcal{G}(\mathbf{R})$ a maximal compact subgroup. Let $X = \mathcal{G}(\mathbf{R})/\mathcal{K}$ be the associated symmetric space, and assume it is a Hermitian symmetric domain. Let $\Gamma \subset \mathcal{G}(\mathbf{Q})$ be a (torsion-free) arithmetic subgroup of finite covolume, so in particular $\Gamma\backslash X$ is a complex manifold locally isomorphic to X. Then $\Gamma\backslash X$ has a *unique* structure of a quasiprojective complex algebraic variety. □

Recall our connected Shimura datum (G, X^+). Decomposing $G_{\mathbf{R}} = \prod G_i, u = \prod u_i$ as in the proof of the proposition above, we have written

$$X^+ = \prod_{u_i=1} G_i^{\mathrm{ad}}(\mathbf{R})^+/Z_{G_i^{\mathrm{ad}}(\mathbf{R})^+}(u_i) = \prod_{u_i=1} G_i^{\mathrm{ad}}(\mathbf{R})/\mathcal{K}_i = \prod_{\text{all } i} G_i^{\mathrm{ad}}(\mathbf{R})/\mathcal{K}_i = G^{\mathrm{ad}}(\mathbf{R})/\prod \mathcal{K}_i,$$

where each $\mathcal{K}_i$ is a maximal compact subgroup of $G_i^{\mathrm{ad}}(\mathbf{R})$ (possibly equal to the whole factor!).

So we may apply the Baily–Borel theorem to the group $\mathcal{G} = G^{\mathrm{ad}}$ and the maximal compact subgroup $\mathcal{K} = \prod \mathcal{K}_i$.

Let $\Gamma \subset G(\mathbf{Q})$ be a torsion-free congruence subgroup of $G(\mathbf{Q})$ arising as $G(\mathbf{Q}) \cap K$ like above. Let $\overline{\Gamma}$ be its image in $G^{\mathrm{ad}}(\mathbf{Q})$. Baily–Borel says that $\overline{\Gamma}\backslash X^+$ is algebraic (uniquely!) provided that $\overline{\Gamma}$ is arithmetic. It is not so hard to see that the image of an arithmetic group under an isogeny is arithmetic. The center of $G(\mathbf{Q})$ is finite, so in fact $\Gamma = \overline{\Gamma}$ is torsion-free.

The reason I am mentioning these facts is to point out that when we have an inclusion $\Gamma \hookrightarrow \Gamma'$, the natural projection $\Gamma\backslash X^+ \to \Gamma'\backslash X^+$ is actually algebraic. The follows from the uniqueness statement in Baily–Borel. The map is finite (as Γ has finite index in Γ' by arithmeticity), so the algebraic structure on $\Gamma'\backslash X^+$ induces an algebraic structure on $\Gamma\backslash X^+$ making the (*a priori* merely analytic) map in question algebraic. As the algebraic structure on the source is *unique*, we see that as $\Gamma = K \cap G(\mathbf{Q})$ varies, in fact the locally symmetric spaces $\mathrm{Sh}_K^0(G, X^+)$ form an inverse system of complex algebraic varieties.

3. Congruence subgroups and adelic description of connected Shimura varieties

3.1. **Congruence subgroups.** Let $G_{/\mathbf{Q}}$ be reductive. Fix a faithful representation $G \to \mathrm{GL}_n$. A **congruence subgroup** of $G(\mathbf{Q})$ is a subgroup Γ containing a finite-index subgroup of the form $\Gamma(N) := G(\mathbf{Q}) \cap \ker(\mathrm{GL}_n(\mathbf{Z}) \to \mathrm{GL}_n(\mathbf{Z}/N\mathbf{Z})$ for some N.

3.1.1. **Proposition.** This notion of a congruence subgroup is independent of the choice of faithful representation; i.e. it is intrinsic to the $\mathbf{Q}$-group G. The intersection $K \cap G(\mathbf{Q})$ for a compact open subgroup $K \subset G(\mathbf{A}_f)$ is a congruence subgroup Γ_K. Moreover, at least when G is semisimple, simply connected, and of noncompact type,[3] every congruence subgroup arises in this way.

Here $\mathbf{A}_f$ denotes the ring of finite adeles.

The adelic description in this proposition could also be adopted as a suitable definition, but it's worthwhile to make contact with the traditional definition in terms of congruence conditions on matrix entries. In the sequel, we'll work with the adelic description, which by the proposition is more general.

3.1.2. **Remark.** Recall how to topologize the adelic points of an affine variety $X_{/\mathbf{Q}}$. Choose a closed embedding $X \hookrightarrow \mathbf{Aff}^N$ into affine space, give the $\mathbf{A}_f$-points $\mathbf{A}_f^N$ of affine space the product topology induced by the usual restricted product topology on the adeles, and then give $X(\mathbf{A}_f)$ the subspace topology. As an exercise, you can check this is independent of the chosen affine embedding. One neat way to do this is explained in an expository note by Brian [2, Prop. 2.1]; he uses a "universal" affine embedding. Alternatively, you can argue by hand: a choice of affine embedding determines a compact subset "$X(\mathbf{Z}_p)$" $:= X(\mathbf{Q}_p) \cap \mathbf{Z}_p^N \subset X(\mathbf{Q}_p) \subset \mathbf{Q}_p^N$ and an identification of topological spaces $X(\mathbf{A}_f) = \prod' X(\mathbf{Q}_p)$, where the restricted product is with respect to the subsets "$X(\mathbf{Z}_p)$". Granting (or checking) that the topology on $X(\mathbf{Q}_p)$ induced from $\mathbf{Q}_p^N$ is actually independent of the embedding, it follows that the adelic topology is also independent of embedding. This is because the polynomials relating the coordinates for two different

[3] FIXME: maybe also in general? Regardless, this is the case we actually care about!

affine embeddings of X have coefficients with only finitely many primes occuring in the denominators. This means that the subsets "$X(\mathbf{Z}_p)$" determined by the two embeddings will agree for almost all primes p, and thus the correspoding restricted direct products are homeomorphic.

