Math 116: Homework N9

Due on Tuesday, December 3, on Gradescope

1. Prove that the quadric
   \[ S = \{ z_1^2 + z_2^2 = 1 \} \subset \mathbb{C}^2 \]
   is conformally equivalent to the cylinder \( \mathbb{C}/\mathbb{Z} \), where \( \mathbb{Z} \) acts by translations \( z \mapsto z + ik \).
   \textit{Hint:} \( \sin^2 t + \cos^2 t = 1 \).

2. Let \( \Lambda = \Lambda(\omega_1, \omega_2) \) be a lattice and \( \wp : \mathbb{C} \to \mathbb{C}P^1 \) the corresponding Weierstrass function.
   a) Prove that the expression
   \[ \zeta(z) = \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus 0} \left( \frac{1}{z - \lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right). \]
   defines a meromorphic function on \( \mathbb{C} \) such that for all \( z \in \mathbb{C} \) we have
   (i) \( \zeta'(z) = -\wp(z) \);
   (ii) \( \zeta(-z) = -\zeta(z) \);
   (iii) \( \zeta(z + \omega_1) = \zeta(z) + \eta_1; \quad \zeta(z + \omega_2) = \zeta(z) + \eta_2. \)

b) Show that \( \eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i. \)
   \textit{Hint:} Consider a parallelogram \( P \) with vertices
   \[ \frac{1}{2}(-\omega_1 - \omega_2), \frac{1}{2}(-\omega_1 + \omega_2), \frac{1}{2}(\omega_1 - \omega_2), \frac{1}{2}(\omega_1 + \omega_2). \]
Compute integral $\int_{\partial P} \zeta(z) dz$ (where $P$ is oriented as the boundary of the parallelogram $P$) in two different ways:

one, using the residue theorem, and

the other one directly by combining contributions from opposite sides of $P$ and taking into account that $\zeta(z + \omega_j) - \zeta(z) = \eta_j$, $j = 1, 2$.

3. a) Show that the function

$$\theta(z) = \prod_{1}^{\infty} (1 + h^{2n-1}e^z)(1 + h^{2n-1}e^{-z})$$

is holomorphic on $\mathbb{C}$ if $|h| < 1$.

b) Show that

$$\theta(z + 2 \log h) = h^{-1}e^{-z}\theta(z).$$

4. Consider a sequence of distinct complex numbers $a_n \in \mathbb{C}$, $n = 1, 2, \ldots$, such that

$$\sum_{1}^{\infty} \frac{1}{|a_n|} < \infty,$$

a) Show that there exists an entire function $g : \mathbb{C} \to \mathbb{C}$ which has simple zeroes at the points $a_n$, $n = 1, 2, \ldots$, and nowhere else.

b) Let $A_n$ be any sequence of complex numbers. Show that there exists an entire function $f : \mathbb{C} \to \mathbb{C}$ such that

$$f(a_n) = A_n, \quad n = 1, 2, \ldots.$$  

**Hint.** Let $g(z)$ be the function constructed in a). Show that

$$\sum_{1}^{\infty} g(z) \frac{e^{\gamma_n(z-a_n)}}{z-a_n} \frac{A_n}{g'(a_n)}$$

converges for some choice of numbers $\gamma_n$.  

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5. Suppose that $\sum_{1}^{\infty} |a_n|^2 < \infty$. Show that the convergence of the product $\prod_{1}^{\infty} (1 + a_n)$ is equivalent to the convergence of the series $\sum_{1}^{\infty} a_n$.

6. Show that the product
\[
\prod_{1}^{\infty} \cos \frac{z}{2^k}
\]
converges to $\frac{\sin z}{z}$.

Hint: $\sin 2z = \sin z \cos z$.

Each (sub)problem is 10 points.