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Part I

Complex Analysis Basics
Chapter 1

Linear algebra

1.1 Complex numbers

The space $\mathbb{R}^2$ can be endowed with an associative and commutative multiplication operation. This operation is uniquely determined by three properties:

- it is a bilinear operation;
- the vector $(1, 0)$ is the unit;
- the vector $(0, 1)$ satisfies $(0, 1)^2 = -(1, 0)$.

The vector $(0, 1)$ is usually denoted by $i$, and we will simply write 1 instead of the vector $(1, 0)$. Hence, any point $(a, b) \in \mathbb{R}^2$ can be written as $a + bi$, where $a, b \in \mathbb{R}$, and the product of $a + bi$ and $c + di$ is given by the formula

$$(a + bi)(c + di) = ac - bd + (ad + bc)i.$$ 

The plane $\mathbb{R}^2$ endowed with this multiplication is denoted by $\mathbb{C}$ and called the set of complex numbers. The real line generated by 1 is called the real axis, the line generated by $i$ is called the imaginary axis. The set of real numbers $\mathbb{R}$ can be viewed as embedded into $\mathbb{C}$ as the real axis.
Given a complex number \( z = x + iy \), the numbers \( x \) and \( y \) are called its real and imaginary parts, respectively, and denoted by \( \text{Re} \, z \) and \( \text{Im} \, z \), so that \( z = \text{Re} \, z + i \text{Im} \, z \).

For any non-zero complex number \( z = a + bi \) there exists an inverse \( z^{-1} \) such that \( z^{-1}z = 1 \). Indeed, we can set
\[
z^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.
\]

The commutativity, associativity and existence of the inverse is easy to check, but it should not be taken for granted: it is impossible to define a similar operation any \( \mathbb{R}^n \) for \( n > 2 \).

Given \( z = a + bi \in \mathbb{C} \) its conjugate is defined as \( \overline{z} = a - bi \). The conjugation operation \( z \mapsto \overline{z} \) is the reflection of \( \mathbb{C} \) with respect to the real axis \( \mathbb{R} \subset \mathbb{C} \). Note that
\[
\text{Re} \, z = \frac{1}{2}(z + \overline{z}), \quad \text{Im} \, z = \frac{1}{2i}(z - \overline{z}).
\]

Let us introduce the polar coordinates \( (r, \phi) \) in \( \mathbb{R}^2 = \mathbb{C} \). Then a complex number \( z = x + yi \) can be written as \( r \cos \phi + ir \sin \phi = r(\cos \phi + i \sin \phi) \). This form of writing a complex number is called, sometimes, trigonometric. The number \( r = \sqrt{x^2 + y^2} \) is called the modulus of \( z \) and denoted by \( |z| \) and \( \phi \) is called the argument of \( \phi \) and denoted by \( \text{arg} \, z \). Note that the argument is defined only mod \( 2\pi \). The value of the argument in \( [0, 2\pi) \) is sometimes called the principal value of the argument. When \( z \) is real than its modulus \( |z| \) is just the absolute value. We also not that \( |z| = \sqrt{|z|^2} \).

An important role plays the triangle inequality
\[
|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|.
\]

**Exponential function of a complex variable**

Recall that the exponential function \( e^x \) has a Taylor expansion
\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots.
\]

We then define for a complex \( z \) the exponential function by the same formula
\[
e^z := 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \ldots.
\]
One can check that this power series absolutely converging for all \( z \) and satisfies the formula

\[
e^{z_1 + z_2} = e^{z_1} e^{z_2}.
\]

In particular, we have

\[
e^{iy} = 1 + iy - \frac{y^2}{2!} + i\frac{y^3}{3!} - \frac{y^4}{4!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{2k!} + i \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}.
\]

But \( \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{2k!} = \cos y \) and \( \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!} = \sin y \), and hence we get Euler’s formula

\[
e^{iy} = \cos y + i \sin y,
\]

and furthermore,

\[
e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y),
\]

i.e. \( |e^{x+iy}| = e^x \), \( \arg(e^z) = y \).

In particular, any complex number \( z = r(\cos \phi + i \sin \phi) \) can be rewritten in the form \( z = re^{i\phi} \). This is called the exponential form of the complex number \( z \).

Given \( z_1 = r_1 e^{i\phi_1}, z_2 = r_2 e^{i\phi_2} \) we get

\[
z_1 z_2 = (r_1 e^{i\phi_1})(r_2 e^{i\phi_2}) = r_1 r_2 e^{i(\phi_1 + \phi_2)},
\]

i.e. when multiplying complex numbers their moduli are being multiplied and arguments added (but be aware that arguments are defined mod 2\( \pi \)).

Note that

\[
(e^{i\phi})^n = e^{i n \phi},
\]

and hence if \( z = re^{i\phi} \) then \( z^n = r^n e^{i n \phi} = r^n (\cos n\phi + i \sin n\phi) \).

Note that the operation \( z \mapsto iz \) is the rotation of \( \mathbb{C} \) counterclockwise by the angle \( \frac{\pi}{2} \). More generally a multiplication operation \( z \mapsto zw \), where \( w = \rho e^{i\theta} \) is the composition of a rotation by the angle \( \theta \) and a radial dilatation (homothety) in \( \rho \) times.

\[\text{Convergence of power series will be discussed later in Chapter 5}\]
Exercise 1.1.  
1. Compute $\sum_0^n \cos k\theta$ and $\sum_1^n \sin k\theta$. 
2. Compute $1 + \left(\frac{n}{4}\right) + \left(\frac{n}{8}\right) + \left(\frac{n}{12}\right) + \ldots$.

1.2 Complex linear function from the real perspective

A linear map $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is by definition is required to satisfy two conditions:

\[ F(z_1 + z_2) = F(z_1) + F(z_2); \]
\[ F(\lambda z) = \lambda F(z), \]

where $z_1, z_2, z$ are any vectors from $\mathbb{R}^2$ and $\lambda \in \mathbb{R}$ is a real number. Any such map is a multiplication by a $2 \times 2$-matrix:

\[ F(z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \]

A linear function of one complex variable is a linear map $F : \mathbb{C} \rightarrow \mathbb{C}$ satisfies in addition the condition

\[ F(\lambda z) = \lambda F(z), \quad \text{for any complex number } \lambda. \quad (1.2.1) \]

Any such function has to satisfy $F(z) = F(1)z = kz$, where $k = a + ib = F(1)$. Equivalently,

\[ F(x + iy) = (a + ib)(x + iy) = ax - by + i(ay + bx). \]

Thus, viewing a complex number $z = x + iy$ as a vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ we get

\[ F(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \]

In other words, we proved the following

Lemma 1.2. A real linear map $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is complex linear map $\mathbb{C} \rightarrow \mathbb{C}$ if and only if $a = d$ and $b = -c$. 

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In particular, the matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the matrix of multiplication by $i$.

Note that
\[
\det \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a^2 + b^2 = |c|^2, \quad \text{where} \ c = a + ib.
\] (1.2.2)

In other words, the (real) determinant of the matrix of the multiplication by a complex number $c$ is equal to $|c|^2$.

We can also view a real linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ as a map $\mathbb{R}^2 \to \mathbb{C}$, i.e. as a complex-valued linear (in a real sense) function $F(x, y) = f_1(x, y) + if_2(x, y)$. If $F$ was given by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then
\[
f_1(x, y) = ax + by, f_2(x, y) = cx + dy.
\]
We also have
\[
F(x, y) = f_1(x, y) + if_2(x, y) = ax + by + i(cx + dy) = (a + ic)x + (b + id)y = Ax + By.
\] (1.2.3)

Note that $x = \frac{1}{2}(z + \bar{z}), y = -\frac{i}{2}(z - \bar{z})$, and hence
\[
F(x, y) = Ax + By = \frac{A}{2}(z + \bar{z}) - \frac{Bi}{2}(z - \bar{z}) = \frac{1}{2}(A - iB)z + \frac{1}{2}(A + iB)\bar{z} = \alpha z + \beta \bar{z},
\] (1.2.4)
where we denoted $\alpha := \frac{1}{2}(A - iB), \beta := \frac{1}{2}(A + iB)$. Note that the function $l_1(z) = \alpha z$ is complex linear, while the function $l_2(z) = \beta \bar{z}$ is complex anti-linear, which means that it is linear in the real sense, but satisfied the condition $l_2(\lambda z) = \bar{\lambda} l_2(z)$.

If $F$ is a complex linear map, then $\bar{F}$ is anti-linear and vice versa. In particular, every complex anti-linear map $F$ has the form $F(z) = az$ for a complex number $a$.

The following lemma summarizes the above discussion.

**Lemma 1.3.** Any linear in the real sense map $F : \mathbb{C} \to \mathbb{C}$ can be uniquely written as a sum $F = F_1 + F_2$, where $F_1$ is complex linear and $F_2$ is complex anti-linear.

If $F$ is given by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $F_1(z) = az$, $F_2(z) = \beta \bar{z}$, where $\alpha := \frac{1}{2}(A - iB), \beta := \frac{1}{2}(A + iB)$.

$A = a + ic, B = b + id$. 

---

2 Complex-valued linear (in the real sense) functions on $\mathbb{R}^2$ form a 2-dimensional complex vector space. Formulas (1.2.3) and (1.2.4) say that the pairs of functions $(x,y)$ as well as the pair $(z, \bar{z})$ form a basis of this space.
Chapter 2

Holomorphic functions

2.1 Differentiability and the differential

For any point \( z = (x, y) \in \mathbb{R}^2 \) we denote by \( \mathbb{R}^2_a \) the space \( \mathbb{R}^2 \) with the origin, shifted to the point \( a \).
Though the parallel transport allows one to identify spaces \( \mathbb{R}^2 \) and \( \mathbb{R}^2_a \) it will be important for us to think about them as different spaces.

Let \( U \) be a domain in \( \mathbb{R}^2 \). A vector-valued function \( f : U \to \mathbb{R}^2 \), where \( U \subset \mathbb{R}^2 \) a domain in \( \mathbb{R}^2 \), is called \textit{differentiable} at a point \( a \in U \) if near the point \( a \) in can be well approximated by a linear function. More precisely, if there exists a linear map \( A : \mathbb{R}^2_a \to \mathbb{R}^2 \) such that

\[
 f(a + h) - f(a) = A(h) + o(\|h\|)
\]

for any sufficiently small vector \( h = (h_1, h_2) \in \mathbb{R}^2 \), where the notation \( o(t) \) stands for any vector-valued function such that \( \frac{o(t)}{t} \to 0 \). The linear function \( A \) is called the \textit{differential} of the function \( f \) at the point \( a \) and is denoted by \( d_a f \). In other words, \( f \) is differentiable at \( a \in U \) if for any \( h \in \mathbb{R}^2_a \) there exists a limit

\[
 d_a f(h) = A(h) = \lim_{t \to 0} \frac{f(a + th) - f(a)}{t},
\]

and the limit \( A(h) \) \textit{linearly} depends on the vector \( h \). By identifying \( \mathbb{R}^2_a \) and with \( \mathbb{R}^2 \) via the parallel transport we can associate with the linear map \( d_a f \) its matrix \( J_a(f) \), called the \textit{Jacobi matrix} or
derivative of the map $f$. If we denote by $u(x,y)$ and $v(x,y)$ the coordinate functions of the map $f$, i.e.

$$f(x,y) = (u(x,y), v(x,y))$$

then

$$J_a(f) = \begin{pmatrix}
\frac{\partial u}{\partial x}(a) & \frac{\partial u}{\partial y}(a) \\
\frac{\partial v}{\partial x}(a) & \frac{\partial v}{\partial y}(a)
\end{pmatrix}.$$  

The function $f$ is called differentiable on the whole domain $U$ if it is differentiable at each point of $U$.

### 2.2 Holomorphic functions, Cauchy-Riemann equations

Let us now view the map $f : U \to \mathbb{R}^2$ as a complex valued function $f(z) = u(z) + iv(z)$, $z = x + iy$.

The function $f$ is called differentiable at the point $a$ in a complex sense, or holomorphic at $a$ if the differential $d_a f$ is a complex linear map. The following theorem lists equivalent definitions of holomorphicity.

**Theorem 2.1.** The function $f = u + iv : U \to \mathbb{C}$ is holomorphic at a point $a \in U$ if one of the following equivalent conditions is satisfied:

1. $f(a + h) - f(a) = ch + o(|h|)$, for a complex number $c$;

2. There exists a limit

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}, \quad \text{denoted by} \quad f'(a)$$

and called the complex derivative of $f$ at the point $a$;

3. $f$ is differentiable in the real sense and the following Cauchy–Riemann equations are satisfied at the point $a$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$  

(2.2.1)
(4) \( f \) is differentiable in the real sense and
\[
\frac{\partial f}{\partial \bar{z}}(a) := \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) + i \frac{\partial f}{\partial y}(a) \right) = 0.
\]

**Proof.** Statement (1) is just a reformulation of the fact that the differential \( d_a f \) is a complex linear map. Equivalence (1) and (2) is straightforward. According to Lemma 1.2, condition (3) just means that the Jacobi matrix is a matrix of a complex linear map.

To deal with condition (4) let us recall that according to Lemma 1.3, for any differentiable at \( a \) function \( f \) we can decompose the linear map \( d_a f \) into a complex linear and anti-linear:
\[
d_a f = \partial_a f + \bar{\partial}_a f,
\]
so that we have \( \partial_a f(h) = \alpha h, \bar{\partial}_a f(h) = \beta \bar{h} \). Rephrasing Lemma 1.3, we have
\[
\alpha = \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) - i \frac{\partial f}{\partial y}(a) \right) h + \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) + i \frac{\partial f}{\partial y}(a) \right) \bar{h}.
\]

If we introduce the notation
\[
\frac{\partial f}{\partial z}(a) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) - i \frac{\partial f}{\partial y}(a) \right),
\]
\[
\frac{\partial f}{\partial \bar{z}}(a) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) + i \frac{\partial f}{\partial y}(a) \right),
\]
then we can write
\[
d_a f(h) = \frac{\partial f}{\partial z}(a) h + \frac{\partial f}{\partial \bar{z}}(a) \bar{h}.
\]
Hence, \( f \) is holomorphic at a point \( a \) if and only if
\[
\frac{\partial f}{\partial \bar{z}}(a) = 0,
\]
and this condition is just another form of the Cauchy-Riemann equations (2.2.1).

It is important to note that \( f \) is holomorphic at \( a \), i.e. \( \frac{\partial f}{\partial \bar{z}}(a) = 0 \) then \( f'(a) = \frac{\partial f}{\partial \bar{z}}(a) \).

Note that if \( f \) is holomorphic at the point \( a \) then the determinant of the real Jacobi matrix is given by
\[
\det J(f)(a) = \begin{vmatrix}
\frac{\partial u}{\partial x}(a) & \frac{\partial u}{\partial y}(a) \\
\frac{\partial v}{\partial x}(a) & \frac{\partial v}{\partial y}(a)
\end{vmatrix} = |f'(a)|^2.
\]
see formula (1.2.2)

The function \( f : U \rightarrow \mathbb{C} \) is called holomorphic in \( U \) if it is holomorphic at every point of \( U \). This is equivalent to the condition \( \frac{\partial f}{\partial \overline{z}} = 0 \) in \( U \).

The following proposition summarizes property of complex differentiation which are analogous to the corresponding facts in the real case.

**Proposition 2.2.**  (1) If \( f, g \) are holomorphic at \( a \in \mathbb{C} \) then \( f \pm g \) and \( fg \) are holomorphic at \( a \) and

\[
(f \pm g)'(a) = f'(a) \pm g'(a), \quad (fg)'(a) = f'(a)g(a) + f(a)g'(a);
\]

if \( g(a) \neq 0 \) then \( \frac{f}{g} \) is holomorphic at \( a \) and

\[
\left( \frac{f}{g} \right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)};
\]

(2) If \( f \) is holomorphic at \( a \) and \( g \) is holomorphic at \( f(a) \) then the composition \( g \circ f \) is holomorphic at the point \( a \) and

\[
(g \circ f)'(a) = g'(f(a))f'(a).
\]

The proof of (1) repeats the corresponding proofs in the real case, while (2) is the chain rule with an additional observation that a composition of complex linear maps is itself complex linear.

Proposition 2.2 implies, for instance that \( (z^n)' = nz^{n-1} \) for any integer \( n \).

According to our definition of a holomorphic function it is not even clear whether this function is \( C^1 \)-smooth, i.e. whether its derivative continuously depends on a point of the domain. It turns out that this is automatically true, which is the subject of the following theorem.

**Theorem 2.3** (H. Looman, D. Menchoff). Every holomorphic function in a domain \( U \) is of class \( C^1 \), i.e. its derivative continuously depends on the point of \( U \).

The proof of this theorem is given in Section 4.4.1 below.

We will assume in what follows the conclusion of this theorem, i.e. that a holomorphic function is of class \( C^1 \) and will show that this in turn implies that a holomorphic function is infinitely differentiable, and moreover analytic, i.e it is equal to the sum of its converging Taylor series expansion in a neighborhood of each point of \( U \).
2.3 Complex derivative and directional derivatives

Let $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is an open domain, is a complex valued differentiable in a real sense, but not necessarily holomorphic function. Recall that for any point $a \in U$ and a vector $v \in \mathbb{C}$ the directional derivative $\frac{\partial f}{\partial v}(a)$ by definition is the value of the differential $d_a f$ on the vector $v$, i.e.

$$\frac{\partial f}{\partial v}(a) = d_a f(v) = \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t}.$$ 

In particular,

$$\frac{\partial f}{\partial x}(a) = d_a f(1), \quad \frac{\partial f}{\partial y}(a) = d_a f(i).$$

The derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ cannot be interpreted as partial derivatives of the function $f$. However, it is possible, though maybe not necessarily very insightful, to interpret them as directional derivatives of a function of two complex variables, as it is explained below.

Let us write $z = x + iy$, $w = u + iv$. Consider vector fields $T = \frac{1}{2}(1, -i)$ and $S = \frac{1}{2}(1, i)$.

**Lemma 2.4.** For a complex valued function $f : U \to \mathbb{C}$ consider the function

$$F(z, w) = f(z) + if(w), \quad (z, w) \in U \times U \subset \mathbb{C}^2$$

defined on the domain $U \times U \subset \mathbb{C}^2$. Then

$$\frac{\partial f}{\partial z}(a) = d_{(a,a)} F(T), \quad \frac{\partial f}{\partial \bar{z}}(a) = d_{(a,a)} F(S).$$

**Proof.** We have

$$d_{(a,a)} F(T) = \frac{1}{2} (d_a f(1) - d_a f(i)) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) - i \frac{\partial f}{\partial y}(a) \right) = \frac{\partial f}{\partial z}(a).$$

$$d_{(a,a)} F(S) = \frac{1}{2} (d_a f(1) + d_a f(i)) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) + i \frac{\partial f}{\partial y}(a) \right) = \frac{\partial f}{\partial \bar{z}}(a).$$

If $f$ is holomorphic at $a$ then for any vector $v \in \mathbb{C}$ we have

$$\frac{\partial f}{\partial v}(a) = d_a f(v) = f'(a)v,$$
where we identify $v$ with a complex number $v \in \mathbb{C}$ via a parallel transport. For instance,

$$\frac{\partial f}{\partial x}(a) = d_a f(1) = f'(a) \cdot 1 = f'(a),$$

$$\frac{\partial f}{\partial y}(a) = d_a f(i) = i f'(a).$$

In this case $\frac{\partial f}{\partial z}(a) = f'(z) = \frac{\partial f}{\partial x}(a) \text{ and } \frac{\partial f}{\partial z}(a) = 0.$
Chapter 3

Differential 1-forms and their integration

3.1 Complex-valued differential 1-forms

Let us first recall some basics of the theory of real differential forms. For our purposes we will need only 1-forms on domains in $\mathbb{R}^2$. By definition a differential 1-form $\lambda$ on a domain $U \subset \mathbb{R}^2$ is a field of linear functions $\lambda_z : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Thus, a differential 1-form is a function of arguments of 2 kind: of a point $z \in U$ and a vector $h \in \mathbb{R}^2$. It depends linearly on $h$ and arbitrarily (but usually continuously and even smoothly) on $z$, i.e. we have

$$\lambda_z(h) = a_1(z)h_1 + a_2(z)h_2,$$

where $h_1, h_2$ are Cartesian coordinates of $h \in \mathbb{R}^2$. Any differential 1-form can be multiplied by a function ("a field of scalars"): $(f\lambda)_z(h) = f(z)\lambda_z(h)$.

Given a real-valued function $f : U \rightarrow \mathbb{R}^2$ on $U$ its differential $df$ is an example of a differential form:

$$d_z(f)(h) = \frac{\partial f}{\partial x}(z)h_1 + \frac{\partial f}{\partial y}(z)h_2.$$  

In particular differentials $dx$ and $dy$ of the coordinate functions $x, y$ are also differential 1-forms, and any other differential form can be written as a linear combination of $dx$ and $dy$:

$$\lambda = Pdx + Qdy,$$
where \( P, Q : U \to \mathbb{R} \) are functions on the domain \( U \). In particular, we have

\[
d f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.
\]

A differential 1-form \( \lambda \) is called \textit{exact} if \( \lambda = df \). The function \( f \) is called the \textit{primitive} of the 1-form \( \lambda \). The primitive is defined uniquely up to adding a constant.

Not every closed differential 1-form \( \lambda = Pdx + Qdy \) is exact. The necessary condition for exactness is that \( \lambda \) is \textit{closed} which by definition means \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \). The necessity of this condition for exactness follows from the mixed derivatives equality (assuming that the coefficients \( P, Q \) are \( C^1 \)-smooth). Indeed, if \( P = \frac{\partial f}{\partial x} \) and \( Q = \frac{\partial f}{\partial y} \). Then

\[
\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.
\]

On the other hand the closedness of \( \lambda \) is not sufficient for its exactness, as it is demonstrated by an example of a 1-form \( d\phi \) on \( \mathbb{R}^2 \setminus 0 \) (written in polar coordinates), or

\[
d\phi = \frac{1}{x^2 + y^2} (xdy - ydx)
\]

in Cartesian coordinates. We will discuss a bit later the precise argument for that, but it is intuitively clear that the primitive of this form is not a univalent function on \( \mathbb{R}^2 \setminus 0 \). On the other hand, as we will see below, \textit{any closed 1-form on} \( \mathbb{R}^2 \), \textit{or more generally on any simply connected domain in} \( \mathbb{R}^2 \) \textit{is exact}.

We will also consider complex-valued differential 1-forms. A \( \mathbb{C} \)-valued differential 1-form is a field of \( \mathbb{C} \)-valued linear in the real sense functions, or simply it is an expression \( \alpha + i\beta \), where \( \alpha, \beta \) are usual real-valued differential 1-forms. All usual operations on complex valued 1-forms are defined in the same way as for real-valued forms, and in addition such forms can be multiplied not only by real-valued but also by complex-valued functions.

Note that a complex-valued function (or 0-form) on a domain \( U \subset \mathbb{C} \) is just a map \( f = u + iv : U \to \mathbb{C} \). Its differential \( df \) is the same as the differential of this map, but it also can be viewed as a \( \mathbb{C} \)-valued differential 1-form \( df = du + idv \).
Example 3.1.

\[ dz = dx + idy, \quad d\bar{z} = dx - idy, \quad zdz = (x + iy)(dx + idy) = xdx - ydy + i(xdy + ydx), \]

We have

\[ dx = \frac{1}{2}(dz + d\bar{z}), \quad dy = -\frac{i}{2}(dz - d\bar{z}). \]

Hence, any complex valued 1-form \( \lambda \) can be written as a linear combination of forms \( dz \) and \( d\bar{z} \):

\[ \lambda = f dz + gd\bar{z}, \]

which is a decomposition of \( \lambda \) into a sum of complex linear and complex anti-linear parts.

Lemma 3.2. The form \( \lambda = f dz + gd\bar{z} \) is closed if and only if

\[ \frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial z}. \]

Proof.

\[ \lambda = f dz + gd\bar{z} = f(dx + idy) + g(dx - idy) = (f + g)dx + i(f - g)dy. \]

The closedness of \( \lambda \) means by definition that

\[ \frac{\partial(f + g)}{\partial y} = \frac{\partial(i(f - g))}{\partial x}, \]

which is equivalent to

\[ \frac{\partial f}{\partial y} - i\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y} - i\frac{\partial g}{\partial x}. \]

Dividing both parts by \((-i)\) we get

\[ \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}, \]

and hence

\[ \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y} \right) = \frac{\partial g}{\partial z}. \]

\[ \blacksquare \]

Let us express the differential \( df \) of a complex valued function \( f \) as a combination of differential forms \( dz = dx + idy \) and \( d\bar{z} = dx - idy \) parts.
Lemma 3.3. For any complex valued function \( f = u + iv \) we have
\[
d f = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}.
\]
In particular, when \( f \) is holomorphic we have
\[
d f = \frac{\partial f}{\partial z} dz = f'(z) dz.
\]

Proof. We have
\[
d f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) dz + \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) d\bar{z} = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}.
\]

3.2 Holomorphic 1-forms

A complex-valued 1-form \( \lambda \) is called holomorphic if \( \lambda = f dz \) for a holomorphic function \( f \).

Lemma 3.4. The form \( f dz \) is closed in a domain \( U \) if and only if the function \( f \) is holomorphic in \( U \).

Proof. According to Lemma 3.2 closedness of \( f dz \) is equivalent to \( \frac{\partial f}{\partial \bar{z}} = 0 \), which, in turn, is equivalent to the holomorphicity of \( f \).

Example 3.5. The holomorphic form \( \frac{dz}{z^n} \), \( n \geq 1 \), on \( \mathbb{C} \setminus 0 \) is always closed. It is exact if and only if \( n > 1 \).

Indeed, If \( n > 1 \) then
\[
\frac{dz}{z^n} = d\left( \frac{1}{(1-n)z^{n-1}} \right),
\]
i.e. \( \frac{dz}{z^n} \) is exact. If \( n = 1 \) we have in polar coordinates
\[
\frac{dz}{z} = \frac{d(r e^{i\phi})}{r e^{i\phi}} = \frac{e^{i\phi} dr + ire^{i\phi} d\phi}{r e^{i\phi}} = \frac{dr}{r} + id\phi = d(ln r) + id\phi,
\]
but we already discussed above that the form \( d\phi \) is not exact.
3.3 Integration of differential 1-forms along curves

Curves as paths

A path, or parametrically given curve in a domain $U \subset \mathbb{R}^2 : [a, b] \rightarrow U$. We will assume in what follows that all considered paths are differentiable. Given a differential 1-form $\alpha = Pdx + Qdy$ in $U$ we define the integral of $\alpha$ over $\gamma$ by the formula

$$\int_{\gamma} \alpha = \int_{a}^{b} \gamma^{*} \alpha.$$

Denoting the coordinate functions of $\gamma(t)$ by $x(t)$ and $y(t)$ (i.e. $\gamma(t) = (x(t), y(t))$) the pull-back differential form $\gamma^{*} \alpha$ is by definition equal to

$$\gamma^{*} \alpha = P(x(t), y(t))dx(t) + Q(x(t), y(t))dy(t) = (P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t))dt,$$

so that

$$\int_{\gamma} \alpha = \int_{a}^{b} \gamma^{*} \alpha = \int_{a}^{b} (P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t))dt.$$

An important property of the integral of a differential 1-form is that it does not depend on the parameterization of the curve.