Proof of Proposition 3.1.1. First we tackle independence of the choice of representation. This is in fact a consequence of the previous remark, i.e. a consideration of the denominators of polynomials relating the matrix entries of two different faithful representations of G. Concretely, view G as a subgroup of GL_n via our original choice of representation, and consider another faithful representation $\mathbf{Aff}^{n^2+1} \supset G \stackrel{\rho}{\hookrightarrow} \mathrm{GL}_m \hookrightarrow \mathbf{Aff}^{m^2+1}$. This yields m^2+1 polynomials $\{\delta\} \cup \{f_{ij}\}_{1\le i,j\le m} \subset \mathbf{Q}[x_{ab}, \Delta]_{1\le a,b\le n}$ in n^2+1 indeterminates, expressing the $m \times m$ matrix entries of $\rho(g)$ and $\det(\rho(g^{-1}))$ in terms of the $n \times n$ matrix entries of g and $\det g^{-1}$. By an extremely simple linear change of coordinates on $\mathbf{Aff}^{n^2+1}$ and $\mathbf{Aff}^{m^2+1}$, we can replace f_{ij}, δ by other polynomials which express $[\rho(g)-1]_{ij}$ and $\det(\rho(g^{-1}))-1$ in terms of $[g-1]_{ab}$ and $\det g^{-1}-1$. Let $\{c_\alpha\} \subset \mathbf{Q}^\times$ be the set of nonzero coefficients of the f_{ij} and δ, and let N be the product of their denominators. If $g \in \Gamma(N)$ then each $[g-1]_{ab}$ is an integer divisible by N and $\det g^{-1}-1 = 0$. So each $f_{ij}([g-1]_{ab}, \det g^{-1}-1)$ is actually an integer, i.e. $\rho(g) \in \mathrm{Mat}_m(\mathbf{Z})$. On the other hand, $\delta([g-1]_{ab}, \det g^{-1}-1)$ is also an integer, so $\det \rho(g)^{-1} \in \mathbf{Z}$. By the same argument applied to g^{-1}, we also get $\det\rho(g) \in \mathbf{Z}$. So $\det\rho(g) \in \mathbf{Z}^\times$, and thus $\rho(g) \in \mathrm{GL}_m(\mathbf{Z})$. This show $\rho(\Gamma(N)) \subset \rho(G(\mathbf{Q})) \cap \mathrm{GL}_m(\mathbf{Z})$, where the righthand side is $\Gamma_\rho(1)$, the "standard" arithmetic subgroup of $G(\mathbf{Q})$ defined with respect to the representation ρ. Precisely the same reasoning shows $\rho(\Gamma(NM)) \subset \Gamma_\rho(M)$ for any $M \ge 1$. Now just a little thought should convince you that although the notions of principal congruences subgroups may differ between ρ and our original representation, a subgroup of $G(\mathbf{Q})$ *contains* a principal congruence subgroup in one sense if and only if it contains such a subgroup in the other sense. Thus "congruence subgroup" is well-defined.[4,5]

We now turn to the adelic description of congruence subgroups. The principal congruence subgroup $\Gamma(N) = G(\mathbf{Q}) \cap \ker(\mathrm{GL}_n(\mathbf{Z}) \to \mathrm{GL}_n(\mathbf{Z}/N\mathbf{Z}))$ arises as $G(\mathbf{Q}) \cap K(N)$ for a compact open subgroup $K(N) \subset G(\mathbf{A}_f)$ defined by $K(N) = G(\mathbf{A}_f) \cap \ker(\mathrm{GL}_n(\widehat{\mathbf{Z}}) \to \mathrm{GL}_n(\mathbf{Z}/N\mathbf{Z}))$; concretely $K(N) = \prod K_p$ with

$$K_p = \begin{cases} G(\mathbf{Q}_p) \cap \mathrm{GL}_n(\mathbf{Z}_p) \subset \mathrm{GL}_n(\mathbf{Q}_p), & p \nmid N \\ G(\mathbf{Q}_p) \cap \ker(\mathrm{GL}_n(\mathbf{Z}_p) \to \mathrm{GL}_n(\mathbf{Z}/p^r\mathbf{Z})), & p^r || N. \end{cases}$$

Use $(id, 1/\det)$ to embed $G \to \mathrm{GL}_n \to \mathrm{Mat}_n \times \mathbf{Aff}^1 \approx \mathbf{Aff}^{n^2+1}$. With respect to this affine embedding, we see that $G(\mathbf{A}_f)$ is the direct product of the $G(\mathbf{Q}_p)$s restricted with respect to the open subgroups "$G(\mathbf{Z}_p)$" $= G(\mathbf{Q}_p) \cap \mathrm{GL}_n(\mathbf{Z}_p)$, since $\mathrm{GL}_n(\mathbf{Z}_p) \subset \mathrm{GL}_n(\mathbf{Q}_p)$ is precisely the set of those matrices whose "coordinates" (under $(id, 1/\det)$) in $\mathbf{Q}_p^{n^2+1}$ are all integral. In particular, since $K_p =$ "$G(\mathbf{Z}_p)$" for almost all p, it follows that $K(N)$ is open. Since K_p is compact for all p, $K(N)$ is compact. Finally, $K(N) \cap G(\mathbf{Q})$ is clearly equal to $\Gamma(N)$. So the principal congruence subgroup arises from $K(N)$ in the expected way.

