1. a) A fractional linear transformation \( w = f(z) \) of \( \mathbb{C}P^1 \) is called parabolic if it has exactly one fixed point. Prove that any parabolic transformation with a fixed point \( z_0 \) can be written as

\[
\frac{1}{w - z_0} = \frac{1}{z - z_0} + h, \quad \text{if } z \neq \infty
\]

and in the form

\[
w = z + h \quad \text{if } z_0 = \infty.
\]

b) Prove that any fractional linear transformation \( w = f(z) \) of \( \mathbb{C}P^1 \) with two fixed points \( z_1 \) and \( z_2 \), \( z_1 \neq z_2 \) can be written in the form

\[
\frac{w - z_1}{w - z_2} = k \frac{z - z_1}{z - z_2}, \quad \text{if } z_1, z_2 \neq \infty
\]

and in the form

\[
w - z_1 = k(z - z_1) \quad \text{if } z_2 = \infty.
\]

A transformation \( f \) with two fixed points is called hyperbolic if \( k > 0 \), elliptic if \( |k| = 1, k \neq 1 \), and loxodromic if \( k = re^{i\theta} \), where \( r \neq 1 \) and \( \theta \neq 0 \) (mod 2\( \pi \)).
c) Prove that a transformation $f(z) = \frac{az+b}{cz+d}$, $f \neq \text{Id}$, with $ad - bc = 1$ is

$$
\begin{align*}
\text{elliptic} & \quad \text{if } a + d \in \mathbb{R}, |a + d| < 2; \\
\text{hyperbolic} & \quad \text{if } a + d \in \mathbb{R}, |a + d| > 2; \\
\text{parabolic} & \quad \text{if } a + d \in \mathbb{R}, |a + d| = 2; \\
\text{loxodromic} & \quad \text{if } \text{Im}(a + d) \neq 0.
\end{align*}
$$

Hint: $a + d$ is the trace of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. 

2. a) Let $U = \{z \in \mathbb{C}; 0 < \text{Im}z < 1\}$. Suppose that the group $\mathbb{Z}$ acts on $U$ by translations: $z \mapsto z + k$ for $k \in \mathbb{Z}$. Prove that $U/\mathbb{Z}$ is conformally equivalent to an annulus $A$ and compute its conformal modulus $m(A)$.

b) Let the group $\mathbb{Z}$ acts on $\mathbb{H}$ by multiplication: $z \mapsto e^{k}z$. Prove that $\mathbb{H}/\mathbb{Z}$ is conformally equivalent to an annulus $A$ and compute its conformal modulus $m(A)$.

3. In the following problems find the Laurent expansion and compute the annuli of convergence:

a) $f(z) = e^{z+\frac{1}{2}}$ near $z = 0$;

b) $\frac{1}{(z^2+1)^2}$ near $z = i$ and near $z = \infty$.

4. Let $f : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C}$ be an injective holomorphic function. Suppose that its Laurent expansion has the form

$$f(z) = z + \sum_{-\infty}^{-1} c_k z^k.$$ 

Prove that

$$\sum_{n=1}^{\infty} n |c_{-n}|^2 \leq 1.$$ 

What is the geometric meaning of this inequality?

5. Let $U \subset \mathbb{C}$ be a bounded connected domain and $f : U \to U$ a holomorphic map. Suppose that $0 \in U$, $f(0) = 0$ and $f'(0) = 1$. Prove that $f(z) = z$ for all $z \in U$. 

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Hint: Compute the first non-zero coefficient in the Taylor expansion at 0 of the function $f_n(z) - z$, where $f_n := f \circ \cdots \circ f$.

All problems and subproblems are 10 points