**Proposition 3.6.** Let a path $\overline{\gamma}$ be obtained from $\gamma : [a, b] \rightarrow U$ by a reparameterization, i.e. $\overline{\gamma} = \gamma \circ \phi$, where $\phi : [c, d] \rightarrow [a, b]$ is an orientation preserving diffeomorphism. Then $\int_{\overline{\gamma}} \alpha = \int_{\gamma} \alpha$.

Thus the integral $\int_{\gamma} \alpha$ depends only on the curve $\gamma$ as an oriented submanifold and not on a particular parameterization which is compatible with the orientation. For instance, given a unit circle $S^1 = \{|z| = 1\}$ oriented counter-clockwise we can compute $\int_{S^1} d\phi = 2\pi$. Indeed, the circle can be parameterized by the angular coordinate $\phi \in [0, 2\pi]$, and this parameterization is compatible with the counter-clockwise orientation. Hence, $\int_{S^1} d\phi = \int_{0}^{2\pi} d\phi = 2\pi$.

**Exercise 3.7.** Compute $\int_{S^1} \frac{dz}{z}$
Solution. Let us parameterize the circle by polar coordinates: \( z = e^{i\phi}, \phi \in [0, 2\pi] \). Then \( \frac{dz}{z} = d(\ln r) + id\phi \) and \( \int_{S^1} dz = \int d(\ln r) + \int_0^{2\pi} d\phi = 0 + 2\pi i = 2\pi i \). Here we used the fact that the integral of an exact form \( d(\ln r) \) over the circle \( S^1 \) is 0, see Theorem 3.8 below.

3.4 Integrals of closed and exact differential 1-forms

**Theorem 3.8.** Let \( \alpha = df \) be an exact 1-form in a domain \( U \subset \mathbb{C} \). Then for any path \( \gamma : [a, b] \to U \) which connects points \( A = \gamma(a) \) and \( B = \gamma(b) \) we have

\[
\int_{\gamma} \alpha = f(B) - f(A).
\]

In particular, if \( \gamma \) is a loop then \( \oint_{\gamma} \alpha = 0 \).

Similarly for an oriented curve \( \Gamma \subset U \) with boundary \( \partial \Gamma = B - A \) we have

\[
\int_{\Gamma} \alpha = f(B) - f(A).
\]

**Proof.** We have \( \int_{\gamma} df = \int_{a}^{b} \gamma^* df = \int_{a}^{b} d(f \circ \gamma) = f(\gamma(b)) - f(\gamma(a)) = f(B) - f(A) \). ■

It turns out that closed forms are *locally exact*. A domain \( U \subset V \) is called *star-shaped* with respect to a point \( a \in V \) if with any point \( x \in U \) it contains the whole interval \( I_{a,x} \) connecting \( a \) and \( x \), i.e. \( I_{a,x} = \{a + t(x - a); \ t \in [0, 1]\} \). In particular, any convex domain is star-shaped.

**Proposition 3.9.** Let \( \alpha = Pdx + Qdy \) be a closed 1-form in a star-shaped domain \( U \subset \mathbb{R}^2 \). Then it is exact.

**Proof.** Define a function \( F : U \to \mathbb{R} \) by the formula

\[
F(x,y) = \int_{I_{a,z}} \alpha, \quad z = (x,y) \in U,
\]

where the intervals \( I_{a,z} \) are oriented from \( a \) to \( z \).
We claim that \( dF = \alpha \). Let us choose \( a \) as the origin. Then \( I_{0, z} \) can be parameterized by

\[ t \mapsto tz, \quad t \in [0, 1]. \]

Hence,

\[
F(z) = \int_{\rightarrow I_{0, z}} \alpha = \int_{0}^{1} P(tx, ty)dt + Q(tx, ty)dy = \int_{0}^{1} (xP(tx, ty) + yQ(tx, ty))dt
\]  

(3.4.1)

Taking partial derivatives of the integral with respect to \( x \) and \( y \) we get

\[
\frac{\partial F}{\partial x}(x, y) = \int_{0}^{1} \left( tx\frac{\partial P}{\partial x}(tx, ty) + ty\frac{\partial Q}{\partial x}(tx, ty) \right) dt + \int_{0}^{1} P(tx, ty)dt;
\]

\[
\frac{\partial F}{\partial y}(x, y) = \int_{0}^{1} \left( tx\frac{\partial P}{\partial y}(tx, ty) + ty\frac{\partial Q}{\partial y}(tx, ty) \right) dt + \int_{0}^{1} Q(tx, ty)dt
\]

By our assumption the form \( \alpha \) is closed, and hence \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \). Using this we can further write

\[
\frac{\partial F}{\partial x}(x, y) = \int_{0}^{1} \left( tx\frac{\partial P}{\partial x}(tx, ty) + ty\frac{\partial P}{\partial y}(tx, ty) \right) dt + \int_{0}^{1} P(tx, ty)dt;
\]

\[
= \int_{0}^{1} t\frac{\partial P}{\partial t}(tx, ty)dt + \int_{0}^{1} P(tx, ty) = tP(x, y) \bigg|_{0}^{1} - \int_{0}^{1} P(tx, ty)dt + \int_{0}^{1} P(tx, ty)dt = P(x, y);
\]

\[
\frac{\partial F}{\partial y}(x, y) = \int_{0}^{1} \left( tx\frac{\partial Q}{\partial x}(tx, ty) + ty\frac{\partial Q}{\partial y}(tx, ty) \right) dt + \int_{0}^{1} Q(tx, ty)dt
\]

\[
= \int_{0}^{1} t\frac{\partial Q}{\partial t}(tx, ty)dt + \int_{0}^{1} Q(tx, ty) = tQ(x, y) \bigg|_{0}^{1} - \int_{0}^{1} Q(tx, ty)dt + \int_{0}^{1} Q(tx, ty)dt = Q(x, y);
\]

Thus

\[ dF = Pdx + Qdy = \alpha \]
Given a differential 1-form $\alpha = Pdx + Qdy$ we will define $\int_{\Gamma} |\alpha|$ as

$$\int_{\Gamma} |\alpha| := \int_{a}^{b} |P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)| \, dt,$$

where $(x(t), y(t), t \in [a, b])$ is a parameterization of $\Gamma$. Unlike $\int_{\Gamma} \alpha$ the integral $\int_{\Gamma} |\alpha|$ is non-negative and does not depend on the orientation of $\Gamma$. Clearly, we have

$$\left| \int_{\Gamma} \alpha \right| \leq \int_{\Gamma} |\alpha|,$$

and

$$\int_{\Gamma} |\alpha + \beta| = \int_{\Gamma} |\alpha| + \int_{\Gamma} |\beta|.$$
Chapter 4

Cauchy integral formula

4.1 Stokes/Green theorem

Given a bounded domain \( U \subset \mathbb{C} \) with a smooth (or piece-wise smooth boundary) we will always orient it boundary \( \partial U \) as follows. For each point \( p \in \partial U \) take an outward normal vector \( \nu \). Then \( i\nu \) is tangent to \( \partial U \) and defines its orientation. For instance, suppose \( A \) is the annulus \( 1 \leq |z| \leq 2 \). Its boundary is the union of two circles: \( S_1 = \{|z| = 1\} \) and \( S_2 = \{|z| = 2\} \). Then \( A \) induces on the outer circle \( S_2 \) the counter-clockwise orientation, and on the inner circle \( S_1 \) the clockwise orientation.

The fundamentally important fact about integration of 1-forms is the following theorem which belongs to George Green and it is a special case of a more general result, called Stokes’ theorem (which was not actually proved by George Stokes!)

**Theorem 4.1.** Let \( U \subset \mathbb{C} \) be a bounded domain with a piecewise smooth boundary \( \partial U \), and \( \alpha = Pdx + Qdy \) a differential 1-form on \( U \) with coefficients which are \( C^1 \)-smooth in \( U \) and continuous in the closure \( \overline{U} \). Let us orient the curve \( \partial U \) as the boundary of \( U \). Then

\[
\int_{\partial U} Pdx + Qdy = \iint_{U} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.
\]

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**Corollary 4.2.** Suppose a 1-form $\alpha$ is closed in a domain $U$. Then

$$\int_{\partial U} \alpha = 0.$$ 

Indeed, closedness of $\alpha = Pdx + Qdy$ just means that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

### 4.2 Area computation

If for $\alpha = Pdx + Qdy$ we have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ in the closure $\overline{U}$ of the domain $U$, then Green’s formula yields

$$\int_{\partial U} Pdx + Qdy = \iint_{U} dxdy = \text{Area}(U).$$ 

For instance this is the case for $\alpha = xdy$, $-ydx$ or $\frac{1}{2}(xdy - ydx)$. Another example of such 1-form $\alpha$ is the form

$$\alpha = -\frac{i}{2}zd\overline{z} = -\frac{i}{2}(x - iy)(dx + idy) = \frac{1}{2}(xdy - ydx) - \frac{i}{2}d(xy).$$

**Proposition 4.3.** Let $\Gamma \in \mathbb{C}$ be a piecewise smooth curve, $\Omega \supset \Gamma$ is neighborhood and $f : \Omega \to \mathbb{C}$ a holomorphic function such that $f(\Gamma) \subset \mathbb{C}$ bounds a domain $U \subset \mathbb{C}$. Suppose $f(\Gamma)$ is oriented as the boundary of $U$ and $\Gamma$ is oriented accordingly and $f$ preserves these orientations. Then

$$\text{Area}(U) = -\frac{i}{2} \int_{\Gamma} \overline{f(z)} f'(z)dz.$$ 

**Proof.** Using Green’s formula together with the change of variable formula we get

$$\text{Area}(f(U)) = -\frac{i}{2} \int_{\overline{f(\Gamma)}} \overline{z}dz = -\frac{i}{2} \int_{\overline{f(\Gamma)}} \overline{f(z)}df(z) = -\frac{i}{2} \int_{\Gamma} \overline{f(z)} f'(z)dz.$$
4.3 Cauchy theorem and Cauchy integral formula

**Corollary 4.4** (Cauchy theorem). Let $f$ be a function, holomorphic in the domain $U$ and continuous up to the boundary. Then

$$\int_{\partial U} f(z)dz = 0.$$ 

**Proof.** According to Lemma 3.4 the holomorphic differential 1-form $f(z)dz$ is closed in $U$. ■

**Example 4.5.** Let $U \in \mathbb{C}$ be any domain such that $0 \in U$. Then

$$\int_{\partial U} \frac{dz}{z^n} = \begin{cases} 2\pi i, & n = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Indeed, for $n > 1$ the 1-form $\frac{dz}{z^n}$ is exact in $U \setminus 0$,

$$\frac{dz}{z^n} = d \left(\frac{1}{(1-n)z^{n-1}}\right),$$

and the integral of an exact form over any closed loop is equal to 0.

If $n = 1$ consider a disc $D_\epsilon = \{|z| < \epsilon\} \subset U$. Then according to Corollary 4.2

$$0 = \int_{\partial(U \setminus \{D_1\})} \frac{dz}{z} = \int_{\partial U} \frac{dz}{z} - \int_{\partial D_\epsilon} \frac{dz}{z}.$$

But we already computed that $\int_{\{|z| = \epsilon\}} \frac{dz}{z} = 2\pi i$, and therefore

$$\int_{\partial U} \frac{dz}{z} = 2\pi i.$$

**Theorem 4.6** (Cauchy integral formula). Suppose that $f : \overline{U} \rightarrow \mathbb{C}$ is a continuous function which is holomorphic in $U$. Then for any $u \in U$ we have

$$\frac{1}{2\pi i} \int_{\partial U} \frac{f(z)dz}{z - u} = f(u).$$
Proof. Let \( D_\delta(u) \) denote the disc \( \{|z-u|<\epsilon\} \) centered at \( u \), where \( 0<\delta<|u|\). The function \( \frac{f(z)}{z-u} \) is holomorphic in \( U\setminus u \), and therefore according to the Cauchy theorem 4.4 we have

\[
\int_{\partial(U\setminus D_\delta(u))} \frac{f(z)dz}{z-u} = 0.
\]

Hence,

\[
\int_{\partial U} \frac{f(z)dz}{z-u} = \int_{\partial D_\delta(u)} \frac{f(z)dz}{z-u} = \int_{|w|=\delta} \frac{f(u+w)dw}{w}.
\]

The function \( f \) is continuous at the point \( u \). Hence, for any \( \epsilon \) there exists \( \delta>0 \) such that if \( |w|\leq \delta \) then

\[
|f(u+w)-f(u)|<\epsilon.
\]

Hence,

\[
\left| \int_{|w|=\delta} \frac{f(u+w)dw}{w} - \int_{|w|=\delta} \frac{f(u)dw}{w} \right| \leq \int_{0}^{2\pi} \frac{|f(u+w)-f(u)|\delta d\phi}{\delta} \leq 2\pi\epsilon.
\]

Note that according to Example 4.5

\[
\int_{|w|=\delta} \frac{f(u)dw}{w} = f(u) \int_{|w|=\delta} \frac{dw}{w} = 2\pi if(u).
\]

Thus

\[
\left| \int_{\partial U} \frac{f(z)dz}{z-u} - 2\pi i f(u) \right| \leq 2\pi\epsilon
\]

for any \( \epsilon > 0 \). But the left-hand side is independent of \( \epsilon \), and therefore

\[
\frac{1}{2\pi i} \int_{\partial U} \frac{f(z)dz}{z-u} = f(u).
\]

As the first application of the Cauchy integral formula we prove the infinite differentiability of a holomorphic function.
Corollary 4.7. Any holomorphic in a domain $U$ function is infinitely differentiable at every point. Its derivatives can be computed by the formula

$$f^{(k)}(u) = \frac{k!}{2\pi i} \int_{\partial U} \frac{f(z)dz}{(z-u)^{k+1}}.$$  

Proof. The variable $u$ enters the integral $\int_{\partial D} \frac{f(z)dz}{z-u}$ as a parameter. The integrand $\frac{f(z)dz}{z-u}$ is differentiable with respect to the parameter, and hence the integral itself is differentiable with respect to $u$ and we can compute the derivative $f'(u)$ by the differentiating the integral with respect to the parameter, i.e.

$$f'(u) = \frac{d}{du} \left( \frac{1}{2\pi i} \int_{\partial U} \frac{f(z)dz}{z-u} \right) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(z)dz}{z-u}.$$

Applying the same argument to the integral $\int_{\partial U} \frac{f(z)dz}{(z-u)^2}$ we compute $f''(u)$, etc. \[\blacksquare\]

Corollary 4.8 (Cauchy inequality). Let $f : U \rightarrow \mathbb{C}$ be a holomorphic function. Suppose that for a point $z_0$ the closed disc $\overline{D}_r(z_0) = \{ |z-z_0| \leq r \}$ is contained in $U$. Then

$$|f^{(n)}(z_0)| \leq \frac{Mn!}{r^n},$$

where $M := \max_{|z-z_0|=r} |f(z)|$.

Proof. By the Cauchy integral formula we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{|\zeta|=r} \frac{f(z_0 + \zeta) d\zeta}{\zeta^{n+1}}.$$

Therefore,

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{M r d\theta}{r^{n+1}} = \frac{Mn!}{r^n}.$$  

\[\blacksquare\]
4.4 Integral criterion for exactness

**Theorem 4.9.** Let \( \alpha = P \, dx + Q \, dy \) be a differential form with continuous coefficients in a domain \( U \subset \mathbb{C} \). Suppose that for any piecewise smooth loop \( \gamma : [a, b] \to U, \gamma(a) = \gamma(b) \), we have \( \int_{\gamma} \alpha = 0 \). Then the form \( \alpha \) is exact.

**Proof.** We can assume that \( U \) is connected. Otherwise the same argument can be repeated for each connected component. Choose any point \( a \in U \). For any other point \( z \in U \) choose a path \( \gamma_z \) connecting \( a \) to \( z \) and oriented from \( a \) to \( z \). Define

\[
 f(z) := \int_{\gamma_z} \alpha.
\]

Then \( f(z) \) is independent of the choice of the connecting path \( \gamma_z \). Indeed, any other choice differs by an integral over a loop, which is by assumption is equal to 0. We claim that \( f \) is differentiable and \( df = \alpha \). Indeed, for any point \( z = (x, y) \in U \) and a small \( t \) we have

\[
 f(x + t, y) - f(x, y) = \int_{l_t} \alpha,
\]

where \( l_t \) is a straight interval connecting the point \( (x, y) \) with the point \( (x + t, y) \). Hence

\[
 f(x + t, y) - f(x, y) = \int_{l_t} \alpha = \int_{x}^{x+t} P(u)du.
\]

Hence

\[
 \frac{\partial f}{\partial x}(x, y) = \lim_{t \to 0} \frac{f(x + t, y) - f(x, y)}{t} = \frac{d}{dt} \left( \int_{x}^{x+t} P(u)du \right) = P(x, y).
\]

Similarly, we get

\[
 \frac{\partial f}{\partial y}(x, y) = Q(x, y).
\]

Thus the function \( f \) has both partial derivatives, which are by our assumption are continuous. This
implies differentiability of $f$. Indeed, for any point $a = (x, y)$ and a vector $h = (h_1, h_2)$ we have

\[
f(a + h) - f(a) = (f(x + h_1, y + h_2) - f(x + h_1, y)) + (f(x + h_1, y) - f(x))
\]

\[
= Q(x + h_1, y)h_2 + o(h_2) + P(x, y)h_1 + o(h_1) = (Q(x, y) + o(h_1))h_2 + P(x, y)h_1 + o(h_1)
\]

\[
= P(x, y)h_1 + Q(x, y)h_2 + (o(h_1)h_2 + o(h_1) + o(h_2)) = P(x, y)h_1 + Q(x, y)h_2 + o(|h|).
\]

And this is by definition means that $f$ is differentiable and that $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = Pdx + Qdy = \alpha$.

Remark 4.10. The analysis of the above proof shows that it is enough to assume that $\int_\gamma \alpha = 0$ only for piece-wise linear loops formed with only horizontal and vertical intervals. Moreover, to prove local exactness, or equivalently closedness of the form $\alpha$ it is sufficient to prove the statement only for boundaries of rectangulars. We will use this remark in the proof of the Looman-Menchoff’s theorem in Section 4.4.1 below.

4.4.1 Proof of Theorem 2.3

Green’s theorem required the form to be $C^1$-smooth. Hence, in order to deduce from it Cauchy’s theorem we had to assume $C^1$-smoothness of a holomorphic function. We prove below Looman-Menchoff’s theorem which shows that the $C^1$-condition follows from holomorphicity, i.e. complex differentiability at every point of the domain. The key in the proof is the following Lemma which is due to R. Narasimhan, see Section 1.6 in the book “Complex Analysis on One Variable” by R. Narasimhan and Y. Nievergelt, Birkhäuser, 2001.

Lemma 4.11. Suppose the function $f$ is holomorphic in the domain $U$, i.e. it is differentiable in the complex sense at every point of the domain $U$. Let $Q \subset U$ be any rectangular. Then $\int_{\partial Q} f(z)dz = 0$.

Proof. Denote the perimeter of the rectangular $Q$, i.e. the total length of its sides by $L$. Note that the diameter $d$ of the rectangular, i.e. the maximal distance between its points satisfies the the inequality $2d \leq L$. 37
Suppose the converse, i.e. that \( \left| \int_{\partial Q} f(z)dz \right| = a > 0 \). Let us partition the rectangular \( Q \) into 4 rectangulars \( Q_1, Q_2, Q_3 \) and \( Q_4 \) of perimeter \( \frac{L}{2} \). Then

\[
\int_{\partial Q} f(z)dz = \sum_{j=1}^{4} \int_{\partial Q_j} f(z)dz. \tag{4.4.1}
\]

Indeed, each side \( S \) of \( Q_j \) which is not contained in \( \partial Q \) is also a side of another rectangular \( Q_{j'} \), \( j' \neq j \). As the part of boundaries of \( Q_j \) and \( Q_{j'} \) the side \( S \) has opposite orientations, and hence the integrals over all parts of boundaries of \( Q_j \) which are not in \( \partial Q \) cancel, and hence we get equality (4.4.2). Then

\[
a = \left| \int_{\partial Q} f(z)dz \right| \leq \sum_{j=1}^{4} \left| \int_{\partial Q_j} f(z)dz \right|. \tag{4.4.2}
\]

Hence, there exists \( j_0 \in \{1, 2, 3, 4\} \) we have

\[
\left| \int_{\partial Q_{j_0}} f(z)dz \right| \geq \frac{a}{4}.
\]

Now we partition the rectangular \( Q_{j_0} \) into 4 rectangulars \( Q_{j_01}, Q_{j_02}, Q_{j_03}, Q_{j_04} \) of perimeter \( \frac{L}{4} \) and again conclude that there exists \( j_1 \in \{1, 2, 3, 4\} \) such that

\[
\left| \int_{\partial Q_{j_{01}}} f(z)dz \right| \geq \frac{a}{16}.
\]

Continuing this process we find a sequence of rectangulars

\[
Q_{j_0} \supset Q_{j_{01}} \supset \cdots \supset Q_{j_{0j_1-j_k}} \supset \ldots,
\]

such the perimeter of \( \partial Q_{j_{0j_1-j_k}} \) is equal to \( \frac{L}{2^k} \) and

\[
\left| \int_{\partial Q_{j_{0j_1-j_k}}} f(z)dz \right| \geq \frac{a}{4^k}. \tag{4.4.3}
\]
The intersection

\[ Q_{j_0} \cap Q_{j_0,j_1} \cap \cdots \cap Q_{j_0,j_1\cdots j_k} \cap \cdots \]

consists of a unique point \( z_0 \in Q \). The function \( f \) is holomorphic at \( z_0 \) and hence for any \( \epsilon > 0 \) there exists \( K \) such that for \( k > K \) and any point \( z \in \partial Q_{j_0,j_1\cdots j_k} \) we have

\[
|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \epsilon|z - z_0| < \frac{\epsilon L}{2^k},
\]

because \( z - z_0| \leq \frac{L}{2} < L \). Note that

\[
\int_{\partial Q_{j_0,j_1\cdots j_k}} (f(z_0) + f'(z_0)(z - z_0))dz = 0,
\]

because the inhomogeneous linear function \( f(z_0) + f'(z_0)(z - z_0) \) is holomorphic and \( C^1 \)-smooth. Therefore, (4.4.5) implies that

\[
\left| \int_{Q_{j_0,j_1\cdots j_k}} f(z)dz \right| \leq \frac{\epsilon L^2}{4^k}.
\]

Choosing \( \epsilon < \frac{\alpha}{L^2} \) we get a contradiction with (4.4.3). \( \blacksquare \)

**Proof.** [Proof of Theorem 2.3] In view of Theorem 4.9 and Remark 4.10 Lemma 4.11 implies that the form \( f(z)dz \) is locally exact, i.e. in a neighborhood of each point there exists a function \( g(z) \) such that \( dg(z) = f(z)dz \), and hence \( g \) is holomorphic. But then we can apply Corollary 4.7 to conclude that the function \( f = g' \) is holomorphic as well. \( \blacksquare \)
Chapter 5

Convergent power series and holomorphic functions

5.1 Recollection of basic facts about series

Let us recall that a series \( \sum_{k=0}^{\infty} b_k \), where \( b_j \) are complex numbers, is called converging if there exists a finite limit of partial sums

\[
S := \lim_{N \to \infty} S_N = \lim_{N \to \infty} \sum_{k=0}^{N} b_k.
\]

In this case we write \( \sum_{k=0}^{\infty} b_k = S \). A necessary and sufficient condition for convergence is given by the Cauchy criterion:

For any \( \epsilon > 0 \) there exists \( N \) such that for any \( n \geq N \) and \( m > 0 \) we have \( \left| \sum_{k=n}^{n+m} b_k \right| < \epsilon \).

A series \( \sum_{k=0}^{\infty} b_k \) is called absolutely converging if the series \( \sum_{k=0}^{\infty} |b_k| \) is converging. Absolute convergence implies convergence, as it immediately follows from the Cauchy criterion and the inequality

\[
\left| \sum_{k=n}^{n+m} b_k \right| \leq \sum_{k=n}^{n+m} |b_k|.
\]

An important tool for establishing an absolute convergence (or divergence) is the following comparison criterion:
Lemma 5.1. Let $\sum a_n$ and $\sum b_n$ be two series such that $a_n, b_n \geq 0$. Suppose that there exists $N$ such that for $n \geq N$ we have $a_n \leq b_n$. Then if $\sum b_n$ is converging then so does $\sum a_n$, and if $\sum a_n$ is diverging then so does $\sum b_n$.

5.2 Power series

A power series is a series of the form $\sum_{n=0}^{\infty} a_n z^n$, $a_n, z \in \mathbb{C}$.

A remarkable fact about power series is the existence of a radius of convergence.

Proposition 5.2. For any power series $\sum_{n=0}^{\infty} a_n z^n$ there exists $R$ (which could be $0$ or $\infty$) such that for $|z| < R$ the power series is absolutely converging and for $|z| > R$ it is diverging. The radius of convergence $R$ can be computed by the following formula (due to Jacques Hadamard):

$$\frac{1}{R} = \limsup |a_n|^{\frac{1}{n}}.$$ 

Proof. The proof follows from the comparison with a geometric series $\sum r^n$ which converges when $r < 1$ and diverges when $r \geq 1$. Indeed, for any $r < R$ we have $|a_n| < \frac{1}{r^n}$ for a sufficiently large $n$. Hence, if $|z| < r$ then $|a_n| |z|^n < \left(\frac{|z|}{r}\right)^n$, and therefore the power series $\sum_{n=0}^{\infty} a_n z^n$ is absolutely converging due to the comparison with the geometric series $\sum_{n=0}^{\infty} \left(\frac{|z|}{r}\right)^n$. But $r$ is any number $< R$, and hence $\sum_{n=0}^{\infty} a_n z^n$ is absolutely converging for all $|z| < R$. If $|z| > R$ then there exists infinitely many $n$ such that $|a_n| |z|^n > \frac{1}{|z|}$, and hence for these values of $n$ we have $|a_n| |z|^n > 1$. This implies that $\sum_{n=0}^{\infty} a_n z^n$ is diverging because the common term of a converging series must converge to 0.

The disc $\{|z| < R\}$ is called the disc of convergence.

Exercise 5.3. Verify the following statements.

1. The radius of convergence of the geometric series $\sum_{n=0}^{\infty} z^n$ is equal to 1. On the boundary of $\{|z| = 1\}$ of the disc of convergence the series diverges at every point.

2. The radius of convergence of the geometric series $\sum_{n=0}^{\infty} \frac{z^n}{n}$ is also equal to 1. However, the behavior on the boundary of the disc of convergence is different: the series is convergent at every point except $z = 1$. 42
3. The radius of convergence of the exponential series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is equal to $\infty$, i.e. the series is absolutely converging on the whole $\mathbb{C}$.

4. The radius of convergence of the series $\sum_{n=0}^{\infty} n!z^n$ is equal to 0, i.e. the series is divergent for any $z \neq 0$.