Any compact open subgroup $K \subset G(\mathbf{A}_f)$ contains some $K(N)$ (with finite index), so $K \cap G(\mathbf{Q})$ contains $\Gamma(N)$ (with finite index), and is thus a congruence subgroup.

Conversely, a general congruence subgroup Γ contains some $\Gamma(N)$ with finite index. If $g_1, \dots, g_r$ are coset-representatives, then $K = \bigcup g_i K(N)$ is a compact open subgroup of $G(\mathbf{A}_f)$ such that $K \cap G(\mathbf{Q}) = \Gamma$, as desired. The only thing that needs checking is that K is actually a subgroup. In fact, K is the closure of $\Gamma = \bigcup g_i \Gamma(N)$. To see this, we use the following

3.1.3. **Theorem** (Strong Approximation Theorem). For connected semisimple, simply connected G of non-compact type, $G(\mathbf{Q})$ is dense in $G(\mathbf{A}_f)$. □

So $\Gamma(N) = G(\mathbf{Q}) \cap K(N)$ is dense in $K(N)$. Thus $g_i K(N) = g_i\overline{\Gamma} = \overline{g_i\Gamma}$, and hence $K = \overline{\Gamma}$ is a group. □

We should say a word about Strong Approximation, since it will come up again later. The theorem can be attributed (I think?) in its greatest generlity to Platonov (number field case; Prasad did function fields). In any event, there's a good explanation in [6, Ch. 7]. The most interesting hypothesis seems to be the one about "noncompact type", since as we've seen this is tied up with axiom SV3 and hence the negative

[4] The same argument shows that S-congruence subgroups and S-arithmetic subgroups (those commensurable with $\Gamma(1)$) are defined independent of the choice of faithful representation.

[5] The should be a more conceptual proof of this result using the interpretation of a choice of faithful representation as giving rise to a flat $\mathbf{Z}$-model $\mathcal{G}$ of G by taking the Zariski closure of G in $\mathrm{GL}_{n/\mathbf{Z}}$. But I couldn't figure it out.

curvature of the associated Hermitian symmetric space. It seems (to me) to be tricky to distill into just a few words exactly how this hypothesis is used in the proof.

3.2. **Adelic description of $\Gamma\backslash X$.** We now assume (G, X^+) is a connected Shimura datum. We also impose the *additional* hypothesis that G is simply connected.

By Lemma 2.1.3, the Strong Approximation Theorem applies to G. This lets us give an adelic description to the connected Shimura variety $\mathrm{Sh}^0(G, X^+)$ as follows. Recall that we have the arithmetic quotient

$$\mathrm{Sh}^0_K(G, X^+) := \Gamma\backslash X^+, \qquad \Gamma = \Gamma_K = K \cap G(\mathbf{Q}),$$

and that we are interest in the inverse limit of these quotients as K ranges through compact open subgroups $K \subset G(\mathbf{A}_f)$.[6] We first describe this for fixed K.

Consider the product $X^+ \times G(\mathbf{A}_f)$, where the second factor has its adelic topology. There is an evident map from X^+ to this product given by $x \mapsto (x, 1)$. There are alo some group actions:

- $G(\mathbf{Q})$ has a left action on $X^+ \times G(\mathbf{A}_f)$. It has an action upon X^+ via the $G(\mathbf{R}) = G(\mathbf{R})^+$-action on the Hermitian symmetric domain X^+. *Note that we are using Cartan's theorem, quoted in Brian's lectures, to the effect that the real Lie group attached to a simply connected semisimple algebraic* $\mathbf{R}$*-group is* ***connected!***[7] It has an action on $G(\mathbf{A}_f)$ via the diagonal embedding $G(\mathbf{Q}) \hookrightarrow G(\mathbf{A}_f)$. Combining these gives the desired left action on the product.
- A compact open subgroup K of the finite-adelic points of G acts upon $X^+ \times G(\mathbf{A}_f)$ on the right, by acting trivially on the first factor, and by right-translation on the second factor.

3.2.1. **Proposition.** The map $x \mapsto (x, 1)$ induces a homeomorphism $\mathrm{Sh}^0_K(G, X^+) = G(\mathbf{Q})\backslash(X^+ \times G(\mathbf{A}_f))/K$.

Proof. Using that $\Gamma_K = K \cap G(\mathbf{Q})$, it is an elementary exercise to check that the map in question is set-theoretically injective. For surjectivity, we must appeal to Strong Approximation. Since $G(\mathbf{Q})$ is dense in $G(\mathbf{A}_f)$, and K is open, we have $G(\mathbf{A}_f) = G(\mathbf{Q})K$; that is, any $g \in G(\mathbf{A}_f)$ can be written $g = \gamma k, \gamma \in G(\mathbf{Q}), k \in K$. So a typical element of the double-coset space that is the target of our map, namely the double coset $[x, g]$ of (x, g) is the same as $[x, \gamma k] = [x, \gamma] = [\gamma^{-1}x, 1]$, and thus gets hit by the map.

