Math 116: Practice Midterm Exam

Solutions

1. Use Abel’s theorem (Problem 2 from Homework 2) to compute

\[ \sum_{n=0}^{\infty} \frac{\sin(2n+1)\phi}{2n+1}, \quad 0 < \phi < \pi. \]

Solution.

\[ \sum_{n=0}^{\infty} \frac{\sin(2n+1)\phi}{2n+1} = \text{Im} \left( \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1} \right), \quad z = e^{i\phi}. \]

Choose \( r \in (0, 1) \) and denote \( u = re^{i\phi} \). According to Abel’s theorem

\[ \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1} = \lim_{r \to 1} \sum_{n=0}^{\infty} \frac{u^{2n+1}}{2n+1} = \lim_{r \to 1} S(u). \]

For \( |u| < 1 \) the series absolutely converges, and hence one can differentiate it termwise.

\[ S'(u) = \sum_{n=0}^{\infty} u^{2n} = \frac{1}{1 - u^2}. \]

Integrating \( \frac{du}{1-u^2} \) we get \( S(u) = \frac{1}{2} \log \frac{1+u}{1-u} \). We note that for \( \phi \in (0, \pi) \) we have \( \frac{1+u}{1-u} \in \{ \text{Im}z > 0 \} \), and hence log is defined. There exists a limit

\[ \lim_{r \to 1} S(u) = \frac{1}{2} \log \left( \frac{1+ e^{i\phi}}{1- e^{i\phi}} \right) = \frac{1}{2} \log \left( \frac{i \sin \phi}{1- \cos \phi} \right) = \frac{\pi}{4}. \]
Hence,
\[ \sum_{n=0}^{\infty} \frac{\sin(2n+1)\phi}{2n+1} = \frac{\pi}{4}. \]

**Remark** To establish convergence of the series \( \sum_{n=0}^{\infty} \frac{\sin(2n+1)\phi}{2n+1} \) one may use the so-called Dirichlet's test: the partial sums \( \sum_{n=1}^{N} \frac{\sin(2n+1)\phi}{2n+1} \) are bounded when \( \phi \in (0, \pi) \), and the sequence \( \frac{1}{2n+1} \) is monotonically decreasing.

2. Compute
\[ \int_{0}^{2\pi} \frac{d\phi}{(a + b \cos \phi)^2}, \quad a > b > 0. \]

**Solution.** Set \( z = e^{i\phi} \). Then \( \cos \phi = \frac{z + z^{-1}}{2} \). Then
\[
\frac{1}{(a + b \cos \phi)^2} = \frac{4}{(2a + bz + bz^{-1})^2} = \frac{4z^2}{(bz^2 + 2az + b)^2},
\]
and \( dz = ie^{i\phi}d\phi = izd\phi \). Hence,
\[
\int_{\partial D_1(0)} \frac{zdz}{(bz^2 + 2az + b)^2} = \int_{0}^{2\pi} \frac{i \cdot z^2 d\phi}{(bz^2 + 2az + b)^2} = \frac{i}{4} \int_{0}^{2\pi} \frac{4z^2 d\phi}{(bz^2 + 2az + b)^2} = \frac{i}{4} \int_{0}^{2\pi} \frac{d\phi}{(a + b \cos \phi)^2},
\]
i.e.
\[
\int_{0}^{2\pi} \frac{d\phi}{(a + b \cos \phi)^2} = -4i \int_{\partial D_1(0)} \frac{dz}{(bz^2 + 2az + b)^2}.
\]

The poles of the function \( \frac{z}{(bz^2 + 2az + b)} \) are at the points
\[ z_{1,2} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}, \]
and only one of the poles,
\[ z_1 = \frac{-a + \sqrt{a^2 - b^2}}{b}, \]
is inside the unit disc. We have
\[
\frac{z}{(bz^2 + 2az + b)^2} = \frac{g(z)}{(z - z_1)^2}, \quad g(z) = \frac{z}{b^2(z - z_2)^2}.
\]
Then the residue at the point \(z_1\) is equal
\[
\text{Res}_{z_1} \left( \frac{z}{(bz^2 + 2az + b)^2} \right) = g'(z_1) = \frac{-ai}{(a^2 - b^2)^{\frac{3}{2}}}.
\]
Thus,
\[
\int_0^{2\pi} \frac{d\phi}{(a + b \cos \phi)^2} = 2\pi i \frac{-ai}{(a^2 - b^2)^{\frac{3}{2}}} = \frac{2\pi a}{(a^2 - b^2)^{\frac{3}{2}}}.
\]
3. Determine the number of solutions in each quadrant of the equation
\[
2z^4 - 3z^3 + 3z^2 - z + 1 = 0.
\]

**Solution.** First we verify that there are no zeroes on coordinate axis. Indeed, if \(z = iy\) then
\[
f(z) = 2z^4 - 3z^3 + 3z^2 - z + 1 = 2y^4 - 3y^2 + 1 + i(3y^3 - y).
\]
The imaginary part vanishes at \(y = 0, \frac{1}{\sqrt{3}}\), but the real part does not vanish at these points. We also note that the real part is positive outside of the intervals \((-1, -\frac{1}{\sqrt{2}})\) and \((\frac{1}{\sqrt{2}}, 1)\). It is important to note that \(\frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}}\). This implies that the total variation of the argument of \(f(z)\) when moving from 0 to \(\pm\infty\) is 0. We also can verify that \(f\) is positive on the real line. This implies that the variation of the argument along the real axis is also 0.

For each quadrant consider a contour consisting of axes and quarter of the circle of a large radius \(R\).

When we moving counterclockwise, the argument does not change when we move along the axes, and changes as the argument of \(z^4\) when we move along the quarter of the circle, which is \(2\pi\). Hence, the total change of the argument is \(2\pi\) which means that each quadrant has exactly one zero.
4. Find a biholomorphism \( f \) which maps the disc \( D_R = \{ |z| < R \} \) onto the half-plane \( \{ \Re z > 0 \} \) and such that \( f(R) = 0, f(-R) = \infty, f(0) = 1 \).

**Solution.** The fractional linear function \( f(z) = \frac{az+b}{cz+d} \) must have a 0 at \( R \) and a pole at \( -R \). Hence, it must have a form \( c\frac{z-R}{z+R} \). The equation \( f(0) = 1 \) yields \( c = -1 \). Hence,

\[
    f(z) = \frac{R-z}{z+R}.
\]

We verify that it maps the circle \( \partial D_R = \{ z = \Re e^{it} \} \) onto the imaginary axis:

\[
    \frac{R - \Re e^{it}}{\Re e^{it} + R} = \frac{(1 - e^{it})(1 + e^{-it})}{|1 + e^{it}|^2} = \frac{-2i \sin t}{|1 + e^{it}|^2} \in \{ \Re z = 0 \}.
\]

5. Let \( U = \mathbb{C} \setminus \{ z; \Re z \leq 0, \Im z = 0 \} \). Find a holomorphic function \( f : U \to \mathbb{C} \) such that \( \arg f(z) = \phi + r \sin \phi \) for \( z = re^{i\phi} \).

**Solution.** We have \( \log f(z) = \ln |f(z)| + i \arg f(z) \). Let us reconstruct the holomorphic function \( \log f(z) \) whose imaginary part is equal to \( \arg z + \Im z \), which is a harmonic function. We know that \( \arg z \) is the imaginary part of \( \log z \) and \( \Im z \) is, of course, the imaginary part of \( z \). Hence, \( \log f(z) = \log z + z + C, \ C \in \mathbb{R} \) and hence

\[
    f(z) = C_1 ze^z, \ C_1 = e^C.
\]