Exercise 5.4 (Operations on converging power series). Suppose power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ are converging for $|z| < R$. Denote $A(z) := \sum_{n=0}^{\infty} a_n z^n$, $B(z) := \sum_{n=0}^{\infty} b_n z^n$. Then the series $\sum_{n=0}^{\infty} (a_n + b_n)z^n$ and $\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) z^n$ are also converging for $|z| < R$ and

$$A(z) + B(z) = \sum_{n=0}^{\infty} (a_n + b_n)z^n,$$

$$A(z)B(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) z^n.$$

Proposition 5.5. Suppose the series $f(z) := \sum_{n=0}^{\infty} a_n z^n$ is converging in the disc $D_R := \{ |z| < R\}$. Then

a) The series $\sum_{n=1}^{\infty} na_n z^{n-1}$ is also converging in $D_R$ and $\sum_{n=1}^{\infty} na_n z^{n-1} = f'(z)$. Thus, the sum of a power series is holomorphic in the disc of its convergence.

b) The series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$ is converging in $D_R$ to a function $F(z)$ such that $F'(z) = f(z)$.

Proof. First observe that according to the Hadamard criterion the power series $\sum_{n=0}^{\infty} a_n z^n$, $\sum_{n=1}^{\infty} na_n z^{n-1}$ and $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$ have the same radius of convergence. Indeed,

$$\lim \sup (n|a_n|)^{\frac{1}{n}} = \lim \sup |a_n|^\frac{1}{n} = \lim \sup \left( \frac{|a_n|}{n+1} \right)^{\frac{1}{n}}.$$

Let us prove that $\sum_{n=1}^{\infty} na_n z^{n-1} = f'(z)$. Recall an identity

$$\frac{u^n - v^n}{u - v} = u^{n-1}v + u^{n-2}v^2 + \cdots + uv^{n-1}.$$
Differentiating both sides with respect to \( v \) we get
\[
-nv^{n-1}(u-v) + u^n - v^n \quad \frac{(u-v)^2}{(u-v)^2} = u^{n-1} + 2u^{n-2}v + \cdots + (n-1)uv^{n-2}.
\]

Taking \( u = z + h, v = z \in D_{R-\epsilon} \) we get
\[
\left| \frac{(z+h)^n - z^n}{h} - nz^{n-1} \right| \leq \left| h \right| \left( (R-\epsilon)^{n-1} + 2(R-\epsilon)^{n-1} + \cdots + (n-1)(R-\epsilon)^{n-1} \right)
\]
\[
= \frac{n(n-1)(R-\epsilon)^{n-1}|h|}{2}.
\]

Hence, we have
\[
\left| \frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} na_n z^{n-1} \right| \leq \sum_{n=1}^{\infty} |a_n| \left| \left( \frac{(z+h)^n - z^n}{h} - nz^{n-1} \right) \right|
\]
\[
\leq \sum_{n=1}^{\infty} |a_n| \frac{n(n-1)(R-\epsilon)^{n-1}|h|}{2} \leq C(\epsilon)|h|.
\]

Hence, \( \frac{f(z+h) - f(z)}{h} \) converges to \( \sum_{n=1}^{\infty} na_n z^{n-1} \) when \( h \to 0 \).

To prove b) we apply a) to the series \( F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} \).

5.3 Analytic vs holomorphic

A function \( f : U \to \mathbb{C} \) is called analytic if in a neighborhood of any point \( z_0 \in U \) it can be presented as a sum of a converging power series.

**Lemma 5.6.** Given an analytic function \( f : U \to \mathbb{C} \), then the coefficients of its power expansion are equal to its Taylor coefficients, i.e for any point \( z_0 \) and a sufficiently small \( \epsilon > 0 \) we have
\[
f(z_0 + u) = f(z_0) + f'(z_0)u + \frac{f''(z_0)}{2}u^2 + \cdots + \sum_{0}^{\infty} \frac{f^{(n)}(z_0)}{n!} u^n + \cdots,
\]
Indeed, according to Proposition 5.5 a converging power series can be differentiated term-wise in the disc of its convergence. Hence, if \( f(z_0 + u) = \sum_0^\infty a_n u^n \) then

\[
\begin{align*}
  f(z_0) &= a_0, \\
  f'(z_0 + u) &= \sum_1^\infty na_{n-1} u^{n-1}, \quad \text{and hence} \\
  f'(z_0) &= a_1. \quad \text{Continuing this process we get} \\
  a_n &= \frac{f^{(n)}(z_0)}{n!}.
\end{align*}
\]

**Theorem 5.7.** The notions of a holomorphicity and analyticity are equivalent.

**Proof.** According to Proposition 5.5 any analytic function is holomorphic. To see the converse, take any point \( z_0 \in U \) and choose \( r > 0 \) such that the closed disc \( \overline{D} = \{ |z - z_0| \leq r \} \) of radius \( r \) centered at \( z_0 \) is contained in \( U \). Changing the variable \( u := z - z_0 \) we can express the function \( f(u) \) in the open disc \( D = \{ |u| < r \} \) by the Cauchy formula

\[
  f(u) = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(\zeta) d\zeta}{\zeta - u}.
\]

We have

\[
  \frac{f(\zeta)}{\zeta - u} = \frac{f(\zeta)}{\zeta} \frac{1}{1 - \frac{u}{\zeta}} = \frac{f(\zeta)}{\zeta} \sum_0^\infty u^n \zeta^n.
\]

Let us prove that the power series in the right hand side can be integrated term-wise and thus we get

\[
  \int_{|\zeta| = r} \frac{f(\zeta) d\zeta}{\zeta - u} = \sum_0^\infty \left( \int_{|\zeta| = r} \frac{f(\zeta) d\zeta}{\zeta^{n+1}} \right) u^n. \tag{5.3.1}
\]

Then for any \( u \in D, |u| < r \) we have

\[
  \left| \frac{f(\zeta)}{\zeta^{n+1}} \right| \leq \frac{M_r}{r^{n+1}},
\]

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where we denoted \( M_r := 2\pi \max_{|\zeta|=r} |f(\zeta)| \), and hence the power series in the right-hand side absolutely converges in \( D \). Let us choose any \( \rho < r \). Then for any \( u \in D, |u| \leq \rho \) we have

\[
\left| \int_{|\zeta|=r} \frac{f(\zeta)d\zeta}{\zeta - u} - \sum_{n=0}^{N} \left( \int_{|\zeta|=r} \frac{f(\zeta)d\zeta}{\zeta^{n+1}} \right) u^n \right| \leq \sum_{N+1}^{\infty} \left( \int_{|\zeta|=r} \frac{f(\zeta)d\zeta}{\zeta^{n+1}} \right) |u|^n \leq \frac{M_r}{r} \sum_{N+1}^{\infty} (\frac{\rho}{r})^n = \frac{M_r \rho^{N+1}}{r^{N+2}(1 - \frac{\rho}{r})} = \frac{M_r \rho^{N+1}}{r^{N+1}(r - \rho)} \rightarrow 0.
\]

This proves formula (5.3.1) for any \( u \in D \) because \( \rho \) is an arbitrary number \(< r \).

**Remark 5.8.** The above argument also shows that if \( f : U \rightarrow \mathbb{C} \) is a holomorphic function and for \( a \in U \) the disc \( D_r(a) = \{|z - a| < r\} \) is contained in \( U \) then the radius \( R \) of convergence of the Taylor expansion of \( f \) at the point \( a \) satisfies the inequality \( R \geq r \).

**Proposition 5.9.** Let \( f(z) \) be a continuous function in \( U \). Suppose that \( \int_{\gamma} f(z)dz = 0 \) for any piecewise smooth (or piecewise linear) loop \( \gamma \) in \( U \). Then the function \( f \) is holomorphic.

**Proof.** According to Theorem 4.9 the differential 1-form \( f(z)dz \) is exact in \( U \). Hence, \( f(z)dz = dg \), or \( g'(z) = f(z) \). Thus \( g \) is holomorphic and so is \( g' = f \).
Chapter 6

Properties of holomorphic functions

6.1 Exponential function and its relatives

So far the only examples of holomorphic functions we had were polynomials and rational functions \( \frac{P(z)}{Q(z)} \), where \( P, Q \) are polynomials, in the domain where \( Q(z) \neq 0 \). The theorem equating holomorphic and analytic functions allows us to greatly extend the set of examples. We begin in this section with exponential function and its close relatives.

As we already pointed out the exponential function is defined by the formula

\[
f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.
\]

We also define

\[
\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},
\]

\[
\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},
\]

\[
\sinh z = \frac{e^z - e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!},
\]

\[
\cosh z = \frac{e^z + e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.
\]
The radius of convergence of all these series is $\infty$, and hence the above formulas define holomorphic function on the whole $\mathbb{C}$.

**Lemma 6.1.**

1) $e^{z_1 + z_2} = e^{z_1} e^{z_2};$

2) $(e^z)' = e^z;$

3) $e^{iz} = \cos z + i \sin z;$

4) $\cos z = \cosh iz, \sin z = -i \sinh iz;$

First two properties follow from the formulas of multiplication and differentiation of power of series, see Exercise 5.4 and Proposition 5.5(a). Formula 3) follows the comparison of series in the left and right hand sides. Formula 4) follows from 3).

It is also interesting to observe that the exponential function is periodic with the imaginary period $2\pi i$, and that the identity

$$\cos^2 z + \sin^2 z = 1$$

holds for all $z \in \mathbb{C}$ and not only when $z$ is real.

### 6.2 Entire functions

Functions which are holomorphic on the whole $\mathbb{C}$ are called *entire*. The functions $e^z$, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ considered in Section 6.1 are examples of entire functions. The sum of any power series with the infinite radius of convergence is an entire holomorphic function. Remark 5.8 implies that the converse is also true:

*If $f : \mathbb{C} \to \mathbb{C}$ is an entire function, then its Taylor expansion at any point has a infinite radius of convergence.*

**Theorem 6.2** (Liouville’s theorem). *If an entire function is bounded it is a constant.*
Proof. This is a corollary of the Cauchy inequality 4.8. Indeed, suppose $|f(z)| \leq M$, then the inequality 4.8 implies that for any $k$ we have

$$|f^k(0)| \leq \frac{Mk!}{R^k} \text{ for any } R > 0.$$ 

Hence $f^k(0) = 0$ for $k > 0$. Therefore,

$$f(z) = f(0) + f'(0)z + \frac{1}{2}f''(0)z^2 + \cdots = f(0).$$

The following so-called little Picard Theorem significantly strengthen Liouville’s theorem.

**Theorem 6.3** (Little Picard theorem). *If an entire function does not take two distinct values $z_1, z_2 \in \mathbb{C}$ then it is a constant.*

Note that the function $\exp z = e^z$ takes all values except 0. Hence, the statement of the little Picard Theorem is sharp. We will prove this theorem later in the notes.

**Theorem 6.4** (Fundamental theorem of algebra). *Any polynomial $P(z) = a_0 + a_1z + \cdots + a_nz^n$, $a_n \neq 0$, of degree $n > 0$ has a root.*

Proof. Suppose $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then $g(z) := \frac{1}{P(z)}$ is an entire holomorphic function. On the other hand,

$$|P(z)| \geq |z^n| \left( |a_n| - \frac{|a_{n-1}|}{|z|} - \frac{|a_{n-2}|}{|z|^2} - \cdots - \frac{|a_0|}{|z|^n} \right).$$

But $\frac{|a_{n-2}|}{|z|^2} + \cdots + \frac{|a_0|}{|z|^n} \to 0$, and therefore

$$\frac{|a_{n-2}|}{|z|^2} + \cdots + \frac{|a_0|}{|z|^n} \leq \frac{|a_n|}{2}$$

for $|z|$ sufficiently large. But then

$$|P(z)| \geq |z^n| \left( |a_n| - \frac{|a_{n-1}|}{|z|} - \frac{|a_{n-2}|}{|z|^2} - \cdots - \frac{|a_0|}{|z|^n} \right) \geq \frac{|a_n||z|^n}{2}.$$
and therefore
\[ |g(z)| \leq \frac{2}{|a_n||z|^n} \xrightarrow{|z| \to \infty} 0. \]
This implies that the function \( g \) is bounded, and therefore by Liouville’s theorem it is constant. But this contradicts the assumption that the degree \( n \) is positive. \( \blacksquare \)

Theorem 6.4 implies that the polynomial \( P(z) \) of degree \( n \) with complex coefficients has \( n \) roots, counted with multiplicities. Indeed, by Theorem 6.4 there exists at least one root \( z_1 \). Then \( P(z) \) can be divided by \( (z - z_1) \):

\[ P(z) = (z - z_1)P_1(z), \]

where \( \deg P_1 = \deg P - 1 \). If degree of \( P_1(z) \) is still positive one can continue the process and get

\[ P(z) = (z - z_1)(z - z_2)P_2(z). \]

Continuing the process we decompose \( P \) into a product of linear terms:

\[ P(z) = a(z - z_1) \ldots (z - z_n). \]

### 6.3 Analytic continuation

Let us recall that a domain \( U \) is called connected if one cannot present it as the union \( U = U_1 \cup U_2 \) of disjoint non-empty open sets. Equivalently, a disconnected domain is a domain which admits a continuous function on \( U \) which takes exactly two values: 0 and 1.

There is a related notion of path-connectedness. A domain \( U \subset \mathbb{C} \) is called path-connected if for any two points \( A, B \in U \) there exists a continuous path \( \phi : [0, 1] \to U \) such that \( \phi(0) = A, \phi(1) = B \). The notions of connectedness and path connectedness coincide for open sets (for more general sets path connectedness is a stronger notion).

**Lemma 6.5.** Let \( f : U \to \mathbb{C} \) be a holomorphic function on a connected domain \( U \). Suppose that there exists a sequence of distinct points \( z_n \in \mathbb{C}, n = 1, \ldots, \) such that \( f(z_n) = 0 \), and \( \lim_{n \to \infty} z_n = a \in U. \) Then \( f \equiv 0 \) in \( U \).

In other words, zeroes of a holomorphic function are always isolated.
Proof. Denote by $A$ the set of all points $a \in U$ which satisfy the conditions of the lemma, i.e. that there exists a sequence of distinct points $z_n \in \mathbb{C}$, $n = 1, \ldots, \infty$, such that $f(z_n) = 0$, and $\lim_{n \to \infty} z_n = a$. Then by continuity we have $f(a) = 0$ for every $a \in U$. Let us prove that the set $A$ is open. For every $a \in A$ the holomorphic function $f$ can be expanded to a converging power series in a sufficiently small disc centered at $a$:

$$f(a + u) = c_1 u + c_2 u^2 + \ldots.$$  

If $f$ is not identically 0 in a neighborhood of $a$ then there is $k > 0$ such that $c_k \neq 0$ and $c_j = 0$ for all $j < k$. Then

$$f(a + u) = c_k u^k (1 + g(u)),$$

where $g(u) = \frac{c_{k+1}}{c_k} u + \frac{c_{k+2}}{c_k} u^2 + \ldots$.

The function $g$ is holomorphic in a neighborhood of $u = 0$ and we have $g(0) = 0$. Hence, there exists $r > 0$ such that $|g(u)| < \frac{1}{2}$ for $|u| < r$. Therefore,

$$|f(a + u)| \geq \frac{1}{2} |c_k| |u|^k,$$  

for $|u| < r$.

But this implies that $f(z) \neq 0$ provided that $z \neq a$ and $|z - a| < r$. But this contradicts the assumption of existence of a sequence $z_a \to a$ such that $f(z_n) = 0$. Hence, $f$ is identically equal to 0 in a neighborhood of $a$, i.e. $A$ is open.

Suppose that $U \setminus A \neq \emptyset$. Then for any $b \in U \setminus A$ the point $b$ there is a neighborhood $U_b \subset U$ where there is no zeroes of $f$ with a possible exception of $b$. But then $U_b \subset U \setminus A$, and hence $U \setminus A$ is open. But this contradicts the connectedness of $U$, and hence $A = U$, i.e. the function $f$ is equal to 0 identically on $U$.

Given domains $U \subset V \subset \mathbb{C}$ we say that a holomorphic function $f : V \to \mathbb{C}$ is a holomorphic extension of a holomorphic function $g : U \to \mathbb{C}$ if $f|_U = g$. Lemma 6.5 implies that any two holomorphic extensions of $f$ to a bigger domain coincide.

Example 6.6. 1) The radius of convergence of the series $\sum_{0}^{\infty} z^n$ is 1. However the function $f(z) = \frac{1}{1-z}$ provides a holomorphic extension of $\sum_{0}^{\infty} z^n$ from the unit disc $|z| < 1$ to $\mathbb{C} \setminus 1$.  

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6.4 Complex logarithm

Consider a closed differential 1-form $\frac{dz}{z}$ on $\mathbb{C} \setminus 0$. While this form is not exact on $\mathbb{C} \setminus 0$ it becomes exact when restricted to any simply connected subdomain $U \subset \mathbb{C} \setminus 0$. For instance, take $U := \mathbb{C} \setminus R$, where $R$ is the ray $R = \{z \in \mathbb{C}; \Re z \leq 0, \Im z = 0\}$. The primitive of $\frac{dz}{z}$, which is called logarithm (or sometimes the principal branch of the logarithm) and denoted by $\log z$ (or $\ln z$), can be computed by the formula

$$\log z = \int_{\Gamma_z} \frac{dz}{z},$$

where $\Gamma_z$ is any path connecting 1 with the point $z$. Then

$$d \log z = \frac{dz}{z}, \quad \text{or} \quad (\log z)' = \frac{1}{z}.$$

As we already computed above, $\frac{dz}{z} = d \ln r + id\phi$, and therefore

$$\log z = \int_{\Gamma_z} \frac{dz}{z} = \int_{\Gamma_z} d\ln r + i \int_{\Gamma_z} d\phi$$

$$= \log r + i\phi, \quad \phi \in (-\pi, \pi).$$

Thus the real part of the complex logarithm $\log z$ is equal to $\log |z|$, while the imaginary part is equal to $\arg z$.

**Lemma 6.7** (Properties of the logarithm).

1) $e^{\log z} = z$

2) $\log(1 + z) = \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n}$ for $|z| < 1$.

Indeed, $e^{\log z} = e^{\log r + i\phi} = re^{i\phi} = z$. To prove 2) we observe that $d(\log(1 + z)) = \frac{dz}{1+z}$. But $\frac{1}{1+z} = \sum_{n=1}^{\infty} (-1)^n z^n$ for $|z| < 1$. Hence, by integrating both parts of this equality we get $\log(1 + z) = \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n}$ for $|z| < 1$.

When trying to extend $\log z$ to the whole punctured plane $\mathbb{C} \setminus 0$ we get a multivalued function defined up to a multiple of $2\pi i$. In particular, the equality $\log z_1 z_2 = \log z_1 + \log z_2$ holds only up to a multiple of $2\pi i$. 

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More about the logarithm

More generally, given any simply connected domain $U \neq \emptyset$ we define a logarithm branch $\log^U z$ in $U$ as follows. Choose a point $z_0 \in U$ and a path $\delta$ connecting 1 and $z_0$ in $\mathbb{C} \setminus 0$, and for every point $z \in U$ choose a path $\gamma_z$ connecting $z_0$ in $z$ in $U$. Define

$$\log^U z = \int_{\delta \cup \gamma_z} \frac{dz}{z}.$$  

Simply connectedness of $U$ guarantees that the integral is independent of the choice of $\gamma_z$. However, it does depend on $\delta$, and a different choice of $\delta$ changes the value of the logarithm by adding a multiple of $2\pi i$.

**Lemma 6.8.** Let $U \subset \mathbb{C} \setminus 0$ be a simply-connected domain and $U_0 := \mathbb{C} \setminus R = \mathbb{C} \setminus \{z \in \mathbb{C}; \text{Re} z \leq 0, \text{Im} z = 0\}$. Let $\log^U : U \to \mathbb{C}$ be a logarithm branch defined above, and $\ln : U_0 \to \mathbb{C}$ the principal logarithm branch. Then for any point $z \in U \cap U_0$ we have

$$\log^U z = \ln z + 2k\pi i$$

for some integer $k$ (which is locally independent on $z$, but globally may depend on $z$).

Indeed, $\log^U z - \ln z = \int_{\lambda} \frac{dz}{z}$, where $\lambda$ a loop, but $\int_{\lambda} \frac{dz}{z} = i \int_{\lambda} d\phi = 2k\pi i$.

**Corollary 6.9.** $e^{\log^U z} = z$.

Indeed, if $z \in U \setminus R = U \cap U_0$ then according to Lemma 6.8 we have

$$e^{\log^U z} = e^{\ln z + 2k\pi i} = z.$$

If $z_0 \in R$ then the same holds by continuity.

In the situation when $\log^U$ is defined we can also define a branch of the function $z^a$ for any complex number $a$ by the formula

$$z^a = e^{a \log^U z}.$$ 

In particular, if $a = \frac{1}{n}$ then a branch of $z^{\frac{1}{n}} = \sqrt[n]{z}$ defined this way satisfies $(\sqrt[n]{z})^n = z$. 

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6.5 Schwarz reflection principle

The following result provides an interesting case of a holomorphic extension.

**Theorem 6.10** (Schwarz reflection principle). Let \( U \subset \{ \Im z > 0 \} \subset \mathbb{C} \) be a domain in an upper half plane. Suppose an interval \( I = (a, b) \subset \mathbb{R} \subset \mathbb{C} \) is contained in the boundary \( \partial U \). Denote

\[
\hat{U} := U \cup I \cup \overline{U}.
\]

Let \( f : U \to \mathbb{C} \) be a holomorphic function which extends continuously to \( I \) and takes real values on \( I \). Then \( f \) holomorphically extends to \( \hat{U} \).

**Proof.** Define \( f(z) = \overline{f(\overline{z})} \) for each \( z \in \overline{U} \) and extend it by continuity to \( I \). To show that \( f \) is holomorphic consider any piecewise linear loop \( \gamma \subset \hat{U} \). The function \( f \) is holomorphic in \( U \) and \( \overline{U} \) and extends continuously to \( I \) from both sides. Hence, the integral \( \int_{\gamma} f(z) \, dz \) is equal to 0 over loops \( \delta \) in \( U \cup I \) and \( \overline{U} \cup I \). But any loop \( \gamma \) in \( \hat{U} \) is split by the interval \( I \) into several loops, each one is either in \( U \cup I \) or \( \overline{U} \cup I \). Hence, \( \int_{\gamma} f(z) \, dz = 0 \), and by Proposition 5.9 the function \( f \) is holomorphic in \( \hat{U} \).
Chapter 7

Isolated singularities, residues and meromorphic functions

7.1 Holomorphic functions with isolated singularities

Let $U \subset \mathbb{C}$ be an open domain. A closed subset $Z \subset U$ is called discrete if for every point $u \in Z$ there exists an $\epsilon > 0$ such that the disc $D_\epsilon(z) = \{z; |z - u| < \epsilon\}$ is contained in $U$ and has no other points of $Z$ besides $u$. In other words, any point of $u$ has a neighborhood which does not contain other points of $Z$. The set $Z$ could be finite, or countable, but in the latter case all its accumulation points do not belong to $U$. If a discrete set is contained in a compact set then it is finite.

Given a discrete subset $Z \subset U$ a holomorphic function $f : U \setminus Z \to \mathbb{C}$ is sometimes called a function on $U$ with isolated singularities at the points of $Z$.

Some of these singularities could be fictitious, or removable, i.e. the function $f$ can actually be extended to this point as a holomorphic function.

**Theorem 7.1.** Let $a$ be an isolated singularity of a holomorphic function given on $U = \{0 < |z - a| < r\}$. Suppose that $|f(z)| \leq C|z - a|^{-\sigma}$ for $\sigma < 1$. Then $a$ is a removable singularity. In particular if $f$ is bounded near $a$ then $a$ is removable.

**Proof.** Consider the function $g(z) = (z - a)^2 f(z)$ and extend it to the point $a$ by setting $g(a) = 0$. 

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Then \( g \) is differentiable at \( a \) and \( g'(a) = 0 \). Indeed

\[
|g'(a)| = |\lim_{z \to a} \frac{g(z)}{z-a}| \leq \lim_{z \to a} |f(z)||z-a| \leq |z-a|^{1-\sigma} = 0,
\]

if \( \sigma < 1 \). Hence, the function \( g \) is holomorphic in \( \{|z-a| < r\} \) and vanishes at \( a \) together with its first derivative. Hence, \( g(z) = (z-a)^2 h(z) \), where \( h \) is a holomorphic function. But then \( f(z) = h(z) \) in \( U = \{0 < |z-a| < r\} \), and hence \( h(z) \) is the required holomorphic extension of \( f \) to the disc \( \{|z-a| < r\} \).

Non-removable singularities are divided into two types, poles and essential singularities.

A point \( u \) is called a pole of order \( k \) if in a neighborhood of \( u \) the function \( f \) can be written as

\[
f(z) = \frac{g(z)}{(z-u)^k},
\]

where \( g \) is a holomorphic function such that \( g(u) \neq 0 \). A non-removable singularity which is not a pole is called essential. A holomorphic function with isolated singularities which are all non-essential is called meromorphic. For instance, any rational function \( \frac{P(z)}{Q(z)} \), i.e. the ratio of two polynomials is a meromorphic function on \( \mathbb{C} \).

The following theorem characterizes poles among isolated singularities.

**Proposition 7.2.** An isolated singularity \( u \) of \( f \) is a pole if and only if \( |f(z)| \to \infty \) as \( z \to u \).

Thus for an essential singularity \( u \) the modulus \( |f(z)| \) is unbounded near \( u \) but there is no finite or infinite limit \( \lim_{z \to u} |f(z)| \).

**Proof.** If \( u \) is a pole then near \( u \) we can write \( f(z) = \frac{g(z)}{(z-u)^k} \), where \( g \) is a holomorphic function and \( g(u) \neq 0 \). Hence \( |g(z)| \to |g(u)| \neq 0 \) and \( \frac{1}{|z-u|^k} \to \infty \). Hence, \( \lim_{z \to u} |f(z)| = \infty \).

Suppose now that \( |f(z)| \to \infty \) as \( z \to u \). Then \( h(z) := \frac{1}{f(z)} \to 0 \). Thus, \( u \) is a removable singularity for \( h \) and \( u \) is its zero of some order \( k \). Then we can write \( h(z) = (z-u)^k \bar{h}(z) \), where \( \bar{h}(u) \neq 0 \). Hence,

\[
f(z) = \frac{1}{h(z)} = \frac{1}{\bar{h}(z)} (z-u)^k
\]

has a pole of order \( k \) at \( u \).

The following, so called great Picard Theorem is a far-going generalization of the little Picard Theorem, see Theorem 6.3, as well as Proposition 7.2.
Theorem 7.3. If an analytic function $f : U \to \mathbb{C}$ has an essential singularity at a point $u \in U$, then for any neighborhood $\Omega \ni u, \Omega \subset U$, the restriction $f|_{\Omega \setminus \{u\}} : \Omega \setminus u \to \mathbb{C}$ takes all values in $\mathbb{C}$ with a possible exception of one point $a \in \mathbb{C}$.

Remark 7.4. It follows that the exceptional value $a \in \mathbb{C}$ is independent of a neighborhood $\Omega$ if it is small enough, and that on any $\Omega \setminus u$ the function $f$ takes each value in $\mathbb{C} \setminus a$ infinitely many times.

The proof of the great Picard Theorem goes beyond this class. Interested students can read it, e.g. in Chapter 4 of the book “Complex Analysis on One Variable” by R. Narasimhan and Y. Nievergelt, Birkhäuser, 2001.