Having proved the map is bijective we must address its topological properties. Since K is open, $G(\mathbf{A}_f)/K$ is discrete. So the map $X^+ \to (X^+ \times G(\mathbf{A}_f))/K$ induced by $x \mapsto (x, 1)$ is the inclusion of a connected componented, and in particular a homeomorphism onto its image. Especially, it is a continuous, open map. Since the map we care about is induced from this one by the universal property of the topological quotient $\Gamma\backslash X^+$, our map is continuous. The quotient map $(X^+ \times G(\mathbf{A}_f))/K \to G(\mathbf{Q})\backslash(X^+ \times G(\mathbf{A}_f))/K$ is also open, as is the case for the quotient map coming from any continuous group action.[8] It follows formally that the map we care about is also open:

$$\begin{array}{ccc} X^+ & \longrightarrow & (X^+ \times G(\mathbf{A}_f))/K \\ \downarrow & & \downarrow \\ \Gamma\backslash X^+ & \longrightarrow & G(\mathbf{Q})\backslash(X^+ \times G(\mathbf{A}_f))/K \end{array}$$

□

3.3. **Passage to the inverse limit.** As mentioned above, it's convenient to think about the whole inverse system of arithmetic quotients $\mathrm{Sh}^0_K(G, X^+)$ as K varies. We saw earlier that these actually form an inverse system of complex algebraic varieties, so it is a good thing to consider.

Via some annoying (and not particularly enlightening) point-set-topological considerations which we omit (see [5, 4.19]), one deduces from the above description the following

3.3.1. **Proposition.** Let (G, X^+) be a connected Shimura datum with G simply connected. Then the connected Shimura variety has the following adelic description:

$$\mathrm{Sh}^0(G, X^+) = \varprojlim_K \mathrm{Sh}^0_K(G, X^+) \cong G(\mathbf{Q})\backslash(X^+ \times G(\mathbf{A}_f)).$$

---

[6] Although we clearly have an inverse system of *topological spaces* $\mathrm{Sh}^0_K(G, X^+)$s as K varies, it is not tautological that we have an inverse system of complex algebraic varieties. This subtlety will be discussed below.

[7] This theorem is proved in [6, Ch. 7].

[8] Proof: the saturation of an open subset is nothing but the union of its translates under the group action, and is thus open.

4. General Shimura varieties

We now turn to general Shimura data.

4.1. **Complex (and algebraic) structure.** The first thing we want to do is to show that if (G, X) is a Shimura datum, then X is actually something reasonable: namely, a finite disjoint union of Hermitian symmetric domains. This is an elaboration of the case of $G = \mathrm{GSp}_{2n}$ carried out in §3 of Brian's notes:

4.1.1. **Theorem** (Cor. 5.8 of [5])**.** Let (G, X) be a Shimura datum and X^+ a connected component of X. There is a natural structure of connected Shimura datum on the pair $(\mathcal{D}G, X^+)$. The set X is a finite union of Hermitian symmetric domains.

Proof. Recall that X is the $G(\mathbf{R})$-conjugacy class of a homomorphism $h : \mathbf{S} \to G_{\mathbf{R}}$ satisfying (SV1-3). We need to relate this to the $G^{\mathrm{ad}}(\mathbf{R})^+$-conjugacy class of some $h' : \mathbf{S} \to G^{\mathrm{ad}}_{\mathbf{R}}$. Obviously we can take $h' = \mathrm{Ad} \circ h$ where $\mathrm{Ad} : G \to G^{\mathrm{ad}} = G/Z_G$ is the projection. Letting X' be the $G^{\mathrm{ad}}(\mathbf{R})$-conjugacy of h', we have an evident map $X \to X'$ given by composing with Ad. (Note that the target is generally disconnected, since we haven't just taken the $G^{\mathrm{ad}}(\mathbf{R})^+$-conjugacy class.)

Claim: $X \hookrightarrow X'$.

Proof: We have the usual exact sequences:

$$1 \to \mathcal{D}G \to G \to T \to 1$$
$$1 \to Z \to G \overset{\mathrm{Ad}}{\to} G^{\mathrm{ad}} \to 1$$
$$1 \to Z \cap \mathcal{D}G \to Z \to T \to 1$$

If $h_1, h_2 \in X$ and their images are both equal to some $h' \in X'$, then (after possibly replacing h_2 with some $G(\mathbf{R})$-conjugate) $h_1 h_2^{-1}$ is valued in Z. Since h_1 and h_2 are both conjugates of h, and T is commutative, the projections of h_1 and h_2 to T agree. So $h_1 h_2^{-1}$ is valued in $Z \cap \mathcal{D}G$, i.e. the finite center of the derived group $\mathcal{D}G$. As $\mathbf{S}$ is connected and we are in characteristic 0 (so finite groups are étale), it follows that $h_1 = h_2$, proving the claim.

The image of X in X' will be a union of connected components: each component of X' is a $G^{\mathrm{ad}}(\mathbf{R})^+$-orbit and $G(\mathbf{R}) \twoheadrightarrow G^{\mathrm{ad}}(\mathbf{R})$ is surjective, hence the same is true after adding $+$s, so if any point of X' is hit, all of its connected component is hit. It's not hard to see that $X \to X'$ is continuous. So we can really identify X with some of the components of X'.

We now take our connected component X^+ of X and view it as a connected component of X', that is, a $G^{\mathrm{ad}}(\mathbf{R})^+$-conjugacy class of map $h' : \mathbf{S} \to G^{\mathrm{ad}}_{\mathbf{R}}$. We should check that the axioms SV1-3 descend from h to h'. But these axioms only make reference to the adjoint group, so this is essentially tautological. □

4.1.2. **Example.** Let $G = \mathrm{GL}_2$ and let $X = \mathfrak{h}^+ \cup \mathfrak{h}^- = \mathbf{C} \setminus \mathbf{R}$. We identify X with the $\mathrm{GL}_2(\mathbf{R})$-conjugacy class of $h : \mathbf{S} \to \mathrm{GL}_2$ given by $h(a + ib) = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$. The identification matches ${}^g h$ to $g \cdot i$ (action via fractional linear transformations). This is a Shimura datum.

