Problem 1. Consider the function \( g(z) = zf(z) \), so that \( g(z) \) is a holomorphic function from \( \mathbb{D} \) to itself such that \( g(0) = 0 \) and \( g \) extends continuously to \( \partial \mathbb{D} \) with \( g(z) = 1 \) for all \( z \in \partial \mathbb{D} \). But then \( g(z) - 1 \) is a holomorphic function which extends continuously to 0 on \( \partial \mathbb{D} \), and the maximum modulus principle then implies that \( g(z) = 1 \) for all \( z \in \mathbb{D} \), so that \( f(z) = 1/z \) for all \( z \in \mathbb{D} \setminus \{0\} \), contradicting holomorphicity of \( f(z) \).

Problem 2. Our goal is to apply Rouché’s theorem. First, note that
\[
|z^4 - 6z| \geq 5 \text{ whenever } |z| = 1, \text{ and } |z^4 - 6z| \geq 4 \text{ whenever } |z| = 2 \text{ (this follows from the triangle equality). In particular, } |z^4 - 6z| > 3 \text{ for all } z \text{ on } \partial A, \text{ so Rouché’s theorem implies that } z^4 - 6z + 3 \text{ has the same number of zeros as } z^4 - 6z \text{ in } A. \text{ Now } z^4 - 6z = z(z^3 - 6), \text{ and } z^3 - 6 \text{ has exactly three roots, all of modulus } 6^{1/3} \text{ (namely } 6^{1/3}, e^{2\pi i/3}6^{1/3}, \text{ and } e^{4\pi i/3}6^{1/3}), \text{ and } 1 < 6^{1/3} < 2, \text{ so that } z^4 - 6z \text{ has exactly three zeros in } A, \text{ and hence the same is true of } z^4 - 6z + 3.

Problem 3. (a) Notice that
\[
\frac{2xy}{(x^2 + y^2)^2}
\]

and
\[
\frac{2y^2}{(x^2 + y^2)^2} = -2y + 1 - \frac{1}{x^2 + y^2} + \frac{2y^2}{(x^2 + y^2)^2} = -2y + 1 + \frac{y^2 - x^2}{(x^2 + y^2)^2}.
\]

If \( v \) is a harmonic conjugate of \( u \), then the Cauchy-Riemann equations state that \( u_x = v_y \) and \( u_y = -v_x \), so integration shows that
\[
v = 2xy + 5y - \frac{x}{x^2 + y^2} + g(y)
\]
for some differentiable function \( g(y) \) and
\[
v = 2xy - x - \frac{x}{x^2 + y^2} + h(x)
\]
for some differentiable function \( h(x) \). Comparing these expressions shows that \( v = 5y - x + 2xy - \frac{x}{x^2 + y^2} \) is a harmonic conjugate of \( u \).

(b) Compute
\[
v_x = \frac{2x}{x^2 + y^2} + 1
\]
and
\[
v_y = \frac{2y}{x^2 + y^2} - 2.
\]
As above, it follows from this that if \( u \) is such that \( u + iv \) is holomorphic, then
\[
u = 2 \arctan \left( \frac{x}{y} \right) - 2x + g(y)
\]
for some differentiable function \( g(y) \), and
\[
u = -2 \arctan \left( \frac{y}{x} \right) - y + h(x)
\]
for some differentiable function \( h(x) \). Notice that \( \arctan \left( \frac{x}{y} \right) \) and \( -\arctan \left( \frac{y}{x} \right) \) differ only by a constant wherever they are both defined, so that we may take
\[
u = 2 \arctan \left( \frac{x}{y} \right) - 2x - y.
\]

(c) Notice that if \( z = x + iy \), then \((x^2 + y^2)e^z = |z|^2|e^z|\), so the function \( f(z) = z^2e^z \) satisfies the requirement. (Explicitly, \( u = (x^2 - y^2)e^z \cos(y) \) and \( v = 2xye^z \sin(y) \).

**Problem 4.** Let \( \psi(x, y) = \phi(x^2 + y^2) \) be a harmonic function such that \( \phi \) is \( C^2 \). Compute
\[
\frac{\partial^2 \psi}{\partial x^2}(x, y) = \frac{\partial}{\partial x}(2x\phi'(x^2 + y^2)) = 2\phi'(x^2 + y^2) + 4x\phi''(x^2 + y^2)
\]
and similarly
\[
\frac{\partial^2 \psi}{\partial y^2}(x, y) = 2\phi'(x^2 + y^2) + 4y\phi''(x^2 + y^2),
\]
so that
\[ \Delta \psi(x, y) = 4\phi'(x^2 + y^2) + 4(x^2 + y^2)\phi''(x^2 + y^2). \]

If \( \psi(x, y) \) is harmonic, then this implies \( \phi'(a) = -a\phi''(a) \) for all \( a \geq 0 \). This differential equation is satisfied by \( \phi(a) = c\log(a) + d \) for any \( c, d \in \mathbb{C} \), and in fact these are all of the solutions. So all such \( \psi \) are of the form \( \psi(x, y) = c\log(x^2 + y^2) + d \) for \( c, d \in \mathbb{C} \). (This is easily checked to be harmonic directly.) In particular, if \( \psi \) is required to be defined on all of \( \mathbb{R}^2 \), then \( \psi \) must be a constant function.