Exercise 7.5. Deduce the little Picard theorem from the great one.

7.2 Residues

Suppose that a holomorphic function $f : U \setminus u \to \mathbb{C}$ has an isolated singularity at $u$. Take $r > 0$ such that $\overline{D_r(u)} = \{z; |z - u| \leq r\} \subset U$ and define the residue of $f$ at $u$ by the formula

$$\text{Res}_u f = \frac{1}{2\pi i} \int_{|z-u|=r} f(z)dz. \quad (7.2.1)$$

In view of the Cauchy theorem the integral (7.2.1) is independent of the choice of $r$. If singularity is removable, then again the Cauchy theorem implies that $\text{Res}_u f = 0$.

Example 7.6.

$$\text{Res}_0 \frac{1}{z^k} = \begin{cases} 1 & k = 1; \\ 0 & k > 1. \end{cases}$$

Theorem 7.7 (Residue theorem). Let $U$ be a domain with a piecewise smooth boundary $\Gamma := \partial U$ and compact closure. Suppose $f : U \setminus Z \to \mathbb{C}$ be a holomorphic function with the set $Z$ of isolated singularities. Suppose $f$ extends continuously to $\Gamma$. Then

$$\int_{\Gamma} f(z)dz = 2\pi i \left( \sum_{u \in Z} \text{Res}_u f \right).$$
Proof. This is just a reformulation of the Cauchy theorem. First, we observe that in view of compactness of $U$ there could be only finitely many of isolated singularities in $U$ (why?): $Z = \{ u_1, \ldots, u_k \}$. There exist $r_1, \ldots, r_k > 0$ such that the discs $D_{u_j}(r_j) = \{|z - u_j| \leq r_j \}$ are contained in $U$ and do not intersect each other. Hence, the Cauchy theorem yields:

$$
\int_{\Gamma} f(z) \, dz = \sum_{u \in Z} \int_{\partial D_{u}(r_j)} f(z) \, dz = 2\pi i \left( \sum_{u \in Z} \text{Res}_u f \right).
$$

Theorem 7.7 would provide a way of computing contour integrals if we could develop effective methods for computing the residues. The next proposition explains how to compute residues for poles.

**Proposition 7.8.** Suppose $f$ has a pole of order $n$ at a point $u$, i.e. $f(z) = \frac{g(z)}{(z-u)^n}$, where $g$ is a holomorphic function such that $g(u) \neq 0$. Then

$$
\text{Res}_u f = \frac{1}{(n-1)!} g^{(n-1)}(u), \quad \text{where} \quad g(z) = (z-u)^n f(z).
$$

In particular, if $u$ is a pole of order 1 we have

$$
\text{Res}_u f = g(u) = \lim_{z \to u} (z-u)f(z).
$$

Warning: Formula (7.2.3) can only be used if you already know that the pole is simple. In particular, if the limit in (7.2.3) is 0 then this means that the pole *not simple*.

Proof. Consider the Taylor expansion of $g$ at the point $u$:

$$
g(z) = g(u) + g'(u)(z-u) + \cdots + \frac{g^{(n-1)}(u)}{(n-1)!} (z-u)^{n-1} + \frac{g^{(n)}(u)}{n!} (z-u)^n + \ldots.
$$

Hence,

$$
f(z) = \frac{g(z)}{(z-u)^n} = \frac{g(u)}{(z-u)^n} + \frac{g'(u)}{(z-u)^{n-1}} + \cdots + \frac{g^{(n-1)}(u)}{(n-1)!} (z-u) + h(z),
$$

where $h$ is a holomorphic function at a neighborhood of $u$. But then

$$
\text{Res}_u f = \frac{1}{2\pi i} \int_{|z-u|=r} \left( \frac{g(u)}{(z-u)^n} + \frac{g'(u)}{(z-u)^{n-1}} + \cdots + \frac{g^{(n-1)}(u)}{(n-1)!} (z-u) + h(z) \right) dz
$$

$$
= \frac{1}{2\pi i} \int_{|z-u|=r} \frac{g^{(n-1)}(u) dz}{(n-1)!} (z-u) = \frac{1}{(n-1)!} g^{(n-1)}(u).
$$
We will later discuss computation of residues at essential singularities.

### 7.3 Application of the residue theorem to computation of integrals

The residue theorem yields computation of many definite integrals of functions of 1 real variables which is difficult to compute using elementary methods. We consider here 3 examples.

1. **Compute** \( \int_{0}^{\infty} \frac{dx}{1 + x^4} \).

   Consider a meromorphic function \( f(z) = \frac{1}{1 + z^4} \) and compute \( \int_{\Gamma_R} f(z) \, dz \), where
   \[
   \Gamma_R = \partial \{ z; \text{Im} \, z \geq 0, |z| \leq R \}.
   \]

   Then
   \[
   \int_{\Gamma_R} f(z) \, dz = \int_{-R}^{R} \frac{dx}{1 + x^4} + \int_{S_R^+} f(z) \, dz,
   \]
   where \( S_R^+ = \{ z; \text{Im} \, z \geq 0, |z| = R \} \). We have
   \[
   \left| \int_{S_R^+} f(z) \, dz \right| \leq \int_{0}^{\pi} \frac{R \, d\phi}{R^4 - 1} = \frac{\pi R}{R^4 - 1} \rightarrow 0.
   \]

   Therefore,
   \[
   \int_{\Gamma_R} f(z) \, dz \rightarrow_{R \to \infty} = \int_{-\infty}^{\infty} \frac{dx}{1 + x^4}.
   \]

   The function \( f \) has simple poles at the points \( z_1 = e^{i\pi/4}, z_2 = e^{3i\pi/4} \in \{ z; \text{Im} \, z \geq 0, |z| \leq R \} \), provided that \( R > 1 \). The residues of \( f \) at \( z_j, j = 1, 2 \) are equal to
   \[
   \lim_{z \to z_j} \frac{z - z_j}{1 + z^4} = \frac{1}{(1 + z^4)'_{|z=z_j}} = \frac{1}{4z_j^3}.
   \]
Hence,
\[
\int_{\Gamma_R} f(z) \, dz = 2\pi i \left( e^{-3\pi i/4} + e^{-9\pi i/4} \right) = \pi \sin \frac{\pi}{4} = \frac{\pi \sqrt{2}}{2}.
\]

Therefore,
\[
\int_{0}^{\infty} \frac{dx}{1 + x^4} = \frac{1}{2} \lim_{R \to \infty} \int_{\Gamma_R} f(z) \, dz = \frac{\pi \sqrt{2}}{4}.
\]

**Remark 7.9.** Similar techniques applies to integrals \( \int_{-\infty}^{\infty} R(x) \, dx \), when this integral is converging, where \( R(x) = \frac{P(x)}{Q(x)} \) is a rational function.

2. Compute \( \int_{0}^{\infty} \frac{\cos x \, dx}{1 + x^2} \).

Consider a function
\[
f(z) = \frac{e^{iz}}{1 + z^4}.
\]
Denote
\[
U_{R,a} := \{ z; 0 \leq \text{Im} \, z \leq a, |\text{Re} \, z| \leq R \}.
\]
Then
\[
\int_{0}^{\infty} \frac{e^{iz}}{1 + z^4} \, dz = \int_{-R}^{R} \frac{e^{ix}}{1 + x^2} \, dx + \int_{0}^{a} \frac{e^{iR}e^{-y}}{1 + (R + iy)^2} \, dy + \int_{0}^{a} \frac{e^{-iR}e^{-y}}{1 + (-R + iy)^2} \, dy + \int_{-R}^{R} \frac{e^{-a}e^{ix}}{1 + (x + ia)^2} \, dx = I_1 + I_2 + I_3 + I_4.
\]
Then we have
\[
|I_2| \leq \int_{0}^{a} \frac{e^{-y}}{R^2 - 1} \, dy \leq \frac{1}{R^2 - 1} \int_{0}^{\infty} e^{-y} \, dy \leq \frac{1}{R^2 - 1} \to 0 \quad R \to \infty.
\]
Similarly,
\[
I_3 \to 0 \quad R \to \infty.
\]
Furthermore,
\[
|I_4| \leq \frac{2Re^{-a}}{a^2 - 1}.
\]
Choose \( a = \ln R \) then
\[
|I_4| \leq \frac{2}{(\ln R)^2} - 1 \to 0.
\]
Hence,
\[
\int_{-\infty}^{\infty} \frac{e^{ix}}{1 + x^2} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{ix}}{1 + x^2} \, dx = \lim_{R \to \infty} \int_{\partial U_{R,\ln R}} \frac{e^{iz}}{1 + z^2} \, dz = 2\pi \text{Res}_i e^{iz} = \pi.
\]

Therefore,
\[
\int_{0}^{\infty} \frac{\cos x}{1 + x^2} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} \, dx = \frac{1}{2} \text{Re} \left( \int_{-\infty}^{\infty} \frac{e^{ix}}{1 + x^2} \, dx \right) = \frac{\pi}{2e}.
\]

3. Compute \( \int_{0}^{\infty} \frac{\sin x}{x} \, dx \).

Let \( U_{R,\ln R} \) the domain defined in the previous example. Let \( D_\epsilon \) denote the disc \( \{ |z| < \epsilon \} \). Set \( U_{R,\ln R,\epsilon} := U_{R,\ln R} \setminus D_\epsilon \).

Then we have
\[
0 = \int_{\partial U_{R,\ln R,\epsilon}} \frac{e^{iz}}{z} \, dz = \left( \int_{-R}^{\epsilon} \frac{e^{ix}}{x} \, dx + \int_{\epsilon}^{R} \frac{e^{ix}}{x} \, dx \right) - \int_{\partial D_\epsilon \cap \{ \text{Im} \geq 0 \}} \frac{e^{iz}}{z} \, dz + \int_{0}^{\ln R} \frac{e^{ix}}{x} \, dx + \int_{0}^{\ln R} \frac{e^{-ix}}{x} \, dx.
\]

As in Example 2 the last three terms converge to 0 when \( R \to \infty \). On the other hand,
\[
\int_{\partial D_\epsilon \cap \{ \text{Im} \geq 0 \}} \frac{e^{iz}}{z} \, dz \to \pi i \text{Res}_0 \frac{e^{iz}}{z} = \pi i.
\]

Finally,
\[
\int_{-\epsilon}^{\epsilon} \frac{e^{ix}}{x} \, dx + \int_{\epsilon}^{R} \frac{e^{ix}}{x} \, dx = 2 \int_{\epsilon}^{R} \frac{e^{ix}}{x} \, dx \to \int_{0}^{\infty} \frac{e^{ix}}{x} \, dx.
\]

Hence,
\[
\int_{0}^{\infty} \frac{\sin x}{x} \, dx = \frac{1}{2} \text{Im} \left( \int_{0}^{\infty} \frac{e^{ix}}{x} \, dx \right) = \frac{\pi}{2}.
\]
Similar techniques applies to integrals of the form \[ \int_{-\infty}^{\infty} e^{ix}R(x)dx \], (and in particular to integrals \[ \int_{-\infty}^{\infty} (\cos x)R(x)dx \) and \[ \int_{-\infty}^{\infty} (\sin x)R(x)dx \) when this integral is converging, where \( R(x) = \frac{P(x)}{Q(x)} \) is a rational function.

4. Compute the integral
\[
I = \int_{0}^{\pi} \frac{d\theta}{a + \cos \theta}, \quad a > 1.
\]

Denote \( z = e^{i\theta} \) then \( \cos \theta = \frac{z + z^{-1}}{2} \), and hence \( a + \cos \theta = \frac{z^2 + 2az + 1}{2z} \). Denote \( D := \{z; |z| < 1\} \), Consider the integral
\[
J = \oint_{\partial D} \frac{dz}{z^2 + 2a + 1} = \frac{1}{2} \int_{0}^{2\pi} \frac{id\theta}{a + \cos \theta} = i \int_{0}^{\pi} \frac{d\theta}{a + \cos \theta} = iI,
\]
i.e. \( I = -iJ \).

On the other hand, we can apply the residue theorem to compute the integral \( J \). the function \( f(z) = \frac{1}{z^2 + 2az + 1} \) have simple poles at the points \( A_{\pm} := -a \pm i \sqrt{a^2 - 1} \) and we have \( |A_+| < 1 \) and \( |A_-| > 1 \). We have \( \text{Res}_{A_+} f = \frac{1}{A_+ - A_-} = \frac{1}{2 \sqrt{a^2 - 1}} \).

Hence,
\[
I = -iJ = -i(2\pi i)\text{Res}_{A_+} = \frac{\pi}{\sqrt{a^2 - 1}}.
\]

Remark 7.10. The same method applies to integrals of the form
\[
\int_{0}^{2\pi} R(\cos t, \sin t)dt,
\]
where \( R \) is a rational function.
7.4 Complex projective line or Riemann sphere

Consider the space $C^n$ of $n$-tuples $(z_1, \ldots, z_n)$ of complex numbers $z_j \in \mathbb{C}$. This is an example of a complex vector space. One can add vectors and multiply them by complex numbers:

$$(z_1, \ldots, z_n) + (z'_1, \ldots, z'_n) = (z_1 + z'_1, \ldots, z_n + z'_n),$$
$$\lambda(z_1, \ldots, z_n) = (\lambda z_1, \ldots, \lambda z_n), \quad \lambda \in \mathbb{C}$$

Similar to the real case one can projectivise $C^n$. The complex projective space of dimension $n$, denoted $\mathbb{C}P^n$ is defined as the space of all complex lines through the origin. We will need for our purposes mostly the 1-dimensional complex projective space, or as it is called complex projective line. We analyze this notion below.

Any non-zero vector $z = (z_1, z_2) \in \mathbb{C}^2$ generates the 1-dimensional complex subspace, or complex line

$$l_z = \text{Span}(z) = \{\lambda z; \lambda \in \mathbb{C}\} \subset \mathbb{C}^2.$$ 

The line $l_z$ can be viewed as a point of $\mathbb{C}P^1$. Any proportional vector $\bar{z} = \mu z$, $\mu \in \mathbb{C}$, generates the same line: $l_{\bar{z}} = l_z$. Hence, we equivalently can define $\mathbb{C}P^1$ as the space of points in $\mathbb{C}^2 \setminus 0$ up to a complex proportionality.

Let us fix an affine line $L_1 = \{z_2 = 1\} \subset \mathbb{C}^2$. Any line in $\mathbb{C}^2$ through the origin, except the line $l_{(1,0)} = \{z_2 = 0\}$, intersects $L_1$ at exactly one point. Namely, if $l = l_z$ for $z = (z_1, z_2)$ with $z_2 \neq 0$ then it intersects $L_1$ at the point $(u = \frac{z_1}{z_2}, 1)$. So one can view $\mathbb{C}P^1$ as $\mathbb{C}$ with one point added “at infinity”. On the other hand there is nothing special in this point at infinity. If instead we take an affine line $L_2 = \{z_1 = 1\} \subset \mathbb{C}^2$ then any line through the origin except the line $l_{(0,1)}(z_1 = 0)$ intersects $L_2$ in exactly one point. Namely, if $l = l_z$ for $z = (z_1, z_2)$ with $z_1 \neq 0$ then it intersects $L_1$ at the point $(v = \frac{z_2}{z_1}, 1)$. So in $\mathbb{C}P^1 \setminus (l_{(0,1)} \cup l_{(1,0)})$ we have two coordinates $u$ and $v$ related by the formula $u = \frac{1}{v}$.

Thus, $\mathbb{C}P^1 \setminus \{l_{(1,0)}\}$ and $\mathbb{C}P^1 \setminus \{l_{(0,1)}\}$ can be identified with $\mathbb{C}$, and we can say that $\mathbb{C}P^1$ is obtained by gluing two copies of $\mathbb{C}$ along $\mathbb{C} \setminus 0$ using the gluing map $z \mapsto \frac{1}{z}$. It follows that $\mathbb{C}P^1$ is diffeomorphic to the 2-sphere. To see this, let us take the unit sphere $\Sigma = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ and view the coordinate plane $(x, y)$ as $\mathbb{C}$. Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ be the North
South poles of \( \Sigma \). Consider the stereographic projections \( \text{St}_N : \Sigma \setminus S \to \mathbb{C} \) and \( \text{St}_S : \Sigma \setminus N \to \mathbb{C} \) from the North and South poles, respectively. Let us associate with any point \( p \in \Sigma \setminus N \) its complex coordinate \( u = \text{St}_N(p) \), and with any point \( p \in \Sigma \setminus S \) its complex coordinate \( v = \overline{\text{St}_N(p)} \), where the bar denote the complex conjugation. Then one can check that for \( p \in \Sigma \setminus (S \cup N) \) coordinates \( u \) and \( v \) are related by \( u = \frac{1}{v} \), exactly as we had seen above in \( \mathbb{C}P^1 \).

This leads to the following interpretation of \( \mathbb{C}P^1 \). We add to \( \mathbb{C} \) one extra point \( \infty \). Disc complements \( U_r := \{|z| > r\} \) form a system of neighborhoods of \( \infty \). The function \( u = \frac{1}{z} \) can be viewed as a coordinate at this neighborhood which is equal to 0 at infinity. Given a holomorphic function \( g : U_r \to \mathbb{C} \) we say that it extends to \( \infty \) if the function \( g(\frac{1}{u}) \) extends as a holomorphic function to 0. We say that \( g \) has a pole or zero of order \( m \) at infinity if so does the function \( g(\frac{1}{u}) \). For instance, any polynomial \( P(z) = z^n + a_1z^{n-1} + \cdots + a_n \) of degree \( n \) has a pole of order \( n \) at \( \infty \). Indeed, replacing \( z = \frac{1}{u} \) we observe that the function

\[
f(u) := P\left(\frac{1}{u}\right) = \frac{1 + a_1u + a_2u^2 + \cdots + a_nu^n}{u^n}
\]

has a pole of order \( n \) at 0. Similarly, the function \( \frac{1}{P(u)} \) has a zero of order \( n \) at infinity.

In view of the above interpretation the complex projective line \( \mathbb{C}P^1 \) is also sometimes called the Riemann sphere, or extended complex plane \( \mathbb{C} \) and denoted \( \overline{\mathbb{C}} \).

We had already seen that if \( u \in \mathbb{C} \) is a pole of a function \( f \) then \( |f(z)| \to \infty \) as \( z \to u \). But this means that if we interpret \( \mathbb{C} \) as a subset of \( \mathbb{C}P^1 = \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) then we can continuously extend \( f \) to the point \( u \) by setting \( f(u) = \infty \). Moreover, from the point of view of the coordinate \( u = \frac{1}{z} \) the point \( \infty \) has a coordinate \( u = 0 \) and hence it is no different than any other point on \( \mathbb{C}P^1 \). Thus we conclude that meromorphic functions on a domain \( U \subset \mathbb{C} \) are just \( \mathbb{C}P^1 \)-valued holomorphic functions.

If a meromorphic function on \( \mathbb{C} \) has a pole at infinity then it extends to a meromorphic function on \( \mathbb{C}P^1 \), i.e. a holomorphic map \( \mathbb{C}P^1 \to \mathbb{C}P^1 \).

**Theorem 7.11.** Any rational function \( R(z) = \frac{P(z)}{Q(z)} \) is meromorphic on \( \mathbb{C}P^1 \), and conversely any meromorphic function on \( \mathbb{C}P^1 \), i.e. a holomorphic map \( \mathbb{C}P^1 \to \mathbb{C}P^1 \) is rational, i.e. the ratio of 2 polynomials.
Proof. Let $R(z) = \frac{P(z)}{Q(z)}$, where $P(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n$, $Q(z) = b_0z^m + b_1z^{m-1} + \cdots + b_m$, where $a_0, b_0 \neq 0$. Then if $n > m$ then $R(z)$ has a pole of order $n - m$ at $\infty$, if $n < m$ it has a zero of order $m - n$ at $\infty$, and if $n = m$ then $\infty$ is a removable singularity and not a zero.

Conversely, let $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ be a holomorphic function. In view of compactness of $\mathbb{C}P^1$ the function $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ has finitely many poles and zeroes. Let $q_1, q_2, \ldots, q_k \in \mathbb{C} \subset \mathbb{C}P^1$ be zeroes or poles of $f$. Denote by $r_1, \ldots, r_k$ the multiplicity of zeroes and poles assuming them negative for poles. Also $\infty$ could be a pole or zero of multiplicity $r_\infty$.

Consider the rational function

$$R(z) := (z - q_1)^{r_1} \cdots (z - q_k)^{r_k}$$

The functions $R(z)$ and $f(z)$ have zeroes and poles of the same multiplicities at the same points in $\mathbb{C}$. Then $h(z) := \frac{f(z)}{R(z)}$ has no poles and zeroes in $\mathbb{C}$. We argue that $\infty$ is also not a pole and hence $h$ is a non-zero constant $C$ (according to Liouville’s theorem). Indeed, if $\infty$ is a pole then for $\frac{1}{h}$ it is a zero and then $\frac{1}{h}$ has to be a non-zero constant which implies that $h(\infty) \neq 0$. Therefore, $f(z) = CR(z)$ is rational. ■

**Corollary 7.12.** For any meromorphic function $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ the total multiplicity of all zeroes is equal to the total multiplicities of all poles.

**Proof.** According to Theorem 7.11 $f(z) = \frac{P(z)}{Q(z)}$. The total multiplicity of poles of $f$ in the finite part of $\mathbb{C}$ is equal to degree $d(Q)$ of $Q$ while the total multiplicity of zeroes of $f$ in the finite part of $\mathbb{C}$ is equal to the degree $d(P)$ of $P$. As we already seen in the proof of Theorem 7.11 if $d(P) > d(Q)$ then $f$ has a zero at $\infty$ of order $d(P) - d(Q)$, while if $d(Q) > d(P)$ then $f$ has a pole at $\infty$ of order $d(Q) - d(P)$. If $d(P) = d(Q)$ the $\infty$ is neither pole nor zero. In all cases the difference between the total number of zeroes and poles, including $\infty$, is equal to 0. ■

**Exercise 7.13.** Let $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ be a meromorphic function. Prove that the sum of residues of all its poles is equal to 0.
7.5 Argument principle

Theorem 7.14. Let $U$ be a domain with a piecewise smooth boundary and $f: U \to \mathbb{C}P^1$ be a meromorphic function which $C^1$-extends to the boundary $\partial U$ without poles and zeroes on $\partial U$. Let $q_1, \ldots, q_k \in U$ be the zeroes of $f$ of multiplicities $r_1, \ldots, r_k$, and $p_1, \ldots, p_l$ be poles of $f$ of multiplicities $s_1, \ldots, s_l$. Then

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)dz}{f(z)} = \sum_{j=1}^{k} r_j - \sum_{i=1}^{l} s_i.$$

Proof. According to the Cauchy theorem

$$\int_{\Gamma} \frac{f'(z)dz}{f(z)} = \sum_{j=1}^{k} \int_{|z-q_j|=\epsilon} \frac{f'(z)dz}{f(z)} + \sum_{i=1}^{l} \int_{|z-p_i|=\epsilon} \frac{f'(z)dz}{f(z)},$$

where $\epsilon > 0$ is chosen so small that the discs $D_{\epsilon}(q_j)$ and $D_{\epsilon}(p_i)$, $j = 1, \ldots, k, i = 1, \ldots, l$, are pairwise disjoint and are contained in $U$. If $\epsilon$ is small enough then in $D_{\epsilon}(q_j)$ we have

$$f(z) = (z - q_j)^{r_j} g_j(z),$$

where $g_j(z) \neq 0$, and in $D_{\epsilon}(p_i)$ we have

$$f(z) = (z - q_j)^{-s_i} h_i(z),$$

where $h_i(z) \neq 0$. Then

$$\int_{|z-q_j|=\epsilon} \frac{f'(z)dz}{f(z)} = \int_{|z-q_j|=\epsilon} \frac{r_j(z - q_j)^{r_j-1} g_j(z) + (z - q_j)^{r_j} g_j'(z)}{(z - q_j)^{r_j} g_j(z)} dz$$

$$= \int_{|z-q_j|=\epsilon} \frac{r_j dz}{z - q_j} + \int_{|z-q_j|=\epsilon} \frac{g_j'(z)}{g_j(z)} = 2\pi r_j i.$$  

The second integral is equal to 0 because the function $\frac{g_j'}{g_j}$ is holomorphic in the whole disc $D_{\epsilon}(q_j)$. 

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Similarly, we get
\[
\int_{|z-p_j|<\epsilon} \frac{f'(z)dz}{f(z)} = \int_{|z-p_j|=\epsilon} \frac{(-s_i(z-p_j)^{-s_j-1}h_j(z) + (z-p_j)^{-s_j}h_j'(z)dz}{(z-p_j)^{-s_j}h_j(z)}
\]
\[
= \int_{|z-p_j|=-\epsilon} \frac{s_i dz}{z-p_j} + \int_{|z-p_j|=\epsilon} \frac{h_j'(z)}{h_j(z)} = -2\pi s_j i.
\]

Hence,
\[
\frac{1}{2\pi i} \int_{\partial U} \frac{f'(z)dz}{f(z)} = \sum_{i=1}^{k} r_j - \sum_{i=1}^{l} s_j.
\]

\[\blacksquare\]

**Corollary 7.15** (Rouché’s theorem). Let \( U \) be a domain with a piecewise smooth boundary. Let \( f, g : U \to \mathbb{C} \) be holomorphic functions which \( C^1 \)-extend to \( \partial U \). Suppose that \(|g(z)| < |f(z)|\) for all \( z \in \partial U \). Then the holomorphic functions \( f \) and \( f + g \) have the same number of zeroes in \( U \) counted with multiplicities.

**Exercise 7.16.** Prove that it is sufficient to assume that \( f, g \) continuously, and not necessarily \( C^1 \) extend to \( U \).

**Proof.** Consider a 1-parametric family of functions \( f_t := f + tg, t \in [0, 1] \). By assumption
\[
|f_t|_{\partial U} \geq |f|_{\partial U} - t|g|_{\partial U} > 0.
\]
Hence we can apply the argument principle to conclude that the total number \( n_t \) of zeroes of the function \( f_t \) counted with multiplicities is given by the formula
\[
n_t = \frac{1}{2\pi i} \int_{\partial U} \frac{f_t'(z)dz}{f_t(z)}.
\]
This integral takes only integer values, but at the same time it continuously depends on \( t \). Hence it is a constant, which implies that the number \( n_0 \) of zeroes of \( f \) is equal to the number \( n_1 \) of zeroes of \( f + g \).