Let $X^+ = \mathfrak{h}^+$. Then (SL_2, X^+) is a connected Shimura datum.

4.1.3. **Example.** The following was done in Example 3.4 of Brian's notes: Let $G = \mathrm{GSp}(\Psi)$ for a symplectic space (V, Ψ) over $\mathbf{Q}$. Let $X = X^+ \cup X^-$ where $X^{\pm}$ is the set of $\mathbf{C}$-structures J on $V_{\mathbf{R}}$ (equivalently, Hodge structures of type $(-1, 0), (0, -1)$ on $V_{\mathbf{R}}$) such that $\Psi(u, Jv)$ is $\pm$-definite. Then X^+ can be identified with the Siegel upper half space. One can verify that (G, X) is a Shimura datum and $(\mathrm{Sp}(\Psi), X^+)$ is a connected Shimura datum.

Maybe Jeremy and/or Iurie will check some or all of the axioms in a subsequent talk.

4.2. **Adelic description of locally symmetric quotients.** Above we described *connected* Shimura varieties, defined as inverse systems of arithmetic locally symmetric quotients arising from a connected Shimura datum, in terms of adelic double coset spaces.

Inspired by this description we *define* the Shimura variety attached to a Shimura datum as a similar adelic double coset space.

4.2.1. **Definition.** Let (G, X) be a Shimura datum, $K \subset G(\mathbf{A}_f)$ a compact open subgroup. Set

$$\mathrm{Sh}_K(G, X) := G(\mathbf{Q}) \backslash (X \times G(\mathbf{A}_f))/K.$$

Now we no longer need to apply a simple-connectedness hypothesis (as we did in the connected case, where it was necessary to know – via Cartan's theorem – that $G(\mathbf{R})$ was connected) in order to get a $G(\mathbf{Q})$-action on X.

4.2.2. **Remark.** Note that if K_∞ denotes the centralizer of some fixed $h \in X$ inside $G(\mathbf{R})$, then obviously $X = G(\mathbf{R})/K_\infty$. Writing $K^\infty \subset G(\mathbf{A}_f)$ for our compact open subgroup, we see that $\mathrm{Sh}_{K^\infty}(G,X)$ has the alternate description

$$G(\mathbf{Q})\backslash G(\mathbf{A})/(K^\infty K_\infty).$$

To actually work with this thing I think it's more reasonable to keep $G(\mathbf{A})/K_\infty = X \times G(\mathbf{A}_f)$ decomposed as a product.

4.2.3. **Remark.** Note that $G(\mathbf{A}_f)$ acts on $\varprojlim_K \mathrm{Sh}_K(G,X)$ (on the right, via the maps

$$[x,a] \mapsto [x,ag]$$

$$G(\mathbf{Q})\backslash(X \times G(\mathbf{A}_f))/K = \mathrm{Sh}_K(G,X) \to \mathrm{Sh}_{g^{-1}Kg}(X,G) = G(\mathbf{Q})\backslash(X \times G(\mathbf{A}_f))/g^{-1}Kg.$$

4.2.4. We would like to know that the double coset space $\mathrm{Sh}_K(G,X)$ is nothing but a finite disjoint union of connected Shimura varieties. To state the result we require a bit of notation.

4.2.5. **Notation**. Write $G(\mathbf{R})_+ = \mathrm{Ad}^{-1}(G^{\mathrm{ad}}(\mathbf{R})^+)$ and $G(\mathbf{Q})_+ = G(\mathbf{Q}) \cap G(\mathbf{R})_+$ Let $\mathcal{C}$ be a set of coset representatives for $G(\mathbf{Q})_+\backslash G(\mathbf{A}_f)/K$. For $c \in \mathcal{C}$ let $K_c = cKc^{-1}$ and $\Gamma_c = K_c \cap G(\mathbf{Q})_+$.

The result we are aiming for is the following.

4.2.6. **Proposition.** The set $\mathcal{C}$ is finite and for any connected component X^+ of X there is a natural homeomorphism

$$\mathrm{Sh}_K(G,X) \cong \bigsqcup_{c \in \mathcal{C}} \Gamma_c\backslash X^+.$$

4.2.7. **Example.** Let $G = \mathrm{GL}_2, X = \mathfrak{h}^+ \cup \mathfrak{h}^-$, and K a finite index subgroup in $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ (e.g. $K = K_0(N), K_1(N), \ldots$). Then $G(\mathbf{R})_+ = \mathrm{GL}_2(\mathbf{R})^+, G(\mathbf{Q})_+ = \mathrm{GL}_2(\mathbf{Q})^+$. The determinant induces a bijection

$$G(\mathbf{Q})_+\backslash G(\mathbf{A}_f)/K \approx \widehat{\mathbf{Z}}^\times/\det K.$$

For example, if $K = K(N)$ is the principal congruence subgroup of level N, so that $\mathrm{Sh}_K(G,X)$ is the modular curve $Y(N)$ parametrizing elliptic curves with full level-N structure, then the connected components are in bijection with $(\mathbf{Z}/N\mathbf{Z})^\times$. Since K is normal in this case, all Γ_c's are equal to the usual $\Gamma(N) \subset \mathrm{SL}_2(\mathbf{Z})$, and we see that $Y(N) = \bigsqcup_{(\mathbf{Z}/N)^\times} \Gamma(N)\backslash\mathfrak{h}^+$. If $K = K_1(N)$ or $K_0(N)$ then the corresponding Shimura variety is the connected modular curve $Y_1(N)$ or $Y_0(N)$.

We will begin by showing the following:

4.2.8. **Lemma.** $G(\mathbf{Q})_+\backslash G(\mathbf{A}_f)/K$ is finite.

