1. Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be $2\pi$-periodic continuous function.

Define a function $u : \{|z| = 1\} \rightarrow \mathbb{R}$ by the formula $u(\zeta) = \psi(\theta)$ for $\zeta = e^{i\theta}$. Extend $u$ as a harmonic function to $D$ by the Schwarz’s formula

$$u(z) = \text{Re} f(z), \quad f(z) = \frac{1}{2\pi i} \oint_{\partial D} \frac{u(\zeta)(\zeta + z) d\zeta}{(\zeta - z)\zeta}.$$ 

Define

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \psi(\theta) d\theta, \quad a_n = \frac{1}{\pi} \int_0^{2\pi} \psi(\theta) \cos n\theta d\theta, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} \psi(\theta) \sin n\theta d\theta \quad \text{for} \quad n \geq 1.$$

Show the following power expansion for $f$:

$$f(z) = \sum_{0}^{\infty} c_n z^n, \quad |z| < 1, \quad \text{where} \quad c_0 = \frac{1}{2\pi} \int_0^{2\pi} \psi(\theta) d\theta, \quad c_n = \frac{1}{\pi} \int_0^{2\pi} \psi(\theta) e^{-in\theta} d\theta \quad \text{for} \quad n \geq 1.$$

Deduce from it the expansion

$$u(re^{i\phi}) = a_0 + \sum_{1}^{\infty} (a_n r^n \cos n\phi + b_n r^n \sin n\phi), \quad r < 1,$$

where $a_n = \text{Re} c_n, b_n = -\text{Im} c_n$.

2. Let $\zeta = e^{i\theta}, z = re^{i\phi}$. Verify the identity

$$\frac{\zeta + z}{\zeta - z} = \frac{(I - r^2) + i2r \sin(\phi - \theta)}{1 - 2r \cos(\phi - \theta) + r^2}.$$
3. Let the function $u$ and the coefficients $a_n, b_n$ be as in Problem 1. Prove the following formula for the Dirichlet integral $D_U(u) := \iint_U (u_x^2 + u_y^2) \, dxdy$:

$$D_D(u) = \pi \sum_{n=1}^{\infty} n(a_n^2 + b_n^2),$$

provided that the integral and the sum are converging. Here $\mathbb{D}$ is the unit disc.

4. Let $u : \mathbb{R} \to \mathbb{R}$ be a continuous bounded function. Prove that the function

$$U(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} u(t) \, dt$$

define a harmonic function in $\mathbb{H}$ which extends continuously to the closure $\mathbb{H}$ as equal to $u$ on $\partial \mathbb{H} = \mathbb{R}$.

*Hint.* Use the Poisson-Schwarz formula for $\mathbb{D}$ (See Section 10.4.1 in the lecture notes), and then apply a conformal map $\mathbb{H} \to \mathbb{D}$ to transpose this formula from the disc to the upper half plane.

5. Suppose $\Gamma \subset \text{Aut}(S)$ is a discrete subgroup of conformal automorphisms of $S$ acting on $S$. A domain $U \subset S$ is called a fundamental domain for the action of $\Gamma$ if every trajectory of $\Gamma$ intersects the closure $\overline{U}$ and no two points of the same trajectory belong to $U^1$. For instance, the square $\{0 < x, y < 1\}$ is the fundamental domain for the action $z \mapsto z + (m + in)$ of $\mathbb{Z} \oplus \mathbb{Z}$ on $\mathbb{C}$. Note that the choice of a fundamental domain is not unique.

Find the fundamental domain of $\Gamma = \text{PSL}(2,\mathbb{Z}) \subset \text{PSL}(2,\mathbb{R}) = \text{Aut}(\mathbb{H})$ which consists of transformations $z \mapsto \frac{mx+n}{kz+l}$, where $m, n, k, l$ are integers with $ml - kn = 1$. ($\Gamma$ is called the modular group).

*Hint:* Use Theorem 11.17 from the lecture notes. This theorem provides a domain whose closure intersects every trajectory of $\Gamma$. The fact that no trajectory intersects the interior of the domain in more than one point can also be deduced from the proof but requires some additional argument.

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1 Sometimes, a part of the boundary $\partial U$ is added to the fundamental domain to ensure that every trajectory intersects it exactly once.
Each problem is 10 points.