**Corollary 7.17** (Open image theorem). Let \( U \) be a connected open domain and \( f : U \to \mathbb{C} \) a non-constant holomorphic map. Then the image \( f(U) \subset \mathbb{C} \) is open.
Proof. Take $u \in U$. Without a loss of generality we can assume that $u = 0$ and $f(u) = 0$. Let us write a Taylor expansion of $f$ at $u = 0$:

$$f(z) = a_1 z + a_2 z^2 + \ldots$$

Let $a_k$ be the first coefficient which is not 0. Then

$$f(z) = a_k z^k + a_{k+1} z^{k+1} + \cdots = z^k (a_k + h(z)),$$

where $a_k \neq 0$ and $h(0) = 0$. If $r$ is small enough then $|h(z)| \leq \frac{|h(0)|}{2}$ for $|z| \leq r$. Choose $\rho < \frac{|h(0)|}{4}$. We claim that the disc $D_\rho = \{|z| < \rho\}$ is contained in $f(U)$. Indeed, for any point $v \in D_\rho$ the equation $a_k z^k = v$ has exactly $k$ solutions in $D_\rho(u)$. In other words, the function $g(z) = a_k z^k - v$ has $k$ zeros in $D_\rho(0)$. Note that for $z \in D_\rho(0)$ and $v \in D_\rho$ we have

$$|z^k h(z)| = r^k |h(z)| < \frac{|a_k| r^k}{2},$$

while for $|z| = r$ we have

$$|a_k z^k - v| \geq |a_k| r^k - \frac{|a_k| r^k}{4} > |z^k h(z)|.$$

Hence, according to Rouché’s theorem the function

$$f(z) - v = (a_k z^k - v) + z^k h(z)$$

has also $k \geq 1$ zeroes in $D_\rho(u)$, i.e. the point $v$ is in the image $f(U)$. ■

Corollary 7.17 implies (why?)

Corollary 7.18 (Maximum modulus principle). A non-constant holomorphic function $f : U \to \mathbb{C}$ cannot attain the maximum of its modulus $|f(z)|$ at an interior point of $U$.

7.6 Winding number

Consider a loop $\gamma : [0, 1] \to \mathbb{C} \setminus 0$, $\gamma(0) = \gamma(1)$. It winding number $w(\gamma)$ is defined as

$$w(\gamma) := \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z} = \int_{\gamma} d\theta.$$
The winding number depends on the orientation of the loop but not on its parameterization.

The following proposition is a reformulation of the argument principle.

**Proposition 7.19.** Let $U \subset \mathbb{C}$ be a domain with a piecewise smooth boundary and $f : U \to \mathbb{C}$ is a holomorphic function which extends as a $C^1$-function to $\overline{U}$. Let $\gamma_1, \ldots, \gamma_k : [0, 1] \to \partial U$ parameterizing the boundary components of $U$ according to their orientation as boundary components of $U$. Denote

$$\widetilde{\gamma}_j := f \circ \gamma_j, j = 1, \ldots k.$$  

Then

$$\sum_{j=1}^{k} w(\widetilde{\gamma}_j) = n,$$

where $n$ is the number of zeroes of the function $f$ counted with multiplicities.

**Proof.** Using Theorem [7.14] we get

$$n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)dz}{f(z)} = \sum_{j=1}^{k} \frac{1}{2\pi i} \int_{\gamma_j} \frac{df}{f} = \sum_{j=1}^{k} \frac{1}{2\pi i} \int_{f \circ \gamma_j} \frac{dz}{z} = \sum_{j=1}^{k} w(\widetilde{\gamma}_j).$$

\[\blacksquare\]
Chapter 8

Harmonic functions

8.1 Harmonic and holomorphic functions

The Laplace differential operator $\Delta$ on functions on domains in $\mathbb{C}$ is defined as $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, i.e.

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

for any $C^2$-function $f$. The Laplace operator can be rewritten in the complex notation as

$$\Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}, \text{ i.e. } \Delta f = 4 \frac{\partial^2 f}{\partial z \partial \bar{z}}.$$

Indeed,

$$\frac{\partial^2}{\partial z\partial \bar{z}} = \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = \frac{1}{4} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) = \frac{1}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \frac{1}{4} \Delta,$$

because mixed derivative terms cancel.

A real- or complex-valued $C^2$-smooth function $f$ on a domain $U \subset \mathbb{C}$ is called harmonic if $\Delta f = 0$.

Example 8.1. Any (inhomogeneous) linear function $u(x, y) = ax + by + c$ is harmonic. The function $\ln |z|$ is harmonic on $\mathbb{C} \setminus 0$. 

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The notions of a harmonic functions can be extended to much more general setup, in particular to domains in higher dimensional Euclidean spaces. However in the (real) 2-dimensional case the theory of harmonic and holomorphic functions are intertwined in a very special and interesting way.

**Theorem 8.2.** If \( f = u + iv : U \to \mathbb{C} \) is a holomorphic function, then \( f \), and therefore its real and imaginary parts \( u \) and \( v \) are harmonic. Conversely if \( u : U \to \mathbb{R} \) is a harmonic function and the domain \( U \) is simply connected, then there exists a unique up to an additive constant harmonic function \( v : U \to \mathbb{C} \), called harmonic conjugate of \( u \), such that the function \( f = u + iv \) is holomorphic.

**Proof.** Suppose \( f = u + iv : U \to \mathbb{C} \) is a holomorphic function. Then
\[
\Delta f = \Delta u + i\Delta v = 4 \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial \bar{z}} \right) = 0,
\]
because \( \frac{\partial f}{\partial \bar{z}} = 0 \), and hence \( \Delta f = 0 \) and \( \Delta u = \Delta v = 0 \).

Conversely, suppose that \( u \) is harmonic in a simply connected domain \( U \). Then \( g := u_x - iu_y \) is holomorphic. To see this we verify the Cauchy-Riemann equations for \( g \). We have \( \Delta u = u_{xx} + u_{yy} = 0 \), and hence \( (u_x)_x = (-u_y)_y \). We also have \( (u_x)_y = -(-u_y)_x \) due to the equality of mixed derivatives. In view of simply connectedness of \( U \) the holomorphic 1-form is exact, i.e. there exists a holomorphic function \( f = \tilde{u} + i\tilde{v} : U \to \mathbb{C} \) such that \( df = g(z)dz \), or
\[
\frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)(\tilde{u} + i\tilde{v}) = \frac{1}{2} \left( \tilde{u}_x + \tilde{v}_y + i(\tilde{v}_x - \tilde{u}_y) \right) = g(z) = u_x - iu_y.
\]
Taking into account the Cauchy-Riemann equations we get
\[
\frac{1}{2}(\tilde{u}_x + \tilde{v}_y) = \bar{u}_x = u_x,
\]
\[
\frac{1}{2}(\tilde{u}_y - \tilde{v}_x) = \bar{u}_y = u_y.
\]
Hence, \( d\tilde{u} = du \) and therefore \( \tilde{u} = u + C \), and we can choose \( C = 0 \). Thus the function \( v \) is a harmonic conjugate of \( u \), Any two holomorphic functions with the same real part differ by a constant, and hence the harmonic conjugate is defined up to adding a constant. □
Remark 8.3. If \( U \) is not simply connected then the harmonic conjugate may not exist as a univalent function \( v \), while its differential \( dv \) is well defined as a closed 1-form. For instance, consider the harmonic function \( u = \ln |z| \) on \( \mathbb{C} \setminus 0 \), \( z = re^{i\theta} = x + iy \). Then its harmonic conjugate is \( \phi = \arg z \), which is multivalued. At the same time the form \( d\phi = \frac{zdy - ydx}{x^2 + y^2} \) is a well defined closed 1-form.

8.2 Properties of harmonic functions

Theorem 8.2 implies that similarly to the case of holomorphic functions

Corollary 8.4. Two harmonic functions \( u, \tilde{u} : U \to \mathbb{R} \) on a connected domain \( U \) which coincide on a subdomain \( U' \subset U \) coincide in \( U \)

Proof. Take a point \( a \in U' \) and any point \( b \in U \). There exists a simply connected subdomain \( V \subset U \) which contains the points \( a, b \). Indeed, take any embedded path connecting \( a \) and \( b \) and choose its neighborhood as \( V \). Let \( v, \tilde{v} \) be the harmonic conjugate of \( u \) and \( \tilde{u} \) on \( V \). Holomorphic functions \( f := u + iv, \tilde{f} = \tilde{u} + i\tilde{v} \) have the same real parts near the point \( a \), and hence its imaginary parts near \( a \) differ by an additive constant. Hence, by adjusting this constant we can assume that \( f = \tilde{f} \) near \( a \). But then by uniqueness of the holomorphic continuation we have \( f = \tilde{f} \) on \( V \), and in particular \( f(a) = \tilde{f}(a) \) and hence \( u(a) = \Re f(a) = \Re \tilde{f}(a) = \tilde{u}(a) \).

An important fact is that the notion of a harmonic function is invariant with respect to a holomorphic change of coordinate.

Lemma 8.5. Let \( h : U \to \mathbb{C} \) be a \( C^2 \)-function and \( f : \overline{U} \to U \) a holomorphic function. Then

\[
\Delta(h \circ f)(z) = (\Delta h)(f(z))|f'(z)|^2.
\]

In particular, if \( h \) is harmonic then so is \( h \circ f \).

Proof.
\[ \Delta(h \circ f)(z) = 4 \frac{\partial}{\partial \bar{z}} \left( \frac{\partial (h \circ f)}{\partial z}(z) \right) = 4 \frac{\partial}{\partial \bar{z}} \left( \frac{\partial h}{\partial z}(f(z)) f'(z) + \frac{\partial h}{\partial z} \frac{\partial \bar{f}}{\partial \bar{z}} \frac{\partial f'}{\partial \bar{z}}(z) \right) \]
\[ = 4 \frac{\partial}{\partial \bar{z}} \left( \frac{\partial h}{\partial z}(f(z)) f'(z) \right) = 4 \left( \frac{\partial^2 h}{\partial \bar{z} \partial z}(f(z)) \frac{\partial f'}{\partial \bar{z}}(z) + \frac{\partial^2 h}{\partial \bar{z} \partial z} \frac{\partial f}{\partial \bar{z}}(z) f'(z) \right) \]
\[ = \Delta h(f(z)) f'(z) = \Delta h(f(z)) |f'(z)|^2, \]

because \( \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial z} = 0 \) and \( \frac{\partial f}{\partial \bar{z}} = f'(z). \)

**Example 8.6.** If \( f \) is a holomorphic function then \( h(z) = \ln |f(z)| \) is harmonic. Indeed, the function \( h(z) \) is the composition of a holomorphic function \( f \) with a harmonic function \( \ln |z| \).

**Theorem 8.7** (Mean value theorem). For any harmonic function \( h : U \to \mathbb{C} \), any point \( a \in U \) and \( r > 0 \) such that \( \overline{D}_r(a) = \{|z - a| \leq r\} \subset U \) one has
\[ h(a) = \frac{1}{2\pi} \int_0^{2\pi} h(a + re^{it})dt. \]

**Proof.** It is sufficient to assume that \( h \) takes real values (because we can prove the theorem separately for the real and imaginary parts. Take an open slightly larger disc \( D_\rho(a) = \{|z - a| < \rho\} \subset U \), \( \rho > r \). The disc \( D_\rho(a) \) is simply connected. Hence, we can find a harmonic function \( g : D_\rho(a) \to \mathbb{C} \) which is harmonic conjugate to \( h \), i.e. \( f := h + ig \) is a holomorphic function. Then by the Cauchy integral formula we have
\[ f(a) = \frac{1}{2\pi i} \int_{\partial D_\rho(a)} \frac{f(z)dz}{z - a} = \frac{1}{2\pi i} \int_0^{2\pi} f(a + re^{it})i e^{it} \frac{dt}{re^{it}} \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it})dt. \]
Taking the real part of this equality we get the required formula for the function \( h \).

**Corollary 8.8** (Maximum principle for harmonic functions). Let \( h : U \to \mathbb{R} \) be a non-constant harmonic function on a connected domain \( U \). Then it cannot achieve a local maximum at any (interior) point of \( U \).
**Proof.** Suppose that for $a \in U$ we have $u(a) \geq u(a + z)$ for all $|z| < \epsilon$. The function $h$ cannot be constant on $D_\epsilon(a)$, because then it would be a constant by the uniqueness of harmonic continuation, see Corollary 8.4. Therefore there exists $|z_0| < \epsilon$ such that $h(z_0) < h(a)$. But then

$$\frac{1}{2\pi} \int_0^{2\pi} f(a + |z_0|e^{it}) < f(a),$$

which contradicts Theorem 8.7.

**Remark 8.9.** Maximum principle for harmonic functions can also be deduced from the open image theorem for holomorphic functions.

One of the corollaries of the maximum principle is the uniqueness of an extension of a harmonic function from a boundary of a domain.

**Corollary 8.10.** Let $f, g : U \to \mathbb{R}$ be two harmonic functions which extend continuously to the boundary $\partial U$. Suppose that $f|_{\partial U} = g|_{\partial U}$. Then $f = g$ on $U$.

**Proof.** Suppose for $a \in U$ we have $f(a) > g(a)$. Then the maximum of the harmonic function $f - g$ is achieved in an interior point of $U$ which is impossible. ■
Chapter 9

Conformal mappings and their properties

9.1 Biholomorphisms

Let $f : U \to \mathbb{C}$ be a holomorphic function. Recall that the image $V := f(U)$ is an open set. Suppose that $f$ is injective, i.e. $f(z_1) \neq f(z_2)$ for any $z_1, z_2 \in U, z_1 \neq z_2$. Then $f$ can be viewed as a $1-1$, i.e. bijective map $U \to V$.

Lemma 9.1. If $f : U \to V$ is bijective and holomorphic, then the derivative $f'$ never vanishes, and the inverse map $f^{-1} : V \to U$ is also holomorphic.

Proof. Suppose $f'(a) = 0$ for $a \in U$. Then expanding $f$ to a Taylor series near the point $a$ we have

$$f(z) - f(a) = \frac{f''(a)}{2}(z-a)^2 + \cdots = (z-a)^k(c + g(z)),$$

where $k \geq 2$, $c \neq 0$ and $g(z)$ is a holomorphic function such that $g(a) = 0$. Hence, there exists $\epsilon > 0$ such that if $|z-a| \leq \epsilon$ then $|g(z)| < \frac{|c|}{2}$. Take any $b \neq 0, |b| < \frac{|c|e^k}{2}$. Note that the equation $c(z-a)^k = b$ has exactly $k$ solutions in the disk $D_\epsilon(a) = \{|z-a| < \epsilon\}$. For any $z \in \partial D_\epsilon(a)$ we have

$$|c(z-a)^k - b| > \frac{|c|e^k}{2} > \epsilon^k |g(z)| = |(z-a)^kg(z)|.$$

Note that

$$f(z) - (f(a) + b) = (c(z-a)^k - b) + (z-a)^kg(z).$$
Hence, Rouché’s theorem implies that the equation $f(z) = f(a) + b$ has the same number of solutions as the equation $c(z - a)^k = b$, which is $k > 1$, which contradicts to the injectivity of $f$. This proves that $f'(z) \neq 0$ for all $z \in D_\epsilon(a)$.

But then the chain rule implies the the inverse map $h = f^{-1} : V \to U$ is also hoomorphic and $h'(f(z)) = \frac{1}{f'(z)}$.

A bijective holomorphic map $f : U \to V$ is called a biholomorphism. Thus the map $f^{-1} : V \to U$, inverse to a biholomorphism is itself a biholomorphism.

9.2 Conformal mappings

Let us assume that $\mathbb{C} = \mathbb{R}^2$ is endowed with the standard Euclidean metric. Any orientation preserving orthogonal transformation of $\mathbb{R}^2$ is a rotation, i.e in complex notation is given by $z \mapsto e^{i\theta}z$. Note that any complex linear map $\mathbb{C} \to \mathbb{C}$ has the form $z \mapsto cz$, i.e. $z \mapsto re^{i\theta}z$, where $c = re^{i\theta}$. Geometrically this map is characterized by two properties: it preserves the orientation and it preserves all angles. Maps with this properties are called linear conformal. Any linear conformal map is a composition of a rotation with a scaling (homothety) $z \mapsto rz$, and therefore linear conformal maps $\mathbb{R}^2 \to \mathbb{R}^2$ are exactly the same as linear complex maps $\mathbb{C} \to \mathbb{C}$.

A differentiable in the real sense bijective map $f : U \to V$ is called conformal if its differential $d_zf : \mathbb{C}_z \to \mathbb{C}_{f(z)}$ is linear conformal for any $z \in U$.

The above discussion implies that conformal maps $U \to V$ coincide with biholomorphisms $U \to V$.

**Remark 9.2.** A not necessarily linear map $f : U \to \mathbb{R}^2$ is called an isometry if its differential at every point is orthogonal, i.e. preserves the Euclidean metric. However, one can show that any isometry $U \to \mathbb{R}^2$ has to be an affine map: it is a composition of a rotation with a parallel translation. It is a remarkable fact that by relaxing the isometry condition to the conformality condition one greatly enlarges the class of maps.

If there exists a conformal map (or a biholomorphism) $f : U \to V$, then $U$ and $V$ are called
biholomorphic or conformally equivalent.

9.3 Examples of conformal mappings

9.3.1 Unit disc and the upper-half plane

We begin with exploring the inversion operation \( \text{inv} : \mathbb{C} \setminus 0 \to \mathbb{C} \setminus 0 \), which in polar coordinates is given by the formula \( r \mapsto \frac{1}{r} \). In complex notations we have \( \text{inv}(z) = \frac{1}{\bar{z}} \).

**Lemma 9.3.** Image \( \text{inv}(l) \) of the line \( l = \{\text{Im} z = d\} \) is the circle \( \{|z - \frac{i}{2d}| = \frac{1}{2d}\} \).

**Proof.** We have \( \text{inv}(x + id) = \frac{1}{x-id} = \frac{x+id}{d^2+x^2} \). Therefore,

\[
\left| \text{inv}(x + id) - \frac{i}{2d} \right| = \left| \frac{x + id}{d^2 + x^2} - \frac{i}{2d} \right| = \left| \frac{2id^2 + 2dx - id^2 - ix^2}{d^2 + x^2} \right| = \frac{\sqrt{d^4 + x^4 - 2d^2x^2 + 4d^2x^2}}{2d(d^2 + x^2)} = \frac{1}{2d} \]

Therefore, the map \( z \mapsto \frac{1}{z} \) maps the half-plane \( \{\text{Im} z > d\} \) onto the open disc \( \{|z - \frac{i}{2d}| < \frac{1}{2d}\} \).

Consequently,

**Proposition 9.4.** The map

\[
z \mapsto \frac{2}{z+i} + i = \frac{iz + 1}{z + i}
\]

is a biholomorphism between the upper half plane \( \mathbb{H} = \{\text{Im} z > 0\} \) and the unit disc \( \mathbb{D} = \{|z| < 1\} \).

**Exercise 9.5.** Show that

\[
z \mapsto \frac{i - z}{i + z}
\]
defines another conformal equivalence \( \mathbb{H} \to \mathbb{D} \), and that the inverse map \( \mathbb{D} \to \mathbb{H} \) is given by

\[
z \mapsto \frac{1 - z}{1 + z}.
\]

*It is called* Joukovsky’s map.

**Exercise 9.6.** Show that the map \( z \mapsto z + \frac{1}{z} \) is a conformal isomorphism of \( \mathbb{C} \setminus \overline{\mathbb{D}} \) onto \( \mathbb{C} \setminus \{-2 < \text{Re} z < 2; \text{Im} z = 0\} \).
9.3.2 Strips and sectors

The map $z \mapsto e^z$ conformally maps the infinite strip $P_a := \{ -a < \Im z < a \}$, $a < \pi$ onto the sector $S_a := \{ -a < \arg z < a \}$. The strip $P_\pi := \{ -\pi < \Im z < \pi \}$ is mapped by the exponential map onto the domain $\mathbb{C} \setminus \{ \Re z \leq 0, \Im z = 0 \}$, the complement of the negative real ray in $\mathbb{C}$.

The map $z \mapsto z^2$ establishes a biholomorphism between the upper-half plane $\mathbb{H} = \{ \Im z > 0 \}$ and the complement $\mathbb{C} \setminus \{ \Re z \geq 0, \Im z = 0 \}$ of the positive real ray. The map $z \mapsto z^\alpha$ for $0 < \alpha < 1$ establishes a biholomorphism between the the upper-half plane $\mathbb{H}$ and the sector $\{ 0 < \arg z < \alpha \pi \}$.

Consequently, taking compositions of the above biholomorphisms we can establish more conformal equivalences. For instance, the composition $\log(-z^2)$ of the maps $z \mapsto z^2$, $z \mapsto -z$ and $z \mapsto \log z$ maps the upper-half plane $\mathbb{H}$ onto the strip $P_\pi := \{ -\pi < \Im z < \pi \}$.

9.4 Schwarz lemma

The following statement known as the “Schwarz lemma” will be useful for our further study of conformal mappings.

**Theorem 9.7 (Schwarz lemma).** Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map with $f(0) = 0$ ($\mathbb{D}$ denotes the unit disc). Then

(i) $|f(z)| \leq |z|$;

(ii) If for some $z_0 \in \mathbb{D}$ we have $|f(z_0)| = |z_0|$ then $f$ is a rotation;

(iii) $|f'(0)| \leq 1$, and if $|f'(0)| = 1$ then $f$ is a rotation.

**Proof.** The equality $f(0) = 0$ implies that $g(z) := \frac{f(z)}{z}$ is holomorphic. If $|z| = r$ then

$$\left| \frac{f(z)}{z} \right| = \frac{|f(z)|}{r} \leq \frac{1}{r}.$$ 

Hence, applying the maximum modulus principle we conclude that this true for all $|z| < r$, and therefore passing to the limit $r \to 1$ we get $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. If $|f(z_0)| = |z_0|$ then $z_0$ is an
interior maximum point of the function \( g(z) = \frac{f(z)}{z} \), and hence \( f(z) = cz \) and \( |c| = \left| \frac{f(z_0)}{z_0} \right| = 1 \), i.e. \( f \) is a rotation which proves (ii). Finally the inequality \( |f'(0)| \leq 1 \) follows from the Cauchy formula. We also notice that \( f'(0) = \lim_{z \to 0} \frac{f(z)}{z} = g(0) \), and if \( |f'(0)| = |g(0)| = 1 \), then again the maximum modulus principle implies that \( g(z) = c \), with \( |c| = 1 \), and hence \( f(z) = cz \) is a rotation.

\[ \square \]

Corollary 9.8. Let \( f : \mathbb{D} \to \mathbb{D} \) be a conformal equivalence such that \( f(0) = 0 \). Then \( f \) is a rotation, i.e \( f(z) = e^{i\theta}z, \theta \in \mathbb{R} \).

Proof. By 9.7(iii) we have \( |f'(0)| \leq 1 \), and applying 9.7(iii) to \( f^{-1} \) we get \( |(f^{-1})'(0)| = \frac{1}{|f'(0)|} \geq 1 \). Hence, \( |f'(0)| = 1 \), and applying again 9.7(iii) we conclude that \( f \) is a rotation.

\[ \square \]

9.5 Automorphisms of the Riemann sphere, \( \mathbb{C} \), the unit disc and the upper-half plane

Given a domain \( U \) its self-biholomorphisms \( U \to U \) are called automorphisms. Composition of automorphisms are automorphisms and inverse automorphisms are automorphisms as well. Hence, automorphisms of a given domain form a group.

9.5.1 \( GL(n, \mathbb{C}), GL(n, \mathbb{R}), PGL(n, \mathbb{C}), PGL(n, \mathbb{R}) \) and \( PGL_+(n, \mathbb{R}) = PSL(n, \mathbb{R}) \)

The notation \( GL(n, \mathbb{C}) \) and \( GL(n, \mathbb{R}) \) stand for the group of complex and real linear transformations of \( \mathbb{C}^n \) and \( \mathbb{R}^n \), respectively, or equivalently the groups of \( n \times n \) complex and real non-degenerate matrices. These groups are called the general groups of complex and real linear transformations, respectively.

Note that a linear transformation \( A : \mathbb{C}^n \to \mathbb{C}^n \) defines also a transformation of the projective space \( \mathbb{C}P^{n-1} \). Indeed, the transformation \( A \) maps lines to lines. Such transformations of \( \mathbb{C}P^{n-1} \) are called complex projective transformations. Note that two transformations \( A, \tilde{A} : \mathbb{C}^n \to \mathbb{C}^n \) define the same transformation of \( \mathbb{C}P^{n-1} \) if and only if they are proportional: \( \tilde{A} = cA, c \in \mathbb{C} \). Projective
transformations of \( \mathbb{C}P^{n-1} \) form the complex projective linear group \( PGL(n, \mathbb{C}) \). Thus an element of \( PGL(n, \mathbb{C}) \) is an \( n \times n \) complex matrix up to a complex scalar factor.

Similarly, one defines the real projective group \( PGL(n, \mathbb{R}) \) of projective transformations of \( \mathbb{R}P^{n-1} \). An element of this group can be viewed as an \( n \times n \) real matrix up to a real scalar factor. If \( n \) is even, the group \( PGL(n, \mathbb{R}) \) consists of two connected components \( PGL_+(n, \mathbb{R}) \) and \( PGL_-(n, \mathbb{R}) \), of orientation preserving and reversing orientations. Note that \( PGL_+(n, \mathbb{R}) \) is a subgroup of \( PGL(n, \mathbb{R}) \) while \( PGL_-(n, \mathbb{R}) \) is not. If \( n \) is odd then multiplying a matrix by \(-1\) one changes the sign of its determinant and hence \( PGL(n, \mathbb{R}) \) is connected.

The group \( PGL_+(n, \mathbb{R}) \) is also denoted \( PSL(n, \mathbb{R}) \). The notation \( SL(n, \mathbb{R}) \) stands for the special linear group, i.e. the group of \( n \times n \) matrices with determinant 1. The projective special group \( PSL(n, \mathbb{R}) \) is obtained from \( SL(n, \mathbb{R}) \) by identifying matrices \( A \) and \(-A\). Clearly we get the same thing by identifying matrices \( A \) and \(-A\) in \( SL(n, \mathbb{R}) \), or by identifying all proportional matrices in \( PGL_+(n, \mathbb{R}) \). Hence, \( PGL_+(n, \mathbb{R}) = PSL(n, \mathbb{R}) \).

It turns out that the groups \( PGL(2, \mathbb{C}) \) and \( PSL(2, \mathbb{R}) \) can be also interpreted as groups of conformal automorphisms of some special domains. Namely, we will see below that elements of \( PGL(2, \mathbb{C}) \) serve as automorphisms of the Riemann sphere \( \mathbb{C}P^1 \), while elements of \( PSL(2, \mathbb{R}) \) act as automorphisms of upper-half plane \( \mathbb{H} \).

### 9.5.2 Automorphisms of \( \mathbb{C}P^1 \) and \( \mathbb{C} \)

Take an element \( A \in PGL(2, \mathbb{C}) \), i.e. a complex matrix

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

up to a complex scalar factor and associate with it a fractional linear transformation

\[
z \mapsto \frac{az + b}{cz + d}.
\]

**Lemma 9.9.** The function \( z \mapsto f_A(z) = \frac{az + b}{cz + d} \) is a conformal automorphism \( \mathbb{C}P^1 \to \mathbb{C}P^1 \).

**Proof.** The function \( f \) is meromorphic, and hence a holomorphic map \( \mathbb{C}P^1 \to \mathbb{C}P^1 \). So we only need to check that it is bijective. But this follows from the fact that for any \( w \in \mathbb{C} \subset \mathbb{C}P^1 \) we can
uniquely solve the equation \( \frac{az + b}{cz + d} = w \):

\[
z = \frac{dw - b}{(ad - bc)(-cw + a)}
\]

and if \( w = \infty \) then \( z = -\frac{d}{c(ad - bc)} \) if \( c \neq 0 \), and \( z = \infty \) otherwise. ■

**Lemma 9.10.**

\[ f_{AB} = f_A \circ f_B. \]

**Proof.** The above property can be easily verified by the direct computation. However, we present here a more conceptual proof.