It wouldn't be too much a stretch to just take this on faith.

In the earlier example, the finiteness of $\pi_0(\mathrm{Sh}_K(\mathrm{GL}_2, \mathfrak{h}^\pm)$ came from the finiteness of $\widehat{\mathbf{Z}}/\det K$. This indicates the strategy of proof: lift the result from the quotient $T = G/\mathcal{D}G$.

However, I think it will be enlightening to see a proof, because it turns out require some fairly hard facts about the arithmetic of algebraic groups. What follows is a simplified (!) version of the argument in Milne. Milne's argument results in a precise description of the connected components, but is a little bit more annoying.

Proof. We have map $G(\mathbf{Q})_+\backslash G(\mathbf{Q}) \to G^{\mathrm{ad}}(\mathbf{R})^+\backslash G^{\mathrm{ad}}(\mathbf{R})$ with finite kernel, and the target is $\pi_0(G^{\mathrm{ad}}(\mathbf{R}))$, which is finite by Cartan's theorem. So the source is finite, which means we can simply show $G(\mathbf{Q})\backslash G(\mathbf{A}_f)/K$ is finite.

Let $\widetilde{G} \to G$ be an isogeny (with central kernel) where $\widetilde{G}$ has simply connected derived group. Let $\widetilde{K}$ be the compact open preimage of K. The map $\widetilde{G}(\mathbf{Q}) \to G(\mathbf{Q})$ need not be surjective, but its cokernel is contained in a finite Galois cohomology set. Thus the problem for G is reduced to the problem for $\widetilde{G}$ and we may therefore assume $\mathcal{D}G$ is simply connected.

To do this, recall the exact sequences

$$1 \to \mathcal{D}G \to G \xrightarrow{\nu} T \to 1$$
$$1 \to Z \to G \to G^{\mathrm{ad}} \to 1$$

We will study the map

$$G(\mathbf{Q})\backslash G(\mathbf{A}_f)/K \to \nu(G(\mathbf{Q}))\backslash T(\mathbf{A}_f)/\nu(K).$$

Claim 1: $G(\mathbf{A}_f) \to T(\mathbf{A}_f)$ is surjective and sends compact open subgroups to compact open subgroups.

Sketch of Proof: By the **Kneser–Bruhat–Tits theorem**, $H^1(\mathbf{Q}_p, \mathcal{D}G) = *$.[9] This uses the simple connectness of the derived group. It follows that $G(\mathbf{Q}_p) \twoheadrightarrow T(\mathbf{Q}_p)$ for all p. So it suffices to show that "$G(\mathbf{Z}_p)$" $\twoheadrightarrow$ "$T(\mathbf{Z}_p)$" for almost all p, where these groups are integral points of suitable flat integral models $\mathcal{G}$ and $\mathcal{T}$, the choices of which affect only finitely many primes. Away from finitely many primes, ν extends to a map $\mathcal{G} \to \mathcal{T}$. The kernel $\mathcal{D}G$ of the map on the generic fiber is smooth and connected, so after inverting a few more primes we get a short exact sequence

$$1 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{T} \to 1$$

of $\mathbf{Z}[1/N]$-group schemes. For any prime $p \nmid N$, we can consider the map on $\mathbf{F}_p$-points. The map $\mathcal{G}(\mathbf{F}_p) \to \mathcal{T}(\mathbf{F}_p)$ is surjective by Lang's theorem (vanishing of H^1 for connected algebraic groups over a finite field). This shows that the fiber $Y_P = \nu^{-1}(P)$ over a given $\mathbf{Z}_p$-point of $\mathcal{T}$ has an $\mathbf{F}_p$-point. Since Y_P is smooth, by Hensel's lemma this will lift to a $\mathbf{Z}_p$-point of $\mathbf{G}$ mapping to P.

To prove the claim about the image of a compact open subgroup, note that the property "$G(\mathbf{Z}_p)$" $\twoheadrightarrow$ "$T(\mathbf{Z}_p)$" for almost all p reduces this claim to showing that $G(\mathbf{Q}_p) \to T(\mathbf{Q}_p)$ sends compact open subgroups to compact open subgroups for all p. This is a not-very-hard exercise.

Claim 2: $\nu(G(\mathbf{Q}))$ has finite index in $T(\mathbf{Q})$.

Proof of Claim 2: Let $T^\dagger = T(\mathbf{Q}) \cap \mathrm{im}(\nu : Z(\mathbf{R}) \to T(\mathbf{R}))$. The map $\nu : Z \to T$ is an isogeny, so the map on $\mathbf{R}$-points is closed (because finite) and a submersion over its image, hence open. Thus it induces a surjection $Z(\mathbf{R})^+ \to T(\mathbf{R})^+$. So $T^\dagger$ contains $T(\mathbf{Q})^+$, which has finite index in $T(\mathbf{Q})$. I claim that $T^\dagger \subset \nu(G(\mathbf{Q}))$; this would prove Claim 2. To see this we consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{D}G(\mathbf{Q}) & \longrightarrow & G(\mathbf{Q}) & \longrightarrow & T(\mathbf{Q}) & \longrightarrow & H^1(\mathbf{Q}, \mathcal{D}G) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathcal{D}G(\mathbf{R}) & \longrightarrow & G(\mathbf{R}) & \longrightarrow & T(\mathbf{R}) & \longrightarrow & H^1(\mathbf{R}, \mathcal{D}G) \end{array}$$

The **Hasse principle** for simply connected groups[10] says that $H^1(\mathbf{Q}, \mathcal{D}G) \hookrightarrow \prod_{v \le \infty} H^1(\mathbf{Q}_v, \mathcal{D}G)$. Combined with Kneser–Bruhat–Tits, this shows the righthand vertical map is injective. If $t \in T^\dagger$ then t is in the image under ν of $Z(\mathbf{R}) \subset G(\mathbf{R})$, and hence its image in $H^1(\mathbf{R}, \mathcal{D}G)$ is the basepoint, and therefore the same for its image in $H^1(\mathbf{Q}, \mathcal{D}G)$, so t lifts to $G(\mathbf{Q})$, as desired.

