1. Consider a polynomial

\[ p(z) = z^4 + 2z^2 + 1. \]

Describe it as a branch cover \( \mathbb{CP}^1 \to \mathbb{CP}^1 \) and determine the branching points and their orders.

2. Let \( S \) be a compact Riemann surface and \( f : S \to \mathbb{CP}^1 \) a non-constant meromorphic function.

a) Prove that the total multiplicity of poles of \( f \) is equal to the total multiplicity of its zeroes.

Hint. Let \( A = \{a_1, \ldots, a_N\} \subset S \) be the set of all points in \( S \) where the differential \( df \) vanishes. Denote \( B := f(A) \subset \mathbb{CP}^1 \) and \( \tilde{A} := f^{-1}(B) \subset S \). Verify that

\[ f|_{S \setminus \tilde{A}} : S \setminus \tilde{A} \to \mathbb{CP}^1 \setminus B \]

is a covering map, \( \mathbb{CP}^1 \setminus B \) is connected, and hence all points in \( \mathbb{CP}^1 \setminus B \) have the same number of pre-images.

b) Show that if \( f \) has a unique simple pole then \( f : S \to \mathbb{CP}^1 \) is a biholomorphism.

3. a) Prove that any even (i.e. \( f(-z) = f(z) \)) elliptic function with periods \( \omega_1 \) and \( \omega_2 \) can be presented in the form

\[ f(z) = C \prod_{k=1}^{r} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}, \quad a_k, b_k \in \mathbb{C}. \]
b) How the formula should be corrected if some of $a_k$ or $b_k$ are zeroes?

Remark. Any function can be presented as a sum of an even and odd function:

$$f(z) = \frac{1}{2}(f(z) + f(-z)) + \frac{1}{2}(f(z) - f(-z)) = g(z) + h(z).$$

On the other hand, the ration of two odd functions is odd and $\varphi'(z)$ is odd. Hence,

$$f(z) = g(z) + \frac{h(z)}{\varphi'(z)} \varphi'(z),$$

where $g(z)$ and $\frac{h(z)}{\varphi'(z)}$. Hence, Problem 1 allows us to express any elliptic function via the Weierstrass function and its derivative.

4. Suppose an elliptic function $f : T(\omega_1, \omega_2) = \mathbb{C}/\Lambda(\omega_1, \omega_2) \rightarrow \mathbb{C}P^1$ has simple zeroes $a_1, \ldots, a_r$ and simple poles $b_1, \ldots, b_r$. Prove that

$$\sum_{1}^{r} (a_j - b_j) = n\omega_1 + m\omega_2$$

for some integer $n, m$. Note that $a_j$ and $b_j$ are defined only up to multiples of $\omega_1$ and $\omega_2$.

**Hint:** Consider

$$\int_{\partial P} \frac{zf''(z)}{f(z)} dz,$$

where $P$ is the parallelogram spanned by $\omega_1, \omega_2$ and chosen in such a way that there are no zeroes and poles on $\partial P$.

Each (sub)problem is 10 points.