Recall that \( \mathbb{C}P^1 \) is the space of complex lines in \( \mathbb{C}^2 \), or equivalently the space of pairs \((z_1, z_2) \neq (0, 0)\) of complex numbers up to proportionality \((z_1, z_2) \sim (\lambda z_1, \lambda z_2), \lambda \in \mathbb{C}\). \( \mathbb{C}P^1 \) can be covered by two coordinate charts, with a coordinate \( u = \frac{z_1}{z_2} \), where \( z_2 \neq 0 \) and \( v = \frac{1}{u} = \frac{z_2}{z_1} \), where \( z_1 \neq 0 \).

A matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(n, \mathbb{C}) \) acts on \( \mathbb{C}^2 \) by

\[
z = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}.
\]

This action defines the action on \( \mathbb{C}P^1 \), which in terms of the coordinate \( u \) has the form

\[
u = \frac{z_1}{z_2} \mapsto \frac{zu_1 + bu_2}{cu_1 + du_2} = \frac{au + b}{cu + d},
\]

i.e. it acts exactly by fractional linear transformations. But on \( \mathbb{C}^2 \) the composition of transformation corresponds to multiplication of matrices, and hence so does the composition of fractional linear transformations. ■

It turns out that

**Proposition 9.11.** Any conformal automorphism \( \mathbb{C}P^1 \to \mathbb{C}P^1 \) is a fractional linear transformation.

**Proof.** Holomorphic maps \( \mathbb{C}P^1 \to \mathbb{C}P^1 \), i.e. meromorphic functions, are, according to Theorem 7.11, are rational. Let us analyze when a rational function \( f(z) = \frac{p(z)}{q(z)} \) defines an bijective map
\( \mathbb{C}P^1 \to \mathbb{C}P^1 \). We can assume that the polynomial \( P(z) \) and \( Q(z) \) have no common divisors, which is equivalent to the fact that they do not have any common zeroes. If the degree of the polynomial \( P \) is \( > 1 \) then the function \( f(z) \) has at least two distinct zeroes, or one of the zeroes has multiplicity \( > 1 \). In both cases this implies that \( f \) is not 1-1. Indeed, if there are two distinct zeroes \( z_1, z_2 \) then \( f(z_1) = f(z_2) = 0 \) of \( z_0 \) is a multiple zero then the function \( f \) in a neighborhood of \( z_1 \) can be written as \( c(z - z_1)^k (1 + h(z)) \), where \( k > 1, c \neq 0 \) and \( h(0) = 0 \). Then arguing as in the proof of Lemma 9.1 we conclude that \( f \) is not 1-1 on this neighborhood. Applying the same argument to the function \( \frac{1}{f} \) we show that the degree of \( Q \) is also \( \leq 1 \), and hence \( f \) is fractional linear.

Combining Proposition 9.11 and Lemma 9.9 we get

**Theorem 9.12.** The group \( \text{Aut}(\mathbb{C}P^1) \) of conformal automorphisms of the Riemann sphere is isomorphic to \( \text{PGL}(2, \mathbb{C}) \). The elements of \( \text{PGL}(2, \mathbb{C}) \) act on \( \mathbb{C}P^1 \) by fractional linear transformations.

**Proposition 9.13.** For any 3 points \( z_0, z_1, z_2 \) of \( \mathbb{C}P^1 \) there exists a unique automorphism \( f \in \text{PGL}(2, \mathbb{C}) \) such that \( f(0) = z_0, f(1) = z_1, f(\infty) = z_2 \).

**Proof.** Let \( f(z) = \frac{az + b}{cz + d} \). The required conditions amount to the system of equations on the coefficients of \( f \) (which are given up to a proportionality factor):

\[
\begin{align*}
b &= dz_0, \\
a + b &= z_1(c + d), \\
a &= cz_2.
\end{align*}
\]

We can set \( d = 1 \) and then get \( b = z_0 \) and

\[
\begin{align*}
cz_2 + z_0 &= cz_1 + c, \\
a &= cz_2.
\end{align*}
\]

Thus, \( c = \frac{cz_0}{z_2 - z_1}, a = \frac{(c - z_0)z_2}{z_2 - z_1} \).

Given 4 points \( z_1, z_2, z_3, z_4 \) their *cross ratio* is defined as

\[
(z_1, z_2; z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}.
\]

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For instance, we have
\[(1, z; \infty, 0) = \frac{(\infty - 1)(0 - z)}{(\infty - z)(0 - 1)} = z.\]

**Exercise 9.14.** Prove that in order that two 4-tuples of points were equivalent under a conformal equivalence of \(\mathbb{C}P^1\) it is necessary and sufficient that they had the same cross-ratio.

Any automorphism \(f : \mathbb{C} \to \mathbb{C}\) extends, by the removal of singularities lemma to \(\mathbb{C}P_1\) (why?). Hence, \(f(z) = \frac{az + b}{cz + d}\) and \(f(\infty) = \infty\) implies that \(c = 0\). Therefore, \(f(z) = Az + B\), where \(A = \frac{a}{d}, B = \frac{b}{d}\). In other words,

**Proposition 9.15.** Any automorphism of \(\mathbb{C}\) is a composition of a rotation, a scaling (homothety) and a parallel transport.

### 9.5.3 Automorphisms of \(\mathbb{H}\) and \(\mathbb{D}\)

It turns out that any automorphism of \(\mathbb{H}\) or \(\mathbb{D}\) extends to an automorphism of \(\mathbb{C}P^1\) and hence \(\text{Aut}(\mathbb{H})\) and \(\text{Aut}(\mathbb{D})\) are subgroups of \(\text{Aut}(\mathbb{C}P^1)\) consisting of fractional linear transformations of \(\mathbb{C}P^1\) which map \(\mathbb{H}\) and \(\mathbb{D}\) onto themselves, or as one says, leave them invariant.

**Lemma 9.16.** A fractional linear transformation \(f(z) = \frac{az + b}{cz + d}\) leaves the upper-half plane \(\mathbb{H}\) invariant (i.e. \(f(\mathbb{H}) = \mathbb{H}\)) if and only if \(f \in \text{PSL}(2, \mathbb{R})\), i.e. when the matrix \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) is proportional to a real matrix with a positive determinant.

**Proof.** Indeed, if \(a, b, c, d\) are real and \(\det A = ad - bc > 0\) then for each \(z = x + iy\) with \(y > 0\) we have

\[
\frac{az + b}{cz + d} = \frac{ax + b + iay}{cx + d + icy} = \frac{(ax + b + iay)(cx + d - icy)}{(cx + d)^2 + c^2y^2},
\]

and

\[
\text{Im}\left(\frac{az + b}{cz + d}\right) = \frac{ax + b + iay}{cx + d + icy} = \frac{ay(cx + d) - cy(ax + b)}{(cx + d)^2 + c^2y^2} = \frac{y(ad - bc)}{(cx + d)^2 + c^2y^2} > 0.
\]
We leave it as an exercise to prove the converse, that is if for a fractional linear transformation \( f \) we have \( f(\mathbb{H}) = \mathbb{H} \) then \( f \in \text{PSL}(2, \mathbb{R}) \), i.e. the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is proportional to a real matrix with a positive determinant.

**Corollary 9.17.** For any points \( \alpha, \beta \in \mathbb{H} \) there exists an automorphism \( f : \mathbb{H} \to \mathbb{H} \) such that \( f(\alpha) = \beta \)

**Proof.** It is sufficient to consider the case \( \alpha = i \). Let \( \beta = p + iq \). We have \( q > 0 \). We need to find real \( a, b, c, d \) solving the equation
\[
\frac{ai + b}{ci + d} = p + iq,
\]
or
\[
ai + b = dp - cq + i(cp + dq).
\]
This yields the linear system
\[
\begin{align*}
pd - qc &= b \\
qd + pc &= a.
\end{align*}
\]
The determinant \( p^2 + q^2 > 0 \). Hence, for any \( a, b \) we can solve the system with respect to \( c, d \). For instance, taking \( b = 1, a = 0 \) we find
\[
d = \frac{p}{p^2 + q^2},
\]
\[
c = \frac{-q}{p^2 + q^2}.
\]
Note that
\[
\begin{vmatrix}
0 & 1 \\
-\frac{q}{p^2 + q^2} & \frac{p}{p^2 + q^2}
\end{vmatrix} = \frac{q}{p^2 + q^2} > 0,
\]
and hence the fractional linear transformation belongs to \( \text{PSL}(2, \mathbb{R}) \).

**Remark 9.18.** The statement of the corollary means that \( \text{PSL}(2, \mathbb{R}) \) acts on \( \mathbb{H} \) transitively. One says that a group \( G \) acts transitively on a set \( X \) if for any two points \( x_0, x_1 \in G \) there is a transformation from \( G \) which moves \( x_0 \) to \( x_1 \). As a corollary we get that fractional linear transformations act transitively on \( \mathbb{D} \) as well.
Theorem 9.19. Any conformal automorphism of $\mathbb{D}$ and $\mathbb{H}$ is fractional linear, and hence in the case of $\mathbb{H}$ is given by a matrix from $PSL(2, \mathbb{R})$.

Proof. Let $f : \mathbb{D} \to \mathbb{D}$ be a conformal automorphism. By composing $f$ with a fractional linear transformation $g : \mathbb{D} \to \mathbb{D}$ we can arrange $g(f(0)) = 0$, because fractional linear automorphisms of $\mathbb{D}$ act transitively. Applying Corollary 9.8 we then conclude that $g \circ f$ is a multiplication by $e^{i\theta}$, and hence $f = e^{i\theta}g^{-1}$ is fractional linear. But $\mathbb{H}$ and $\mathbb{D}$ are conformally equivalent via a fractional linear transformation. Hence, any automorphism of $\mathbb{H}$ is fractional linear as well, and by Lemma 9.16 it belongs to $PSL(2, \mathbb{R})$.

Exercise 9.20. Prove that a fractional linear transformation $f(z) = \frac{az+b}{cz+d}$ leaves the unit disc $\mathbb{D}$ invariant (i.e. $f(\mathbb{D}) = \mathbb{D}$) if and only if $f$ has the form

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \bar{\alpha}z}, \quad (9.5.1)$$

where $\alpha \in \mathbb{D}$ and $\theta \in [0, 2\pi)$.

Exercise 9.21. Prove that $\mathbb{C}$ is not conformally equivalent to $\mathbb{H}$ (or $\mathbb{D}$).

Hint: Apply the Liouville theorem for a holomorphic map $\mathbb{C} \to \mathbb{D}$.

9.6 Summary of useful conformal maps

$$\mathbb{D} \to \mathbb{C}P^1 \setminus \overline{\mathbb{D}}$$

$z \mapsto \frac{1}{z}$.

$$\mathbb{H} \to \mathbb{D}$$

$z \mapsto \frac{iz + 1}{z + i}$ or $z \mapsto \frac{i - z}{i + z}$.

General form : $z \mapsto e^{i\theta} \frac{z - a}{z - \overline{a}}$, $a \in \mathbb{H}$, $\theta \in [0, 2\pi)$.

This automorphism sends the point $a$ to the center of the disc.
\[
\mathbb{D} \to \mathbb{H} \\
\quad z \mapsto i\frac{1-z}{1+z}
\]

\[
\mathbb{D} \to \mathbb{D} \\
\quad z \mapsto e^{i\theta} \frac{\alpha - z}{1 - \bar{\alpha}z}, \quad \alpha \in \mathbb{D}, \quad \theta \in [0, 2\pi).
\]

This automorphism sends the point \(\alpha\) to the center of the disc.

**Strip to sector**

\[
z \to e^z \text{ maps a strip } \{0 < \text{Im } z < \alpha\} \text{ onto the sector } 0 < \text{arg } z < \alpha, \quad \alpha \in (0, 2\pi).
\]

In particular, for \(\alpha = \pi\) it maps the strip \(\{0 < \text{Im } z < \pi\}\) onto \(\mathbb{H}\).

\[
\mathbb{H} \to \mathbb{C} \setminus R \\
\quad z \to z^2, \quad \text{where } R = \{\text{Im } z = 0, \text{Re } z > 0\}.
\]

**Sector to \(\mathbb{H}\)**

\[
z \to z^\alpha \text{ maps the sector } 0 < \text{arg } z < \frac{\pi}{\alpha} \text{ onto } \mathbb{H}, \quad \alpha > \frac{1}{2}
\]

**Complement of an interval onto the complement of a disc**

\[
z \mapsto z + \frac{1}{z},
\]

see Exercise [9.3.1]
Chapter 10

Riemann mapping theorem

**Theorem 10.1** (B. Riemann–P. Koebe). Any simply connected domain $U \subset \mathbb{C}$, $U \neq \mathbb{C}$ is conformally equivalent to $\mathbb{D}$. In other words, any two simply connected domains in $\mathbb{C}$ which are different from $\mathbb{C}$ are conformally equivalent.

This theorem is one of the crown achievements of Mathematics of 19th century (except that the first proper proof was given only in 20th century by Koebe).

**Plan of the proof**

Assume that $U \neq \mathbb{C}$.

**Step 1:** Constructing of an injective map $f : U \to \mathbb{D}$. See Proposition [10.7]. Note that by assumption there is at least one point $a \notin U$. If we manage to conformally map $U$ onto a domain missing $D_e(a)$ this would prove the claim (why?).

**Step 2:** Proving that the limit of a uniformly on compact set converging sequence of injective holomorphic maps is injective. See Lemma [10.9].

**Step 3:** Finding a converging sequence of injective maps $f_n : U \to \mathbb{D}$ with $f_n(z_0) = 0$ maximizing the modulus of the derivative $|f'_n(z_0)|$.

See Proposition [10.8]. Note that Step 2 implies that $f$ is injective.
Step 4: Prove that $f$ constructed in Proposition 10.8 is surjective. See Proposition 10.10. This concludes the proof of Theorem 10.1.thm:RMT.

Before proving theorem we need to develop some additional tools.

10.1 Functional analytic background

10.1.1 Arzelá-Ascoli theorem

The Arzelá-Ascoli theorem is a general fact, not specific to the context of holomorphic functions.

Let $K \subset \mathbb{R}^n$ be a compact set, and $f_n : K \to \mathbb{R}^m$ a sequence of continuous maps.

We say that $f_n$ uniformly converges to a map $f : K \to \mathbb{R}^m$ if for any $\epsilon > 0$ there exists an integer $N$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and all $x \in K$.

Lemma 10.2. The uniform limit of a sequence of continuous maps is continuous.

Proof. For any points $a, x \in K$ we have

$$|f(x) - f(a)| = |(f(x) - f_n(x)) + (f_n(x) - f_n(a)) + (f_n(a) - f(a))|$$

$$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|.$$  (10.1.1)

Uniform convergence implies that for any $\epsilon$ there exists $N$ such that $|f(x) - f_n(x)|, |f_n(a) - f(a)| < \frac{\epsilon}{4}$ for $n \geq N$. Here $N$ is independent of $a$ and $x$. On the other hand, $f_N$ is continuous and hence there exists $\delta > 0$ such that $|f_N(x) - f_N(a)| < \frac{\epsilon}{4}$ provided that $|x - a| < \delta$. Hence, (10.1.1) implies that $|f(x) - f(a)| < \epsilon$ if $|x - a| < \delta$, i.e. $f$ is continuous at $a$.

It is straightforward to prove that a uniform convergence is equivalent to the uniform Cauchy property: for any $\epsilon$ there exists $n$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in K$ and all $m, n \geq N$. Thus, according to Lemma 10.2 a uniformly Cauchy sequence of continuous functions always converges to a continuous function.

We say that $f_n$ is uniformly bounded if there exists a constant $C > 0$ such that $|f_n(x)| < C$ for all $n$ and all $x \in K$.  

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We say that $f_n$ is equicontinuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that $|f_n(x) - f_n(y)| < \delta$ for all $n$ and any $x, y \in K$ such that $|x - y| < \delta$.

**Theorem 10.3** (Arzelá-Ascoli). If the sequence $f_n : K \rightarrow X$ is equicontinuous and uniformly bounded, there is a subsequence $f_{n_k}$ which uniformly converges to a continuous function $f : K \rightarrow X$.

**Proof.** The proof follows the standard diagonal argument. We take a sequence of all points in $K$ with rational coordinates. This is a countable set, so we can enumerate them: $x_1, x_2, \ldots \in K$. The sequence $f_k(x_j)$ is bounded. Hence by Bolzano-Weierstrass we can find a converging subsequence $f_{k_1}(x_1)$. Repeating the argument for the sequence $f_{k_1}(x_2)$ we find its converging subsequence $f_{k_2}(x_2)$. Continuing inductively we find subsequences

$$\{f_{k_1}\} \supset \{f_{k_2}\} \supset \ldots$$

such that the subsequence $f_{k_m}$ converges on points $x_1, \ldots, x_m$. We claim that $f_{k_1} \ldots$ converges on the whole compact set $K$. Hence the diagonal subsequence $f_{k_1}$ converges on all rational points $x_m$, $m = 1, \ldots$, i.e. for any $\epsilon > 0$ and any $m$ there exists $N = N_m$ such that $|f_{n_k}(x_m) - f_{k_1}(x_m)| < \epsilon$ for all $n, k \geq N$. The equicontinuity of $f_n$ implies that there exists $\delta > 0$ such that $|f_{n_k}(x) - f_{n_k}(y)| < \epsilon$ for any $n$ and any $x, y \in K$ such that $|x - y| < \delta$ But the set $x_k$ is everywhere dense. Hence, there exists a subsequence $x_{m_j} \rightarrow x$. In particular there exists $J$ such that for $j > J$ we have $|x_{m_j} - x| < \delta$. Thus,

$$|f_{n_k}(x) - f_{k_1}(x)| \leq |f_{n_k}(x) - f_{n_k}(x_{m_j})| + |f_{n_k}(x_{m_j}) - f_{k_1}(x_{m_j})| + |f_{k_1}(x) - f_{k_1}(x_{m_j})| \leq 3\epsilon.$$

for all sufficiently large $n$ and $k$. This means that $f_{n_k}$ satisfies a uniform Cauchy property and hence it uniformly converges to a continuous function. ■

**Corollary 10.4.** Let $U \subset \mathbb{R}^n$ be an open domain. If the sequence $f_n : U \rightarrow X$ is equicontinuous and uniformly bounded on every compact subset $K \subset U$, then there is a subsequence $f_{n_k}$ which uniformly on all compact sets converges to a continuous function $f : U \rightarrow X$.

**Proof.** One can exhaust $U$ by a sequence of compact subsets $K_1 \subset K_2 \subset \ldots U, \bigcup_j K_j = U$, apply Arzelá-Ascoli to choose a subsequence converging on $K_1$, from it choose a subsequence converging on $K_2$ etc. Finally choose a diagonal subsequence. ■
10.1.2 Montel’s theorem

**Theorem 10.5** (Montel’s theorem). *Suppose that a family $\mathcal{F}$ of holomorphic functions $U \to \mathbb{C}$ is uniformly bounded on all compact subsets $K \subset U$. Then one can find a subsequence uniformly converging on all compact sets $K \subset U$.*

**Proof.** A family of uniformly bounded on compact sets holomorphic functions is uniformly continuous on compact sets. Indeed, the Cauchy inequality guarantees (why?) that the family of derivatives $\{f'; f \in \mathcal{F}\}$ is also uniformly bounded, but this implies the uniform continuity via the mean-value theorem.

Hence, we can apply Arzelá-Ascoli to extract a subsequence $f_{kj}$ uniformly converging on compact sets. But the same arguments applies to the derivatives, so we conclude that (possibly after passing to a subsequence), the derivatives $f'_{kj}$ also converge uniformly on compact sets, but then the limit function is holomorphic. ■

**Remark 10.6.** A family $\mathcal{F}$ of holomorphic functions $f : U \to \mathbb{C}$ is called normal, if every sequence in $\mathcal{F}$ contains a subsequence which uniformly converging on all compact subsets of $U$ (but not necessarily to a function from $\mathcal{F}$). Hence, Montel’s theorem says that a uniformly bounded on compact set family is normal.

10.2 Proof of the Riemann mapping theorem

10.2.1 Embedding into $\mathbb{D}$

**Proposition 10.7.** *Let $U \subset \mathbb{C}$ be a simply connected open subset of $\mathbb{C}$, not equal to $\mathbb{C}$. Then there exists an injective holomorphic map $f : U \to \mathbb{D}$.***

**Proof.** Without loss of generality we can assume $0 \notin U$. Choose a point $z_0 \in U$ and define the logarithm branch $\log^U z$. We have $e^{\log^U(z)} = z$, and in particular, the map $\log^U$ is injective.

We claim that there exists $\epsilon > 0$ such that the disc $D_\epsilon(\log^U(z_0) + 2\pi i)$ is contained in $\mathbb{C} \setminus \log^U(U)$. Indeed, if this is not true then there exists a sequence $\log^U(z_n) \in \log^U(U) \cap D_\frac{1}{n}(\log^U(z_0) + 2\pi i)$,
\[ n = 1, 2, \ldots, \text{ and hence } \lim_{n \to \infty} \log^U(z_n) = \log^U(z_0) + 2\pi i. \] Therefore, \( e^{\log^U(z_n)} \to e^{2\pi i + \log^U(z_0)} = z_0. \) But \( e^{\log^U(z_n)} = z_n, \) and therefore \( \lim_{n \to \infty} z_n = z_0. \) But then \( \log^U(z_n) \to \log^U(z_0) \neq \log^U(z_0) + 2\pi i, \) which is a contradiction.

Hence the holomorphic function \( f \) injectively maps \( U \) into the complement of the disc \( D_\epsilon(2\pi i). \) On the other hand, the map \( g(z) = \frac{\epsilon}{z - 2\pi i} \) conformally maps the complement of \( D_\epsilon(2\pi i) \) in the Riemann sphere onto \( \mathbb{D}. \) Hence, composing \( \log^U \) and \( g \) we get the required injective holomorphic map

\[ f(z) = \frac{\epsilon}{\log^U(z) - 2\pi i} \]

of \( U \) into \( \mathbb{D}. \)

**10.2.2 Maximizing the derivative**

In Proposition \[10.7\] we constructed an injective holomorphic map \( f : U \to \mathbb{D} \) such that \( f(z_0) = 0. \) Let us denote by \( \mathcal{F} \) the set of all holomorphic maps with this property. Note that by Cauchy inequality \( |f'(z_0)| \) is uniformly bounded by some constant \( C \) (which depends on the distance of \( z_0 \) to \( \partial U \)) for all \( f \in \mathcal{F}. \) Let \( C_{\text{max}} := \sup_{f \in \mathcal{F}} |f'(z_0)|. \) Thus \( C_{\text{max}} \leq C. \)

**Proposition 10.8.** There exists \( f \in \mathcal{F} \) such that \( |f'(z_0)| = C_{\text{max}}. \)

**Proof.** Take a sequence \( \{f_n\} \) of functions from \( \mathcal{F} \) with \( |f_n'(z_0)| \to C_{\text{max}}. \) The set \( \{f_n\} \) is uniformly bounded on compact subsets (because of Cauchy inequality, see the argument in the proof of Proposition \[10.7\]), and hence it is normal. Therefore, there is a subsequence uniformly on compact sets converging to a function \( f \) which satisfies \( |f'(z)| = C_{\text{max}}. \) We also have \( |f(z)| \leq 1, z \in U, \) and by maxima; modulus principle we conclude that \( |f(z)| < 1. \) In other words, \( f(U) \subset \mathbb{D}. \) Taking into account that it is non-constant (because its derivative at \( z_0 \) is \( \neq 0 \) we conclude from Lemma \[10.9\] that \( f \) is injective, and thus belongs to \( \mathcal{F}. \)

**10.2.3 Preservation of injectivity**

**Lemma 10.9.** Let \( U \subset \mathbb{C} \) be a connected domain and \( f_n : U \to \mathbb{C} \) a sequence of injective holomorphic functions. Suppose that \( f_n \to f \) uniformly on compact sets. Then if \( f \) is not constant then
it is injective as well.

**Proof.** Suppose that \( f \) is not injective, i.e. there are points \( z_1, z_2 \in U, z_1 \neq z_2 \), such that \( f(z_1) = f(z_2) = w \). Consider the map \( g(z) := f(z) - w \) and \( g_n(z) = f_n(z) - f_n(z_1) \). Then \( z_1 \) are zeroes of \( g \). By assumption \( g \) is not a constant, and hence the connectivity of \( U \) implies that zeroes \( z_1 \) and \( z_2 \) are isolated. On the other hand in view of the injectivity of \( f_n \) the function \( g_n \) has a unique zero \( z_1 \). Take \( \epsilon > 0 \) small enough such that \( D_\epsilon(z_2) = \{ |z - z_2| \leq \epsilon \} \subset U \) and inside \( D_\epsilon(z_2) \) there are no other zeroes of \( g \). Hence, according to the argument principle we have

\[
1 = \frac{1}{2\pi i} \int_{\partial D_\epsilon} \frac{g'(z)}{g(z)} \, dz = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\partial D_\epsilon} \frac{g_n'(z)}{g_n(z)} \, dz = 0.
\]

This contradiction implies that \( f(z_1) \neq f(z_2) \) for any \( z_1 \neq z_2 \).

Thus the map \( f \) constructed in Proposition 10.8 is injective. In the next section we will show that the constructed map \( f \) is also surjective, i.e. it is the required biholomorphism \( U \to \mathbb{D} \).

### 10.2.4 Surjectivity

The following proposition completes the proof of the Riemann mapping theorem 10.1.

**Proposition 10.10.** Let \( f : U \to \mathbb{D} \) is a map from \( F \) which satisfies \( |f'(z_0)| = \text{C}_{\text{max}} \). Then \( f(U) = \mathbb{D} \).

**Proof.** Suppose \( f \) is not surjective, i.e. there exists \( a \in \mathbb{D} \) such that \( a \neq f(z) \) for any \( z \in U \). Of course, \( a \neq 0 \), because \( 0 = f(z_0) \). We will use the point \( a \) to construct a holomorphic map \( \overline{f} \in F \) with \( |\overline{f}'(z_0)| > |f'(z_0)| \), which would contradict our assumption that \( |f'(z_0)| = \text{C}_{\text{max}} \). Consider a conformal automorphism \( \psi_a : \mathbb{D} \to \mathbb{D} \) given by the formula

\[
\psi_a(z) = \frac{a - z}{1 - \overline{a}z}.
\]

It interchanges points 0 and \( a \), i.e. \( \psi_a(0) = a \) and \( \psi_a(a) = 0 \). Hence, the inverse fractional linear transformation \( \psi_a^{-1} \) also interchanges the points 0 and \( a \).
Consider the domain \( U' = \psi_a(f(U)) \). Then 0 \( \notin U' \), but \( a \in U' \). The domain \( U' \) is simply-connected because it is biholomorphic to a simply connected domain \( U \). Consider a logarithm branch \( \log^{U'} : U' \to \mathbb{C} \) in \( U' \) and define the \( \sqrt{z} \) branch then
\[
s(z) := e^{\frac{1}{2} \log^{U'}(z)}.
\]
It satisfies \((s(z))^2 = z\). Consider the function
\[
\tilde{f}(z) = \psi_{s(a)} \circ s \circ \psi_a \circ f.
\]
We claim that \( \tilde{f} \in F \) and \( |\tilde{f}'(z_0)| > C_{\text{max}} \). Indeed, we have
\[
\tilde{f}(z_0) = \psi_{s(a)}(s(\psi_a(f(z_0)))) = \psi_{s(a)}(s(\psi_a(0))) = \psi_{s(a)}(s(a)) = 0.
\]
To verify the injectivity of the function \( \tilde{f} \) we observe that it is a composition of injective functions \( f, \psi_a, s \) and \( \sqrt{\psi_a} \).