Combining Claims 1 and 2, we have

$$G(\mathbf{Q})\backslash G(\mathbf{A}_f)/K \overset{1-1}{\twoheadrightarrow} \nu(G(\mathbf{Q}))\backslash T(\mathbf{A}_f)/\nu(K) \overset{finite}{\twoheadrightarrow} T(\mathbf{Q})\backslash T(\mathbf{A}_f)/\nu(K),$$

and $\nu(K)$ is open. So the target is discrete, and it suffices to check that $T(\mathbf{Q})\backslash T(\mathbf{A}_f)$ is compact. Suppose T splits over a number field F. The adjunction morphism $T \to \mathrm{R}_{F/\mathbf{Q}} T_F$ is a closed immersion (see Appendix A of *Pseudo-reductive Groups*). The adelic topology interacts well with closed immersions, so $T(\mathbf{A}_f) \to \mathrm{R}_{F/\mathbf{Q}} T_F(\mathbf{A}_f) = ((\mathbf{A}_f \otimes_{\mathbf{Q}} F)^\times)^{\dim T} = (\mathbf{A}_{F,f}^\times)^{\dim T} = T(\mathbf{A}_{F,f})$ is a closed embedding of topological spaces. It follows that $T(\mathbf{Q})\backslash T(\mathbf{A}_f)$ is a closed subset of $T(F)\backslash T(\mathbf{A}_{F,f})$. This is just a product of $\dim T$ copies of the idéle class group, which is compact by the finiteness of $h(F)$, and Hausdorff. Thus $T(\mathbf{Q})\backslash T(\mathbf{A}_f)$ is compact and we win. $\square$

We now prove Proposition 4.2.6. Recall that this says

$$G(\mathbf{Q})\backslash (X \times G(\mathbf{A}_f))/K = \bigsqcup_{c \in \mathcal{C}} \Gamma_c \backslash X^+.$$

[9] See Ch. 6 of [6] for a proof.

[10] A theorem of Kneser for classical groups, Harder for exceptional groups other than E_8, and of Chernousov for E_8! This is proved in Ch. 6 of [6].

The first step is to relate the lefthand side to a double coset space involvng a Hermitian symmetric domain X^+ that is a connected component of X.

4.2.9. **Lemma.**
$$G(\mathbf{Q})\backslash(X \times G(\mathbf{A}_f))/K = G(\mathbf{Q})_+\backslash(X_+ \times G(\mathbf{A}_f))/K.$$

Proof. By Theorem 4.2.10, $G(\mathbf{Q}) \subset G(\mathbf{R})$ is dense, and X is a homogeneous space under $G(\mathbf{R})$, any point of x can be moved to X^+ by an element of $G(\mathbf{Q})$. So the natural map (from right to left) is surjective. For injectivity, you need to check that $G(\mathbf{R})_+$ is precisely the stabilizer of X^+ for the action $G(\mathbf{R})$ on $\pi_0(X)$. (See [5, 5.7b].) □

In the proof of the lemma, a key role wasplayed by the following deep result.

4.2.10. **Theorem** (Real Approximation)**.** Let G be a connected linear algebraic **Q**-group. Then $G(\mathbf{Q})$ is dense in $G(\mathbf{R})$.

Proof. Like Strong Approximation, this is pretty tricky; see [6, 7.7]. In this form it is apparently due to Kneser. But I have stated only a weak form; the general case[11] applies to an arbitrary number field K in place of **Q**, and asserts the existence of a finite set S of finite places of K such that $G(K)$ is dense in $\prod_{v \notin S} G(K_v))$. The idea of the proof seems to be to bootstrap up from the case of tori.[12] I'll explain that case.

Suppose our torus $G = T$ splits over a number field F and has dimension d. Let $\widetilde{T} = \mathrm{R}_{F/\mathbf{Q}}\mathbf{G}_{\mathrm{m}}^d = "(\underline{F}^\times)^d"$, another torus. Then we have a norm map $\widetilde{T} \to T$. This is built in the same way as the norm map $\underline{F}^\times \to \mathbf{G}_{\mathrm{m}}$, by making it over F and then using Galois descent (exercise). Let $N = \ker(\widetilde{T} \to T)$. The exact sequence
$$1 \to N \to \widetilde{T} \to T \to 1$$
induces a diagram
$$\begin{array}{ccccccc}
\widetilde{T}(\mathbf{Q}) & \xrightarrow{norm} & T(\mathbf{Q}) & \longrightarrow & H^1(\mathbf{Q}, N) & \longrightarrow & 1 \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
\widetilde{T}(\mathbf{R}) & \xrightarrow{norm} & T(\mathbf{R}) & \longrightarrow & H^1(\mathbf{R}, N_{\mathbf{R}}) & \longrightarrow & 1
\end{array}$$
Now $\widetilde{T}(\mathbf{Q}) = (F^\times)^d$ and $\widetilde{T}(\mathbf{R}) = \prod_{v \in \mathrm{arch}_F} (F_v^\times)^d$. So by the "usual" weak approximation theorem, the first vertical map has dense image. Since $\widetilde{T}(\mathbf{R}) \to T(\mathbf{R})$ is a submersion, it has open image, which we denote by $Norms_{\mathbf{R}}$. We know that $norm(\alpha(\widetilde{T}(\mathbf{Q}))) = \beta(norm(\widetilde{T}(\mathbf{Q})))$ is dense in $Norms_{\mathbf{R}}$. So to prove that $\beta(T(\mathbf{Q}))$ is dense in $T(\mathbf{R})$, it suffices to prove that $T(\mathbf{R}) = \beta(T(\mathbf{Q})) \cdot Norms_{\mathbf{R}}$. This is equivalent to saying that anything in $T(\mathbf{R})$ differs from a point of $T(\mathbf{Q})$ by a Norm, or equivalently that γ is surjective (chase the diagram).