**Problem 5.** The area of \( f(\mathbb{D}) \) is \( \int_{f(\mathbb{D})} dxdy \), and by the general change of variables formula from real analysis, we find
\[
\int_{f(\mathbb{D})} dxdy = \int_{\mathbb{D}} |J_f(x, y)|dxdy,
\]
where \( J_f \) is the Jacobian matrix for \( f \). Since \( f \) is holomorphic, we have \( |J_f(x, y)| = |f'(z)|^2 \). Since the series for \( f(z) \) has radius of convergence \( R > 1 \), it converges absolutely and uniformly on \( \mathbb{D} \) and it is therefore sensible to compute
\[
|f'(z)|^2 = (\sum_{m=1}^{\infty} mc_m z^{m-1})(\sum_{n=1}^{\infty} n\bar{c}_n z^{n-1})
= (\sum_{n=1}^{\infty} n^2 |c_n|^2 |z|^{2(n-1)}) + (\sum_{m \neq n} mn c_m \bar{c}_n z^{m-1} z^{n-1})
\]
for all \( z \in \mathbb{D} \). Moreover, because of uniform convergence we may interchange integration and summation to find
\[
\int_{\mathbb{D}} |f'(z)|^2 dxdy = (\sum_{n=1}^{\infty} \int_{\mathbb{D}} n^2 |c_n|^2 |z|^{2(n-1)} dxdy) + (\sum_{m \neq n} \int_{\mathbb{D}} mn c_m \bar{c}_n z^{m-1} z^{n-1} dxdy).
\]
We can rewrite these integrals in polar coordinates. First of all, this gives
\[
\int_{\mathbb{D}} mn c_m \bar{c}_n z^{m-1} z^{n-1} dxdy = \int_{0}^{2\pi} \int_{0}^{1} mn c_m \bar{c}_n r^{m+n-1} e^{i(m-n)\theta} r dr d\theta.
\]
This latter integral is 0 whenever \( m \neq n \) because \( \int_{0}^{2\pi} e^{ik\theta} d\theta = 0 \) whenever \( k \neq 0 \). Thus we obtain (again rewriting in polar coordinates)
\[
\int_{\mathbb{D}} |f'(z)|^2 dxdy = \sum_{n=1}^{\infty} \int_{0}^{2\pi} \int_{0}^{1} n^2 |c_n|^2 r^{2n-1} dr d\theta.
\]
Each of the integrals in this sum may be computed simply, as

\[
\int_0^{2\pi} \int_0^1 n^2 |c_n|^2 r^{2n-1} dr d\theta = \frac{1}{2} \int_0^{2\pi} n|c_n|^2 r^2 nd\theta = \pi n|c_n|^2.
\]

Combining this with the previous calculations yields the final result

\[
\int_{f(D)} dx dy = \pi \sum_{n=1}^{\infty} n|c_n|^2.
\]

**Problem 6.** We will proceed as in the proof of Rouché’s theorem in the notes. Suppose first that \(|f(z)| > 1\) for all \(z \in \partial U\). For each \(t \in [0, 1]\), let \(f_t = f - 1 + t\). By the argument principle, \(n_t\), the number of zeros of \(f\) minus the number of poles, is given by

\[
n_t = \frac{1}{2\pi i} \int_{\partial U} \frac{f_t'(z)}{f_t(z)} dz.
\]

This integral takes on only integer values and it is continuous, so \(n_t\) must be constant, and hence \(n_0 = n_1\). But clearly \(f\) and \(f - 1\) have the same number of poles in \(U\), so this shows that in fact they have the same number of zeroes in \(U\), hence the result.

Now suppose \(|f(z)| < 1\) for all \(z \in \partial U\). Then \(1/f(z)\) is also a meromorphic function on \(U\) which \(C^1\)-extends without singularities to the boundary and \(|1/f(z)| > 1\) for all \(z \in \partial U\). (Here we must assume that \(f(z)\) is non-vanishing on \(\partial U\).) Then the previous case shows that the number of solutions to the equation \(1/f(z) = 1\) in \(U\) is the same as the total multiplicity of zeroes of \(1/f(z)\) in \(U\). But zeroes of \(1/f(z)\) correspond to poles of \(f(z)\), and \(1/f(z) = 1\) if and only if \(f(z) = 1\), so that indeed the number of solutions to the equation \(f(z) = 1\) is equal to the total multiplicity of poles of \(f(z)\) in \(U\), as desired.