To estimate \( |\tilde{f}'(z_0)| \) we define the function \( h : \mathbb{D} \to \mathbb{D} \) by the formula
\[
h(z) = \psi_{s(a)}^{-1}\left((\psi_{s(a)}^{-1}(z))^2\right).
\]
This function satisfies
\[
h(0) = \psi_{s(a)}^{-1}\left((\psi_{s(a)}^{-1}(0))^2\right) = \psi_{s(a)}(a^2) = 0,
\]
but is not injective, because the function \( z \mapsto z^2 \) is not injective. Hence, the Schwarz lemma implies that \( |h'(0)| < 1 \).

On the other hand, we have \( f = h \circ \tilde{f} \). Indeed, consider the diagram
\[
\begin{array}{c}
U \xrightarrow{f} f(U) \xrightarrow{\psi_a} U' \xrightarrow{s} \mathbb{D} \xrightarrow{\psi_{s(a)}} \mathbb{D} \xrightarrow{\tilde{f}} \mathbb{D} \xrightarrow{h^{-1}} \mathbb{D}.
\end{array}
\]
Taking into account that \( \psi_{s(a)}^{-1} \circ \psi_s(a) = \text{Id} \), \((s(z))^2 = z\) and \( \psi_a^{-1} \circ \psi_a = \text{Id} \) we conclude that \( f = h \circ \tilde{f} \).

Using the chain rule we compute \( f'(z_0) = h'(f(z_0)) \cdot \tilde{f}'(z_0) = h'(0) \cdot \tilde{f}'(z_0) \) and hence
\[
|\tilde{f}'(z_0)| \geq |h'(0)| = C_{\text{max}}
\]
because \( |h'(0)| < 1 \). This contradiction concludes the proof of Proposition 10.10, and with it the proof of the Riemann mapping theorem.
10.2.5 Discussion: boundary regularity

Given a simply connected domain $U$ one cannot expect, in general, any control of the boundary behavior of a conformal map $f : \mathbb{D} \to U$ provided by the Riemann mapping theorem, because the boundary of a general domain could be quite terrible. However, it turned out that if boundary is reasonable then the boundary behavior of the conformal map $f$ is reasonable as well.

Let us recall that a curve $\Gamma \subset \mathbb{C}$ is called a $C^k$-submanifold if for any $a \in \Gamma$ there exists $\epsilon > 0$ and a $C^k$-diffeomorphism (i.e. a bijective $C^k$-map $h$ with a $C^k$-inverse) of $D_\epsilon$ onto a neighborhood $U \ni 0$, such that $h(\Gamma) = h(U) \cap \{y = 0\}$.

Note that any 2 conformal equivalences $f, \tilde{f} : \mathbb{D} \to U$ differs by an automorphism of $\mathbb{D}$, which is smooth (and even real analytic) on the boundary. Hence the boundary behavior properties are the same for $f$ and $\tilde{f}$.

**Theorem 10.11.** If the boundary $\partial U$ of a simply connected domain $U$ is a $C^k$-submanifold of $\mathbb{C}$ and $f : \mathbb{D} \to U$ a conformal equivalence. Then $f$ extends a $C^{k-1}$-diffeomorphism $\tilde{f}$ between the closures: $\tilde{f} : \overline{\mathbb{D}} \to \overline{U}$. If $\partial U$ is a $C^0$-submanifold, then $f$ extends to $\overline{\mathbb{D}}$ as a homeomorphism.

The proof of this theorem (which was first proven in a weaker form by P. Painlevé) goes beyond this course.

10.3 Annuli

10.3.1 Conformal classification of annuli

Conformal classification of not simply connected domains is less boring. As an example, we consider this problem for annuli.

Given $r, R > 0, r < R$ The domain $A(r, R) = \{r < |z| < R\}$ is called an annulus.

**Lemma 10.12.** There exists a biholomorphism $h : A(r, R) \to A(r, R)$ which switches the boundary circles, i.e. $h(|z| = r)) = \{|z| = R\}$. 
Proof. This is done by the map \( z \mapsto \frac{R}{r} z \).

Clearly, any two annuli \( A(r, R) \) and \( A(r', R') \) with \( \frac{R}{r} = \frac{R'}{r'} \) are conformally equivalent. Indeed, the required conformal equivalence \( A(r, R) \to A(r', R') \) is the linear map \( z \mapsto \frac{R'}{R} z \). It turns that this sufficient condition together with the one arising from Lemma 10.12 is also necessary.

One can also allow in the definition of an annulus to allow \( r = 0 \) and/or \( R = \infty \). Thus, \( A(0, 1) = \mathbb{D} \setminus 0, \ A(0, \infty) = \mathbb{C} \setminus 0, \ A(1, \infty) = \mathbb{C} \setminus \overline{\mathbb{D}}. \)

Note that the map \( z \mapsto \frac{1}{z} \) establishes a conformal equivalence of \( A(0, 1) \) and \( A(1, \infty) \). However,

**Lemma 10.13.** \( A(0, 1) \) and \( A(0, \infty) \) are not conformally equivalent.

**Proof.** Indeed, suppose \( f : A(0, 1) \to A(0, \infty) \) is a conformal equivalence. Then either \( \lim_{z \to 0} f(z) = 0 \), or \( \lim_{z \to 0} f(z) = \infty \) (why?). In the former case the removal of singularities theorem allows us to extend \( f \) to 0 and hence we get a conformal equivalence \( \mathbb{D} \to \mathbb{C} \), which is impossible. In the latter case we first compose \( f \) with the automorphism \( \mathbb{C} \setminus 0 \to \mathbb{C} \setminus 0 \) given by the function \( z \mapsto \frac{1}{z} \) and then repeat the previous argument.

**Theorem 10.14.** Suppose that \( r \neq 0 \) and \( R \neq \infty \). Then two annuli \( A(r, R) \) and \( A(r', R') \) are conformally equivalent if and only if

\[
\left| \ln \frac{R}{r} \right| = \left| \ln \frac{R'}{r'} \right|.
\]

The quantity \( \left| \ln \frac{R}{r} \right| \) is called the *conformal modulus* of the annulus \( A(r, R) \) and will be denoted by \( \mathfrak{m} (A(r, R)) \).

Before proving this theorem we first we need to discuss *Laurent series*.

### 10.3.2 Laurent series

When studying meromorphic functions we already encountered series containing negative powers. For instance,

\[
e^z = \sum_{n=-2}^{\infty} \frac{z^n}{(n+2)!}, \quad z \neq 0.
\]
Sometimes we have to deal with series containing negative powers all the way up to \(-\infty\). For instance, we have

\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \neq 0. \]

A series of the form

\[ S(z) = \sum_{k=-\infty}^{\infty} a_k z^k \]

is called *Laurent series*. A series is a sum of two series

\[ S_-(z) = \sum_{k=1}^{\infty} a_{-k} z^{-k} \quad \text{and} \quad S_+(z) = \sum_{k=0}^{\infty} a_k z^k, \]

and convergence of \( S(z) \) means convergence of both \( S_\pm(z) \). The series \( S_+(z) \) is the standard power series and it converges in its disc of convergence \( D_R = \{ |z| < R \} \), while \( S_-(z) \) is a power series in the variable \( u = \frac{1}{z} \) and it converges in the disc \( |u| < \frac{1}{r} \) around \( \infty \), or equivalently in the complement of the disc \( D_r = \{ |z| \leq r \} \). Thus if \( r > R \) then the Laurent series \( S(z) \) does not converge anywhere, and if \( r < R \) it (absolutely) converges in the annulus \( A(r, R) = \{ r < |z| < R \} \) and defines a holomorphic function \( S : A(r, R) \to \mathbb{C} \). It turns out that

**Proposition 10.15.** Any holomorphic function \( S : A(r, R) \to \mathbb{C} \) can be presented as the sum of an absolutely converging in \( A(r, R) \) Laurent series, \( S(z) = \sum_{k=-\infty}^{\infty} a_k z^k. \)

**Proof.** This follows from the Cauchy formula. Let us take a slightly smaller annulus \( A'(r', R') \subset A(r, R) \). Then for any \( z \in A(r', R') \) we have

\[
 f(z) = \int_{\partial A'(r', R')} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\partial D_{R'}} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\partial D_r} \frac{f(\zeta)}{\zeta - z} d\zeta.
\]

Changing the variable \( \zeta = \frac{1}{u} \) and \( z = \frac{1}{v} \) in the second integral we get

\[
 \int_{\partial D_r} \frac{f(\zeta)}{\zeta - z} d\zeta = -\int_{\partial \{ |u| < \frac{1}{r} \}} \frac{vf(\frac{1}{u})}{u(u - v)} du.
\]

Therefore,

\[
 f(z) = \int_{\partial D_{R'}} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\partial D_r} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{\partial \{ |u| < \frac{1}{r} \}} \frac{vf(\frac{1}{u})}{u(u - v)} du.
\]

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Arguing as in Theorem 5.7 we can expand the first integral in a power series in $z$ converging for $|z| < R$ and expand the second integral in a power series in $v$ converging for $|v| < \frac{1}{v}$. Changing back $v \mapsto z = \frac{1}{v}$ we get the required Laurent expansion in $z$ variable. ■

We will also need the following formula (due to Green)

**Lemma 10.16.** Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ for $z \in A(r, R)$. Suppose that the map $f$ is injective. Denote by $A(\rho)$, $\rho \in (r, R)$ the area of the domain in $C$ bounded by the curve $f(|z| = \rho)$. Suppose that $f$ sends the circle $|z| = \rho$ oriented as the boundary of the disc $|z| \leq \rho$ to $f(|z| = \rho)$ oriented as the boundary of $C$.

Then

$$A(\rho) = \pi \sum_{n=-\infty}^{\infty} n|a_n|^2 \rho^{2n}.$$  

**Proof.** Using Proposition 4.3 we have

$$A(\rho) = -\frac{i}{2} \int_{|z|=\rho} f(z)f'(z)dz = -\frac{i}{2} \int_{|z|=\rho} \left( \sum_{n=-\infty}^{\infty} \overline{a_n} z^n \right) \left( \sum_{m=-\infty}^{\infty} ma_m z^{m-1} \right)$$

$$= -\frac{i}{2} \sum_{m,n=-\infty}^{\infty} ma_m \overline{a_n} \int_{|z|=\rho} z^{m-1} \overline{z^n} dz = \frac{1}{2} \sum_{m,n=-\infty}^{\infty} ma_m \overline{a_n} \int_{0}^{2\pi} e^{i(m-n)\phi} \rho^{m+n} d\phi = \pi \sum_{n=-\infty}^{\infty} n|a_n|^2 \rho^{2n},$$

because $\int_{0}^{2\pi} e^{ik\phi} d\phi = 0$ unless $k = 0$. ■

**10.3.3 Proof of Theorem 10.14**

Theorem 10.14 follows from the following stronger result.

**Proposition 10.17.** Suppose there exists an injective holomorphic map $f : A(r', R') \to A(r, R)$.

Then

$$m(A(r', R')) \leq m(A(r, R)).$$
Proof. Without loss of generality we can assume that \( r' = r = 1 \). We will view the annuli \( A(1, R) \) and \( A(1, R') \) as a subdomain of the discs \( D_R = \{|z| < R\} \) and \( D_{R'} = \{|z| < R'\} \). Let us denote by \( S_\rho := \{|z| = \rho\} \). Let \( V_\rho \) denote the closed subdomain of \( D_R \) bounded by \( f(S_\rho) \) for \( \rho \in (1, R') \). We can assume that \( V_\rho \subset V_{\tilde{\rho}} \) if \( \rho < \tilde{\rho} \). Otherwise, we can use Lemma 10.12 to switch the boundary components of the annulus \( A(1, R') \). This ensures that the map \( f|_{S_\rho} : S_\rho \to f(S_\rho) \) preserves orientations of \( S_\rho \) and \( f(S_\rho) \) as boundaries of \( D_\rho \) and \( V_\rho \). Hence we can apply formula 10.16 to compute the area \( A(\rho) := \text{Area}(V_\rho) \):

\[
A(\rho) = \pi \sum_{-\infty}^{\infty} n|a_n|^2 \rho^{2n}.
\]

Passing to the limits when \( \rho \to 1 \) and \( \rho \to R' \) and taking into account that

\[
D_1 \subset V_\rho \subset D_R
\]

for any \( \rho \in (1, R') \) we get

\[
\pi \leq \pi \sum_{-\infty}^{\infty} n|a_n|^2 \leq \pi \sum_{-\infty}^{\infty} n|a_n|^2 (R')^{2n} \leq \pi R'^2.
\]

The first inequality then implies that

\[
\pi R'^2 \leq \pi \sum_{-\infty}^{\infty} n|a_n|R^2 \leq \pi \sum_{-\infty}^{\infty} n|a_n|^2 R^{2n}.
\]

But the function \( A(\rho) \) is strictly increasing. Hence the inequality

\[
\pi \sum_{-\infty}^{\infty} n|a_n|^2 (R')^{2n} \leq \pi \sum_{-\infty}^{\infty} n|a_n|^2 R^{2n}
\]

implies \( R' \leq R \), and hence

\[
m(A(r', R')) = \ln R' \leq \ln R \leq m(A(r, R)).
\]

Thus, there is a unique annulus of each finite conformal modulus and exactly two annuli of infinite modulus.
10.4 Dirichlet problem

One of important applications of conformal mappings is for the solution of the following

**Dirichlet problem for harmonic functions.** Given a domain \( U \) and a continuous function \( \phi : \partial U \to \mathbb{R} \) find a harmonic function \( u : U \to \mathbb{R} \) which continuously extends to \( \partial U \) and \( f|_{\partial U} = \psi \).

One can make sense of Dirichlet problem even for discontinuous but integrable functions, where one requires boundary convergence at the points of continuity.

Note that the maximum principle for harmonic functions guarantees that the Dirichlet problem has a unique solution. Indeed, if \( \Delta u = \Delta \overline{u} = 0 \) and \( u|_{\partial U} = \overline{u}|_{\partial U} \) then \( u - \overline{u} \) is harmonic and \( (u - \overline{u})|_{\partial U} = 0 \). Hence, Corollary [8.8] implies that \( \overline{u} = u \).

Thanks to the Riemann mapping theorem, solving Dirichlet problem for simply connected domains can be reduced to solving it for \( D \) and understanding the behavior of the conformal equivalence \( h : U \to D \). Indeed, suppose a conformal equivalence \( h : D \to U \) continuously extends to \( \partial U \) (comp. Theorem [10.11]), then if \( g : D \to \mathbb{R} \) is a solution of the Dirichlet problem for \( D \) for the boundary value \( \psi \circ g : \partial D \to \mathbb{R} \), then, according to Lemma [8.5] the function \( h = g \circ h \) is harmonic and solves the Dirichlet problem for \( U \) with the boundary data \( \psi \). Hence, it is important to solve the Dirichlet problem for the disc \( D \). This is done below via an explicit formula proven by Schwarz, but first written by Poisson.

10.4.1 Poisson integral and Schwarz formula

**Proposition 10.18** (Schwarz’s formula). Let \( u : \overline{D} \to \mathbb{R} \) be a harmonic function on a closure \( \overline{D} \) of the unit disc in \( \mathbb{C} \). Then for any \( a \in D \) we have

\[
 u(a) = \Re \left( \frac{1}{2\pi i} \int_{|\xi|=1} \frac{\xi + a}{\xi - a} u(\xi) \frac{d\xi}{\xi} \right). \tag{10.4.1}
\]

**Remark 10.19.** Note that Schwarz’s formula ([10.4.1]) gives an explicit expression for the holo-
morphic function $f(z)$ whose real part is the harmonic function $u(z)$. Indeed the function
\[ f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta} \]
is holomorphic and according to Schwarz’s formula, $u(z) = \text{Re} f(z)$.

**Proof of Proposition 10.18.** Consider a fractional linear change of coordinates
\[ \zeta = R(v) = \frac{v + a}{1 + \overline{a}v}, \quad a \in \mathbb{D}. \]
Note that $R$ is an automorphism of $\mathbb{D}$ and $R(0) = a$. The function $u(R(v))$ is harmonic in $\mathbb{D}$ and hence the mean value theorem (see Theorem 8.8) for this function asserts:
\[ u(a) = u(R(0)) = \frac{1}{2\pi} \int_0^{2\pi} u(R(e^{i\theta})) d\theta. \]
The integral in the right-hand side can be rewritten as the complex contour integral
\[ u(a) = \frac{1}{2\pi i} \int_{|v|=1} u(R(v)) \frac{dv}{v}. \]
Let us change back the variable $v$ to $\zeta$,
\[ v = R^{-1}(\zeta) = \frac{\zeta - a}{1 - \overline{a} \zeta}. \]
We have
\[ \frac{dv}{v} = \left( \frac{1}{\zeta - a} + \frac{\overline{a}}{1 - \overline{a} \zeta} \right) d\zeta = \left( \frac{\zeta}{\zeta - a} + \frac{\overline{a} \zeta}{1 - \overline{a} \zeta} \right) \frac{d\zeta}{\zeta}. \]
Therefore,
\[ u(a) = \frac{1}{2\pi i} \int_{|v|=1} u(R(v)) \frac{dv}{v} = \frac{1}{2\pi i} \int_{|\zeta|=1} u(\zeta) \left( \frac{\zeta}{\zeta - a} + \frac{\overline{a} \zeta}{1 - \overline{a} \zeta} \right) \frac{d\zeta}{\zeta}. \]
But for $|\zeta| = 1$ we can write $1 = \zeta \overline{\zeta}$, and therefore
\[ \frac{\zeta}{\zeta - a} + \frac{\overline{a} \zeta}{1 - \overline{a} \zeta} = \frac{\zeta}{\zeta - a} + \frac{\overline{a} \zeta}{\zeta - \overline{a} \zeta} = \frac{\zeta}{\zeta - a} + \frac{\overline{a} \zeta}{\overline{\zeta} - \overline{a} \zeta} = \frac{\zeta}{\zeta - a} + \frac{\overline{a}}{\zeta - \overline{a}} \]
\[ = \frac{1 - |a|^2}{|\zeta - d|^2}. \]
Hence, the expression
\[ A := \frac{\zeta}{\zeta - a} + \frac{\bar{a}}{\zeta - \bar{a}} \]
is real, and thus
\[
A = \frac{1}{2} (A + A) = \frac{1}{2} \left( \frac{\zeta}{\zeta - a} + \frac{\bar{a}}{\zeta - \bar{a}} + \frac{\bar{\zeta}}{\bar{\zeta} - a} + \frac{a}{\bar{\zeta} - a} \right) \\
= \frac{1}{2} \left( \frac{\zeta + a}{\zeta - a} + \frac{\bar{\zeta} + \bar{a}}{\bar{\zeta} - \bar{a}} \right) = \text{Re} \frac{\zeta + a}{\zeta - a}.
\]
Combining all the formulas, we get
\[
u(a) = \frac{1}{2\pi i} \int_{|\zeta|=1} u(\zeta) \left( \frac{\zeta}{\zeta - a} + \frac{\bar{a}}{1 - \bar{a} \zeta} \right) d\zeta = \frac{1}{2\pi i} \int_{|\zeta|=1} \text{Re} \left( \frac{\zeta + a}{\zeta - a} u(\zeta) \right) d\zeta.
\]
We note that \( \frac{1}{i} \frac{d\zeta}{\zeta} = d\theta \) is real valued, and hence
\[
\frac{1}{2\pi i} \int_{|\zeta|=1} \text{Re} \left( \frac{\zeta + a}{\zeta - a} u(\zeta) \right) d\zeta = \text{Re} \left( \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\zeta + a}{\zeta - a} u(\zeta) \frac{d\zeta}{\zeta} \right),
\]
and we get the required formula (10.4.1). \( \blacksquare \)

It is useful to rewrite formula (10.4.1) in some different forms.

**Proposition 10.20.** Let \( u : \overline{D} \to \mathbb{R} \) be a harmonic function on a closure \( \overline{D} \) of the unit disc in \( \mathbb{C} \). Then for any \( a \in D \) we have
\[
u(a) = \frac{1}{2\pi i} \int_{|\zeta|=1} \text{Re} \left( \frac{\zeta + a}{\zeta - a} u(\zeta) \right) \frac{d\zeta}{\zeta}
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \text{Re} \left( \frac{e^{i\theta} + a}{e^{i\theta} - a} u(e^{i\theta}) \right) d\theta.
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |a|^2}{|e^{i\theta} - a|^2} u(e^{i\theta}) d\theta.
\]

Equivalently, for \( a = re^{i\phi} \) we have
\[
u(re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \phi) + r^2} u(e^{i\theta}) d\theta.
\]

\[
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\]
The proof is by a straightforward computation using formula (10.4.1). The latter integral called Poisson integral.

Applying formula (10.4.2) to \( u = 1 \) we get

**Corollary 10.21.**

\[
\int_{0}^{2\pi} \frac{1 - |a|^2}{|1 - \overline{a} e^{i\theta}|^2} d\theta = 2\pi
\]

for any \( a \in \mathbb{D} \).

### 10.4.2 Solution of the Dirichlet problem for the unit disc

We will now use Schwarz formula and Poisson integral for solving Dirichlet problem for the unit disc. Let \( \psi : \partial \mathbb{D} \to \mathbb{R} \) be a piece-wise continuous integrable function. Denote

\[
P_{\psi}(z) := \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2} \psi(\zeta) \frac{d\zeta}{\zeta} = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \psi(e^{i\theta}) d\theta \quad (10.4.4)
\]

**Theorem 10.22** (H.A. Schwarz). For any piece-wise continuous integrable function \( \psi : \partial \mathbb{D} \to \mathbb{R} \) we have

- **a)** \( P_{\psi}(z) \) is a harmonic function in \( \mathbb{C} \setminus \partial \mathbb{D} \); moreover, if the function \( \psi \) is equal to 0 in a neighborhood of a point \( e^{i\theta} \in \partial \mathbb{D} \) then the function \( P_{\psi}(z) \) is harmonic in a neighborhood of this point.

- **b)** If \( \psi \) is continuous at a point \( e^{i\theta} \in \partial \mathbb{D} \) then

\[
\lim_{z \to e^{i\theta}, z \in \mathbb{D}} P_{\psi}(z) = \psi(e^{i\theta}),
\]

in particular if \( \psi \) is continuous on \( \partial \mathbb{D} \) that \( P_{\psi}(z) \) extends to a continuous function on \( \mathbb{D} \) which is equal to \( \psi \) on \( \partial \mathbb{D} \).

Thus \( P_{\psi}(z) \) solves the Dirichlet problem for the boundary data \( \psi \).
Before proving Theorem 10.22, let us list some elementary properties of the integral $P_\psi$

**Lemma 10.23.** (1) Given two piecewise continuous functions $\psi_1, \psi_2 : \partial \mathbb{D} \to \mathbb{R}$ we have

$$P_{\psi_1 + \psi_2} = P_{\psi_1} + P_{\psi_2};$$

(2) if $\psi \geq 0$ then $P_\psi \geq 0$;

(3) if $\psi = c$ is a constant then $P_\psi = \psi = c$,

(4) if $c < \psi < C$ for constants $c$ and $C$ then $c < P_\psi < C$.

**Proof.** (1) is straightforward, (2) follows from the fact that $\frac{1 - |z|^2}{|e^{i\theta} - \zeta|^2} > 0$, (3) follows from Corollary 10.21, and (4) is a corollary of (2) and (3).

**Proof of Theorem 10.22**

a) To prove that $P_\psi$ is harmonic we observe that

$$\frac{1 - |z|^2}{|\zeta - z|^2} = \Re \left( \frac{\zeta + z}{\zeta - z} \right),$$

and hence $P_\psi(z)$ is the real part of the holomorphic function

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta}.$$

Hence, $P_\psi(z)$ is harmonic. Note that the function $f(z)$ is holomorphic everywhere except $\partial D$ (why?). Moreover, if the function $\psi$ is equal to 0 in a neighborhood of a point $e^{i\theta_0} \in \partial D$ then the function $f(z)$ is holomorphic in a neighborhood of this point.

b) Suppose that the function $\psi$ is continuous at a point $e^{i\theta_0} \in \partial D$. Without loss of generality we can assume that $\psi(e^{i\theta_0}) = 0$. Indeed, otherwise we can replace $\psi$ by $\psi - \psi(e^{i\theta_0})$ thanks to Lemma 10.23 (3) which implies that the theorem holds for constant functions $\psi$. Given any $\epsilon > 0$ choose $\delta > 0$ so small that $|\psi(e^{i\theta})| < \epsilon$ if $\theta - \theta_0 < \delta$. Choose two complementary arcs $A_1 := \{ |\theta - \theta_0| < \delta \}$ and $A_2 := \partial \mathbb{D} \setminus A_1$.

Let us decompose the function $\psi$ as $\psi_1 + \psi_2$ where $\psi_1|_{A_2} = 0$ and $\psi_2|_{A_1} = 0$. Then $P_\psi = P_{\psi_1} + P_{\psi_2}$. Then the functions $P_{\psi_1}$ and $P_{\psi_2}$ have the following properties:
• (i) $P_{\psi_1}$ is harmonic and hence continuous at interior points of $A_2$ (and $P_{\psi_2}$ is continuous on $A_1$). This follows from part a).

• (ii) $|P_{\psi_1}(z)| < \epsilon$ for $z \in \mathbb{D}$. This is a corollary of Lemma 10.23(4).

• (iii) $P_{\psi_2}|_{A_1} = 0$. Indeed, for $z \in A_1$ we have

$$P_{\psi_2}(z) = \frac{1}{2\pi} \int_{\theta \in (\theta_0 - \delta, \theta_0 + \delta)} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \psi(e^{i\theta}) d\theta.$$ 

But $|z| = 1$ on $\partial D$, and $e^{i\theta} - z \neq 0$ when $\theta \notin (\theta_0 - \delta, \theta_0 + \delta)$, and hence the integrand is equal to 0.

Properties (i) and (iii) imply that there exists $\sigma > 0$ such that if $|z - e^{i\theta_0}| < \sigma$ and $z \in \mathbb{D}$ then $|P_{\psi_2}(z) - P_{\psi_2}(e^{i\theta_0})| = |P_{\psi_2}(z)| < \epsilon$. Thus, for $|z - e^{i\theta_0}| < \sigma$ and $z \in \mathbb{D}$ we have

$$|P_{\psi}(z) - \psi(e^{i\theta})| = |P_{\psi}(z)| \leq |P_{\psi_1}(z)| + |P_{\psi_2}(z)| < 2\epsilon.$$ 

Hence,

$$\lim_{z \to e^{i\theta}, z \in \mathbb{D}} P_{\psi}(z) = \psi(e^{i\theta}).$$

10.4.3 Solving the Dirichlet problem for other domains

As it was already mentioned at the beginning of Section 10.4, the Riemann mapping theorem allows us to transform solutions of the Dirichlet problem from the unit disc to other simply connected domains.