This last surjectivity is a pretty tricky calculation in Galois cohomology, the details of which are in [6, p. 417]. One reduces to studying the corestriction maps on $\widehat{H}^{-1}(-, X_*(N))$ with respect to the inclusion of a decomposition group into $\mathcal{G} = \mathrm{Gal}(F/\mathbf{Q})$. It ultimately boils down to showing that for any complex place v of F, there is some prime p and some prime $\mathfrak{p}$ of F such that $F_{\mathfrak{p}}/\mathbf{Q}_p$ is unramified and the Frobenius automorphism of this extension coincides in $\mathcal{G}$ with complex conjugation in F_v. This is true by Chebotarev. □

Proof of Proposition 4.2.6. By Lemma 4.2.9 it suffices to construct a homeomorphism
$$\bigsqcup_{c \in \mathcal{C}} \Gamma_c\backslash X^+ \to G(\mathbf{Q})_+\backslash(X^+ \times G(\mathbf{A}_f))/K.$$
This is easy: send $\Gamma_c x \mapsto [x, c]$. The proof that this is a homeomorphism is essentially identical to Proposition 3.2.1. □

[11] Apparently due to Sansuc.

[12] FIXME: I'm not sure how hard this reduction is.

4.3. Passage to the inverse limit. By the results above, $\mathrm{Sh}_K(G,X)$ inherits from its connected components a unique structure of quasiprojective complex algebraic variety. In particular, the inverse system $\mathrm{Sh}(G,X)$ of locally symmetric spaces (as K varies) is not just an inverse system of complex manifolds, but of complex varieties.[13]

In fact we can actually make sense of the inverse limit *as (the analytification of) a regular, locally Noetherian* $\mathbf{C}$*-scheme (of infinite type)!* (We could have done this for the connected case, too, but no matter.)

TO BE ADDED (FIXME!)

Setting aside such algebro-geometric considerations, let us remark that the description of the inverse limit $\mathrm{Sh}(G,X)$ as a topological space is slightly more subtle than the case of connected Shimura varieties, due to the presence of a central torus in G. When $Z(\mathbf{Q}) \subset Z(\mathbf{A}_f)$ is *discrete*, then we still get

$$\mathrm{Sh}(G,X) = G(\mathbf{Q})\backslash X \times G(\mathbf{A}_f)$$

[and in fact the right action of $G(\mathbf{A}_f)$ on this big scheme is by algebraic automorphisms]. But the discreteness is actually an additional hypothesis to be imposed upon the Shimura datum (G,X). See [5, 5.28] for the general statement.

4.4. Functoriality of Sh(-,-). In this section we want to make sense of the manner in which a "morphism" of Shimura data (to be defined) gives rise to an actual morphism of Shimura varieties (in the sense of schemes). This will be very important for us: we will get a handle on canonical models for general Shimura varieties by relating them to the symplectic case, which we solve using CM theory. The functoriality is necessary to make sense of the relationship.

4.4.1. **Definition.** A morphism of Shimura data $(G, X \ni h) \to (G', X' \ni h')$ is a morphism $\phi : G \to G'$ such that the $G(\mathbf{R})$-conjugacy class X of $h : \mathbf{S} \to G_{\mathbf{R}}$ maps to the $G'(\mathbf{R})$-conjugacy class X' of $h' : \mathbf{S} \to G'_{\mathbf{R}}$ under post-composition with ϕ. In other words, $\phi_{\mathbf{R}} \circ h$ should belong to X'.

4.4.2. **Proposition** (Deligne).

(1) If ϕ is a morphism of Shimura data, then ϕ induces an inverse system of regular maps $\phi_{K,K'} : \mathrm{Sh}_K(G,X) \to \mathrm{Sh}_{K'}(G',X')$ for $K \subset G(\mathbf{A}_f), K' \subset G'(\mathbf{A}_f)$ compact open subsets such that $\phi(K) \subset K'$.
(2) The inverse system $\{\phi_{K,K'}\}$ intertwines the natural action of $G(\mathbf{A}_f)$ on $\mathrm{Sh}(G,X)$ with the pullback via ϕ to $G(\mathbf{A}_f)$ of the natural action of $G'(\mathbf{A}_f)$ on $\mathrm{Sh}(G',X')$.
(3) If ϕ is a closed subgroup inclusion then for each K there is a $K' \supset \phi(K)$ such that $\phi_{K,K'}$ is a closed immersion.[14]

Sketch. Parts (1) and (2) are formal; part (3) is [3, 1.14-15], but I haven't yet worked through the proof (FIXME). □

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[13]FIXME: There may actually be some subtlety here about "neat subgroups"; it may be necessary to take K sufficiently small to know that each $\Gamma_c\backslash X^+$ satisfies the hypotheses of Baily–Borel...?

[14]If we were careful to construct the scheme structure on the inverse limits, this should say that $\{\phi_{K,K'}\}$ is literally a closed immersion of $\mathbf{C}$-schemes $\mathrm{Sh}(G,X) \to \mathrm{Sh}(G',X')$.