The goal of this supplemental homework is to prove the following formula of C.F. Gauss:

\[
\sum_{k=0}^{n-1} e^{\frac{2\pi ik^2}{n}} = \sqrt{n} \frac{i + i^{1-n}}{i + 1}.
\]

The proof is split into several steps.

Denote

\[ f(z) = \frac{e^{\frac{2\pi iz^2}{n}}}{e^{\frac{2\pi iz}{n}} - 1}. \]

Choose a large \( R > 0 \) and consider a rectangle \( P \) with vertices

\[ v_1 = \frac{n}{2} - iR, \quad v_2 = \frac{n}{2} + iR, \quad v_3 = iR, \quad v_4 = -iR. \]

For a small \( \epsilon > 0 \) consider discs

\[ D_\epsilon(0) := \{ z; |z| < \epsilon \}, \quad D_\epsilon\left(\frac{n}{2}\right) := \{ z; |z - \frac{n}{2}| < \epsilon \}. \]

Denote

\[ C_\epsilon := \partial D_\epsilon(0) \cap \{ z; \text{Re} \, z > 0 \}, \quad C_{\epsilon,n} := \partial D_\epsilon\left(\frac{n}{2}\right) \cap \{ z; \text{Re} \, z < \frac{n}{2} \} \]

and

\[ U_{\epsilon,R} := P \setminus \left( D_\epsilon(0) \cup D_\epsilon\left(\frac{n}{2}\right) \right), \]

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see the Figure. All arcs are assumed oriented as parts of the boundary of the domain $U_{\epsilon,R}$.

1. Using the residue theorem show that

$$\int_{\partial U_{\epsilon,R}} f(z) \, dz = \sum_{0 < k < \frac{n}{2}} e^{\frac{2\pi ik^2}{n}}.$$ 

Show that

$$\sum_{0 < k < \frac{n}{2}} e^{\frac{2\pi ik^2}{n}} = \frac{1}{2} \sum_{0 < k < n, k \neq \frac{n}{2}} e^{\frac{2\pi ik^2}{n}}.$$ 

2. Show that

$$\int_{C_\epsilon} f(z) \, dz \rightarrow_{\epsilon \to 0} = \begin{cases} 1 \quad ; & n \text{ is even;} \\ 0 \quad ; & n \text{ is odd.} \end{cases}$$
3. Show that
\[ \frac{1}{2} \sum_{k=0}^{n-1} e^{\frac{2\pi i k^2}{n}} = \lim_{\epsilon \to 0} \sum_{j=1}^{6} \int_{\gamma_j} f(z)dz. \]

4. Show that
\[ \int_{\gamma_3 \cup \gamma_6} f(z)dz \to 0; \]
\[ \int_{\gamma_4 \cup \gamma_5} f(z)dz = i \int_{\epsilon}^{R} e^{\frac{2\pi y^2}{n}} dy; \]
\[ \int_{\gamma_1 \cup \gamma_2} f(z)dz = i^{1-n} \int_{\epsilon}^{R} e^{\frac{2\pi y^2}{n}} dy. \]

5. Show that
\[ \frac{1}{2} \sum_{k=0}^{n-1} e^{\frac{2\pi i k^2}{n}} = \sqrt{n}(i + i^{1-n}) \int_{0}^{\infty} e^{-2\pi y^2} dy. \]

and plugging into this formula \( n = 1 \) show that
\[ \int_{0}^{\infty} e^{-2\pi y^2} dy = \frac{1}{2(1 + i)}. \]

This concludes the proof of Gauss’ formula.