Indeed, consider a simply connected domain $U$ with a piecewise smooth (non-empty!) boundary. The domain $U$ need not be even compact. Let $\psi : \partial U \to \mathbb{R}$ be a piecewise continuous integrable function. Choose a conformal equivalence $f : U \to \mathbb{D}$, According to Theorem 10.11 it extends continuously to the smooth part of the boundary $\partial U$ and consider the function $\tilde{\psi} := \psi \circ f^{-1} : \partial \mathbb{D} \to \mathbb{R}$. If $\tilde{\psi}$ is integrable on $\partial D$ (if we will see below it is not necessarily the case), then we can consider the solution $u := P_{\tilde{\psi}}$ of the Dirichlet problem for $\mathbb{D}$ with the boundary value $\tilde{\psi}$. In other words, $u$ is harmonic in $\mathbb{D}$, it extends, continuously to the points of $\partial U$, where $\tilde{\psi}$ is
continuous, and coincides there with $\psi$. Then $\tilde{u} := u \circ f$ is the solution of the Dirichlet problem for $U$ with the boundary data $\psi$.

As an example, let us consider the Dirichlet problem for a strip

$$\Omega := \{z \in \mathbb{C}; 0 < \text{Im} \ z < \pi\}.$$ 

The exponential function $\exp(z) = e^z$ maps $\Omega$ conformally onto $\mathbb{H}$, and composing it with a conformal equivalence $g : \mathbb{H} \to \mathbb{D}$, $g(z) = \frac{i - z}{1 + z}$, we get the required biholomorphism

$$f(z) = \frac{i - e^z}{i + e^z}$$ 

of $\Omega$ onto $\mathbb{D}$. We have $f^{-1}(z) = \log \left(\frac{i - z}{1 + z}\right)$

Let $u : \partial \Omega \to \mathbb{R}$ be a continuous bounded function on $\partial \Omega$. Then $\tilde{u}(z) = u(f^{-1}(z))$ is a piecewise continuous function on $\partial D$, and we can use Theorem 10.22 to extend it to a harmonic function

$$P_{\tilde{u}}(z) = \text{Re} \left( \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} u \left( \log \left( \frac{1 - \zeta}{1 + \zeta} \right) \right) \frac{d\zeta}{\zeta} \right)$$ 

on $\mathbb{D}$. Changing back to the strip $\Omega$ we conclude that the function $\Pi_u(z) := P_{\tilde{u}}(f(z))$ is harmonic on $\Omega$. Thus we get

$$\Pi_u(z) = P_{\tilde{u}}(f(z)) = \text{Re} \left( \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{\zeta + f(z)}{\zeta - f(z)} u \left( \log \left( \frac{1 - \zeta}{1 + \zeta} \right) \right) \frac{d\zeta}{\zeta} \right)$$

$$= \text{Re} \left( \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{e^{i(\zeta - 1)} + i(\zeta + 1)}{e^{i(\zeta + 1)} + i(\zeta - 1)} u \left( \log \left( \frac{i(\zeta - 1)}{i(\zeta - 1)} \right) \right) \frac{d\zeta}{\zeta} \right)$$

**Exercise 10.24.** Show that the solution of the Dirichlet problem for the strip $\Omega = \{0 < \text{Re} \ z < \pi\}$ with the boundary data

$$u(x + iy) = f_1(x), \quad u(x) = f_0(x), \quad x \in \mathbb{R},$$

where $f_0, f_1$ are bounded continuous functions can be written in the form

$$\Pi_u(x + iy) = \frac{\sin y}{y} \int_{-\infty}^{\infty} \frac{f_0(x - t)}{\cosh t - \sinh y} dt + \frac{\sin y}{y} \int_{-\infty}^{\infty} \frac{f_1(x - t)}{\cosh t + \sinh y} dt.$$
Chapter 11

Riemann surfaces

11.1 Definitions

A Riemann surface is a 1-dimensional complex manifold. For those, familiar with a notion of a smooth 2-dimensional real manifold the difference is that instead of pair of local coordinates in a coordinate chart, one has 1 complex coordinate and transition maps between two overlapping coordinate chart are required to be holomorphic.

More precisely, a set $S$ is called a Riemann surface, or a 1-dimensional complex manifold if there exist subsets $U_{\lambda} \subset S$, $\lambda \in \Lambda$, where $\Lambda$ is a finite or countable set of indices, and for every $\lambda \in \Lambda$ a map $\Phi_{\lambda} : U_{\lambda} \to \mathbb{C}$ such that

RS1. $S = \bigcup_{\lambda \in \Lambda} U_{\lambda}$.

RS2. The image $G_{\lambda} = \Phi_{\lambda}(U_{\lambda})$ is an open set in $\mathbb{C}$.

RS3. The map $\Phi_{\lambda}$ viewed as a map $U_{\lambda} \to G_{\lambda}$ is one-to-one.

RS4. For any two sets $U_{\lambda}, U_{\mu}, \lambda, \mu \in \Lambda$ the images $\Phi_{\lambda}(U_{\lambda} \cap U_{\mu}), \Psi_{\mu}(U_{\lambda} \cap U_{\mu}) \subset \mathbb{C}$ are open and the map

$$h_{\lambda\mu} := \Phi_{\mu} \circ \Phi_{\lambda}^{-1} : \Phi_{\lambda}(U_{\lambda} \cap U_{\mu}) \to \Phi_{\mu}(U_{\lambda} \cap U_{\mu}) \subset \mathbb{R}^n$$
is a biholomorphism.

Sets $U_\lambda$ are called coordinate neighborhoods and maps $\Phi_\lambda : U_\lambda \to \mathbb{C}$ are called coordinate maps. The pairs $(U_\lambda, \Phi_\lambda)$ are also called local coordinate charts. The maps $h_{\lambda\mu}$ are called transition maps between different coordinate charts. The inverse maps $\Psi_\lambda = \Phi^{-1}_\lambda : G_\lambda \to U_\lambda$ are called (local) parameterization maps. An atlas is a collection $\mathcal{A} = \{U_\lambda, \Phi_\lambda\}_{\lambda \in \Lambda}$ of all coordinate charts.

One says that two atlases $\mathcal{A} = \{U_\lambda, \Phi_\lambda\}_{\lambda \in \Lambda}$ and $\mathcal{A}' = \{U'_\gamma, \Phi'_\gamma\}_{\gamma \in \Gamma}$ on the same Riemann surface $S$ are equivalent, or that they define the same conformal structure on $S$ if their union $\mathcal{A} \cup \mathcal{A}' = \{(U_\lambda, \Phi_\lambda), (U'_\gamma, \Phi'_\gamma)\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ is again an atlas on $X$. In other words, two atlases define the same smooth structure if transition maps from local coordinates in one of the atlases to the local coordinates in the other one are given by smooth functions.

A subset $G \subset S$ is called open if for every $\lambda \in \Lambda$ the image $\Phi_\lambda(G \cap U_\lambda) \subset \mathbb{R}^n$ is open. In particular, coordinate charts $U_\lambda$ themselves are open, and we can equivalently say that a set $G$ is open if its intersection with every coordinate chart is open. By a neighborhood of a point $a \in S$ we will mean any open subset $U \subset S$ such that $a \in U$.

Usually (but not always) it is required that a Riemann surface $S$ satisfy the following additional axiom, called Hausdorff property:

RS5. Any two distinct points $x, y \in S$ have non-intersecting neighborhoods $U \ni x, G \ni y$.

Given two smooth Riemann surfaces $S$ and $\overline{S}$ a map $f : S \to \overline{S}$ is called holomorphic if if for every point $a \in S$ there exist local coordinate charts $(U_\lambda, \Phi_\lambda)$ in $S$ and $(\overline{U}_\lambda, \overline{\Phi}_\lambda)$ in $\overline{S}$, such that $a \in U_\lambda, f(U_\lambda) \subset \overline{U}_\lambda$ and the composition map

$$G_\lambda = \Phi_\lambda(U_\lambda) \xrightarrow{\Psi_\lambda} U_\lambda \xrightarrow{f} \overline{U}_\lambda \xrightarrow{\overline{\Phi}_\lambda} \mathbb{R}^n$$

is holomorphic. In other words, a map is holomorphic, if it is holomorphic when expressed in local coordinates.

A map $f : S \to S'$ is called a biholomorphism if it is holomorphic, invertible, and the inverse is also a holomorphic.
Example 11.1. 1. Any open subset of $\mathbb{C}$ is a Riemann surface.
2. The Riemann sphere $\mathbb{CP}^1$ is a Riemann surface. It can be covered by two coordinate charts with coordinates $z, \zeta \in \mathbb{C}$ which are on the overlap $\mathbb{C} \setminus 0$ are related by $z = \frac{1}{\zeta}$.

11.2 Uniformization theorem (or strong Riemann mapping theorem)

It turns out that the Riemann mapping theorem holds in a stronger form, called uniformization theorem.

Theorem 11.2 (Uniformization theorem). Any simply connected Riemann surface is conformally equivalent to either to $\mathbb{CP}^1$, or $\mathbb{C}$, or $\mathbb{H}$ ($\cong \mathbb{D}$).

The proof of this theorem goes beyond this course. The first rigorous proofs were given by H. Poincaré and P. Koebe in 1907. Since that time many proofs based on different ideas were found.

The significance of the uniformization theorem will become clear in our discussion below. In some sense it will allow us to classify all Riemann surfaces (see Theorem 11.18).

11.3 Riemann surfaces as submanifolds

11.3.1 Affine case

Consider the space $\mathbb{C}^2$ with complex coordinates $(z_1, z_2)$. A function $F : \mathbb{C}^2 \to \mathbb{C}$ is called holomorphic if its differential at each point is a complex linear function. This turns out to be equivalent that $F$ is holomorphic with respect to each of the variables $z_1$ and $z_2$. However we will not need this fact and will not use it\footnote{This fact is in a striking contrast with the real case, where for a differentiability of a function of 2 variables is not sufficient to be differentiable in each of the variable separately.}.
Lemma 11.3. Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a holomorphic function. Denote $S := \{F(z_1, z_2) = 0\}$. Suppose that for each point $p \in S$ at least one of partial derivatives, $\frac{\partial F}{\partial z_1}(p)$ and $\frac{\partial F}{\partial z_2}(p)$ is not equal to 0. Then $S$ is a Riemann surface.

Proof. The holomorphic version of the implicit function theorem (which is proved in an exactly the same way as its real version) asserts that if $p = (a_1, a_2) \neq 0$ then there exists a neighborhood $U \ni a_1$ in $\mathbb{C}$, a neighborhood $V$ of the point $p$ in $\mathbb{C}^2$ and a holomorphic function $h : U_1 \to \mathbb{C}$ such that $h(a_1) = a_2$ and

$$V \cap S = \{z_2 = h(z_1); \ z_1 \in U\}.$$ 

Similarly, if $\frac{\partial F}{\partial z_2}(p) \neq 0$ then $S$ near the point $p$ can be presented as the graph of a function $g$ given on a neighborhood of $a_2 \in \mathbb{C}$. If both partial derivatives are not 0 then the functions $h, g$ are inverse of each other, and they themselves provide transition maps between the coordinates. ■

We call a Riemann surface as in Lemma 11.3 a 1-dimensional complex submanifold, or a smooth (affine) holomorphic curve in $\mathbb{C}^2$. If the defining function $F(z_1, z_2)$ is a polynomial, the the holomorphic curve $S$ is called algebraic.

Lemma 11.4. Any holomorphic curve in $\mathbb{C}^2$ is non-compact.

Proof. The function $f := z_1|_S : S \to \mathbb{C}$ is holomorphic, and hence Re $f$ is harmonic. If $S$ is compact, then $f$ achieves its maximum at a point $p \in S$, but this contradicts the maximum principle for harmonic functions, see Corollary 8.8. ■

Remark 11.5. It is an open problem whether every connected non-compact Riemann surface is conformally equivalent to a holomorphic curve in $\mathbb{C}^2$.

11.3.2 Projective case

The projective plane $\mathbb{C}P^2$ is the space of all complex 1-dimensional subspaces in $\mathbb{C}^3$, i.e. the space of all complex lines in $\mathbb{C}^3$ through the origin. Equivalently, a point in $\mathbb{C}P^2$ can be viewed as a point $z = (z_1, z_2, z_3) \in \mathbb{C}^3 \setminus 0$ up to a complex proportionality coefficient:

$$(z_1, z_2, z_3) \sim (\lambda c_1, \lambda c_2, \lambda c_3), \ \lambda \in \mathbb{C} \setminus 0.$$
$z_1, z_2, z_3$ are called homogeneous coordinates in $\mathbb{C}P^2$ and usually denoted $z_1 : z_2 : z_3$.

The subsets $U_j := \{z_j \neq 0\} \subset \mathbb{C}^3$ give rise to affine coordinate charts $\tilde{U}_j$ in $\mathbb{C}P^2$. Indeed, a point $z = (z_1, z_2, z_3) \in U_3$, viewed as a point in $\mathbb{C}P^2$, is equivalent to $(\frac{z_1}{z_3}, \frac{z_2}{z_3}, 1)$, and hence $(z_{13} = \frac{z_1}{z_3}, z_{23} = \frac{z_2}{z_3})$ can be viewed as coordinates in this neighborhood. Similarly, in $\tilde{U}_1$ we can introduce coordinates $(z_{21} = \frac{z_2}{z_1}, z_{31} = \frac{z_3}{z_1})$, and in $\tilde{U}_2$ coordinates $(z_{12} = \frac{z_1}{z_2}, z_{32} = \frac{z_3}{z_2})$.

We will view $\mathbb{C}^2$ as a subset of $\mathbb{C}P^2$ by identifying it with the affine chart $\tilde{U}_3$ with the affine coordinates $(z_{13}, z_{23})$. Hence $\mathbb{C}P^2 \setminus \mathbb{C}^2 = \{z_1 : z_2 : 0\}$ which is just the projective line, or the Riemann sphere. In other words, we get $\mathbb{C}P^2$ by adding to $\mathbb{C}^2$ a projective line. We recall that we get $\mathbb{C}P^1$ from $\mathbb{C}$ by adding one point, which is of course can be viewed as the 0-dimensional projective space.

Let us recall that a function $F(z_1, z_2, z_3)$ is called homogeneous of degree $d$ if

$$F(\lambda z_1, \lambda z_2, \lambda z_3) = \lambda^d F(z_1, z_2, z_3)$$

for any $\lambda \in \mathbb{C}$. A useful fact about homogeneous function is

**Lemma 11.6** (Euler identity). Let $F(z_1, z_2, z_3)$ is homogeneous of degree $k$. Then

$$\frac{\partial F}{\partial z_1} + \frac{\partial F}{\partial z_2} + \frac{\partial F}{\partial z_3} = kF(z_1, z_2, z_3).$$

**Proof.** Differentiate the identity

$$F(\lambda z_1, \lambda z_2, \lambda z_3) = \lambda^k F(z_1, z_2, z_3)$$

with respect to $\lambda$ and set $\lambda = 1$. \(\blacksquare\)

Suppose that $F$ is a polynomial in variables $z_1, z_2, z_3$. Then it is homogeneous of degree $d$ if all its monomial have degree $d$:

$$F(z_1, z_2, z_3) = \sum_{k+m+n=d} z_1^k z_2^m z_3^n.$$  

For instance, $z_3 z_2^2 - z_1^3 + z_2^3 z_1$ is a homogeneous polynomial of degree 3.

**Lemma 11.7.** If $F(z_1, z_2, z_3)$ is a homogeneous function of any degree $d$, then the equation $F(z_1, z_2, z_3) = 0$ defines a subset $\tilde{S}$ in $\mathbb{C}P^3$. If at every point $p \in \tilde{S}$ we have $\frac{\partial F}{\partial z_j}(p) \neq 0$ for at least one of $j = 1, 2, 3$, then $\tilde{S}$ is compact Riemann surface. The intersection $S_j := \tilde{S} \cap \tilde{U}_j$ is an affine holomorphic curve.
Proof. We have

\[ S_1 := \{ F(1, z_{21}, z_{31}) = 0 \}, \quad S_2 := \{ F(z_{12}, 1, z_{32}) = 0 \}, \quad S_3 := \{ F(z_{13}, z_{23}, 1) = 0 \}. \]

The condition of non-vanishing partial derivatives implies the same condition for \( S_1, S_2, S_3 \), because if the non-vanishing derivative of \( F(z_1, z_2, z_3) \), say at a point \( (z_1, z_2, 1) \), is with respect to \( z_3 \) then the Euler identity gives

\[
\frac{z_1}{\partial z_1} \frac{\partial F}{\partial z_1} + \frac{z_2}{\partial z_2} \frac{\partial F}{\partial z_2} = -\frac{\partial F}{\partial z_3} \neq 0,
\]

and hence one of two other partial derivatives have to not vanish as well. Hence the intersection of \( \bar{S} \) with affine coordinate charts are holomorphic submanifolds, and so does \( \bar{S} \). The surface \( \bar{S} \) is compact because it is a closed subset of \( \mathbb{CP}^2 \) which is compact (why?).

The surface \( \bar{S} \) is called a smooth projective holomorphic curve.

11.4 Linear projective transformations

Any linear maps \( A : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \) induces a map \( \widehat{A} : \mathbb{CP}^n \to \mathbb{CP}^n \). Indeed, \( A(\lambda z) = \lambda z \), \( \lambda \in \mathbb{C}, z \in \mathbb{C}^{n+1} \). In other words, proportional vectors are mapped to proportional vectors. If \( A \) is non-degenerate then \( \widehat{A} \) is called a linear projective transformation of \( \mathbb{CP}^n \), or just a projective transformation. We already had seen that for \( n = 1 \) linear projective transformations of \( \mathbb{C}P^1 \), viewed as the Riemann sphere, are just fractional linear transformation, or as they are called, Möbius transformations. In this section we will concentrate on the case \( n = 2 \).

Points in \( \mathbb{C}P^2 \) correspond to complex lines in \( \mathbb{C}^3 \), while lines in \( \mathbb{C}P^2 \) are by definition the images of planes in \( \mathbb{C}^2 \). Hence each line in \( \mathbb{C}P^2 \) is itself biholomorphic to the projective line \( \mathbb{C}P^1 \). Any projective line \( L \) can be given by a homogeneous equation

\[ \ell(z) = a_1 z_1 + a_2 z_2 + a_3 z_3 = 0. \]

The complement \( U := \mathbb{CP}^2 \setminus L \) can be viewed as an affine chart in \( \mathbb{C}P^2 \). In order to define affine coordinates we pick two other linear functions \( \ell_1(z), \ell_2(z) \) such that \( \ell, \ell_1, \ell_2 \) are linearly independent.
Then
\[ u_1 := \frac{\ell_1}{\ell}, u_2 := \frac{\ell_2}{\ell} \]
are affine complex coordinates in the affine chart \( U \). Every projective line in \( \mathbb{CP}^2 \), different from \( L \) intersects \( U \) in an affine line
\[ A_1 u_1 + A_2 u_2 + A_3 = 0. \]

**Projective pencils.** A *pencil* in \( \mathbb{CP}^2 \) is by definition the set of all lines passing through a given point \( a \in \mathbb{CP}^2 \). We will denote a pencil of lines through \( a \) by \( \Pi_a \). Note that if \( L \) is a projective line and \( a \in L \) then any two lines in \( L_1, L_2 \) from \( \Pi_a \), different from \( L \), intersect the affine chart \( U = \mathbb{CP}^2 \setminus L \) along parallel affine lines. Note that given two distinct lines \( L_1 = \{ \ell_1 = 0 \} \) and \( L_2 = \{ \ell_2 = 0 \} \) in a pencil \( \Pi \) any other lines from this pencil can be given by an equation \( a \ell_1 + b \ell_2 = 0 \) for some \( a, b \in \mathbb{C} \).

Given any 3 lines
\[ L_1 = \{ \ell_1 = 0 \}, L_2 = \{ \ell_2 = 0 \}, L_3 = \{ \ell_3 = 0 \} \]
such that \( \ell_1, \ell_2, \ell_3 \) are linearly independent (or equivalently if the lines \( L_1, L_2, L_3 \) do not belong to one pencil) we say that \( L_1, L_2 \) and \( L_3 \) are in general position.

**Lemma 11.8.** Given 2 triples of lines in general position, there exists a linear projective transformation which sends one triple to another. In particular, any triple is projectively equivalent to the triple \( \{ z_1 = 0 \}, \{ z_2 = 0 \}, \{ z_3 = 0 \} \) of projective coordinate lines.

**Proof.** In \( \mathbb{C}^3 \) this is equivalent to mapping one triple of transverse planes \( P_1, P_2, P_3 \) to another one \( P'_1, P'_2, P'_3 \) by a linear transformation. Let us choose the bases \( v_1, v_2, v_3 \) and \( v'_1, v'_2, v'_3 \) such that
\[ v_1 \in P_1 \cap P_2, v_2 \in P_2 \cap P_3, v_3 \in P_3 \cap P_1, v'_1 \in P'_1 \cap P'_2, v'_2 \in P'_2 \cap P'_3, v'_3 \in P'_3 \cap P'_1. \]
Then the linear transformation which sends the basis \( v_1, v_2, v_3 \) to the basis \( v'_1, v'_2, v'_3 \) sends \( P_1 \) to \( P'_1 \), \( P_2 \) to \( P'_2 \) and \( P_3 \) to \( P'_3 \). ■

**Quadrics.** A *quadric* \( Q \) in \( \mathbb{CP}^2 \) is a complex curve of degree two, i.e. given by a degree two equation
\[ Q = \{ F(z) = A_{11} z_1^2 + A_{22} z_2^2 + A_{33} z_3^2 + 2A_{12} z_1 z_2 + 2A_{13} z_1 z_3 + 2A_{23} z_2 z_3 = 0 \} \]
in the homogeneous coordinates \((z_1 : z_2 : z_3)\). In affine chart a quadric is given by a quadratic equation in 2 variables with linear and constant terms. According to the implicit function theorem the quadric is a smooth, or it is a submanifold of \(\mathbb{C}P^2\), if

\[
\left( \frac{\partial F}{\partial z_1}(z), \frac{\partial F}{\partial z_2}(z), \frac{\partial F}{\partial z_3}(z) \right) \neq 0
\]

for each \(z \in Q\). We can easily compute that

\[
\begin{bmatrix}
\frac{\partial F}{\partial z_1}(z) \\
\frac{\partial F}{\partial z_2}(z) \\
\frac{\partial F}{\partial z_3}(z)
\end{bmatrix}
= A
\begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix},
\]

where

\[
A = \begin{pmatrix}
A_{11} & A_{12} & A_{13} \\
A_{12} & A_{22} & A_{23} \\
A_{13} & A_{23} & A_{33}
\end{pmatrix}
\]

is the matrix of the quadratic form \(F\). Hence, the smoothness condition is equivalent to the condition \(\det A \neq 0\).

**Theorem 11.9.** All smooth quadrics are projectively equivalent.

**Proof.** Choose a point \(a \in Q\). Consider the pencil \(\Pi_a\) of lines through through the point \(a\). Of these lines is tangent to \(Q\) at the point \(a\). We denote it by \(L\). Choose any other line \(L_1 \in \Pi_a\) and any line \(L_2 \not\in \Pi_a\), i.e. \(L_2\) does not pass through the point \(a\). According to Lemma [11.8] there exists a projective linear transformation \(T : \mathbb{C}P^2 \to \mathbb{C}P^2\) such that \(T(L) = \{z_3 = 0\}\), \(T(L_1) = \{z_1 = 0\}\) and \(T(L_2) = \{z_2 = 0\}\).

Consider the affine chart \(U_3 = \{z_3 \neq 0\}(= T(\mathbb{C}P^2 \setminus L))\). To simplify the notation denote the affine coordinates there by

\[
u_1 := z_{13} = \frac{z_1}{z_3}, u_2 := z_{23} = \frac{z_2}{z_3}.
\]

The equation of the affine part \(\tilde{Q} := T(Q) \cap U_3 = T(Q \setminus L)\) in \(U_3\) can be written in the form

\[
a u_2^3 + (b u_1 + c) u_2 + (k u_1^2 + l u_1 + m) = 0.
\]
First, we observe that $\alpha = 0$. Indeed, otherwise we would have 2 points on every line $\{u_1 = \text{const}\}$ which would mean that the intersection of $Q$ with a line of the pencil $P$ consists of 3 points, which is impossible because a quadric cannot intersect a line in more than 2 points.

Second, we argue that $b = 0$. Indeed, suppose $b \neq 0$ and set $u_1 = -\frac{c}{b}$. Then (11.4.1) implies that $ku_1^2 + lu_1 + m = 0$, i.e. $-\frac{c}{b}$ is a root of this quadratic polynomial, which means that

$$ku_1^2 + lu_1 + m = (bu_1 + c)(pu_1 + q),$$

and therefore, (11.4.1) takes the form

$$(bu_1 + c)(u_2 + pu_1 + qu_2) = 0. \tag{11.4.2}$$

But this equation defines two lines

$$\{u_1 = -\frac{c}{b}\} \cup \{u_2 + pu_1 + q\}.$$

But the two lines intersect at the point $(u_1 = -\frac{c}{b}, u_2 = \frac{pc-qb}{b})$, which contradicts to our assumption that the curve $Q$ is smooth. Therefore, (11.4.1) further reduces to

$$cu_2 + ku_1^2 + lu_1 + m = 0, \tag{11.4.3}$$

where $c \neq 0$ (otherwise the curve would reduce to two lines parallel to the $u_2$-axis, and hence intersecting at $a$; this would again contradicts the smoothness assumption). Hence, dividing the equation by $c$ and renaming the constant coefficients we get an equation

$$u_2 = \alpha u_1^2 + \beta u_1 + \gamma = 0, \tag{11.4.4}$$

We have $\alpha \neq 0$. Indeed, otherwise, the equation would be linear, and hence define a line and not a quadric. Finally by making successive linear changes of variables

- $u_1 \mapsto u_1 + \frac{\beta}{2}$ we first kill the coefficient with the linear term $u_1$, thus getting an equation of the form, $u_2 = \alpha u_1^2 + \tilde{\gamma}$;

- $u_2 \mapsto u_2 - \tilde{\gamma}$ getting $u_2 = \alpha u_1^2$, and finally
Thus, any smooth quadric is projectively equivalent to the projectivization of the complex parabola \( u_2 = u_1^2 \).

Exercise 11.10. Following the scheme of the proof of Theorem 11.9 show that any smooth cubic (elliptic) curve \( C \subset \mathbb{C}P^2 \), i.e. a curve given by a homogeneous equation of degree 3, is projectively equivalent to the projectivization of an affine curve given by the equation \( u_2^2 = u_1^3 + pu_1 + q \).

11.5 Quotient construction

An important class of examples is given by the following construction. Let \( S \) be a Riemann surface. A discrete group of biholomorphisms of \( S \) is a finite or countable set \( G \) of biholomorphisms of \( S \) such that

- For any \( f, g \in G \), \( f \circ g \) is in \( G \);
- \( \text{Id} \in G \) and for any \( g \in G \) \( g^{-1} \in G \);
- The trajectory \( Gx = \{ g(x); \ g \in G \} \) (called an orbit) of any point \( x \in S \) is a discrete set.

The first two properties just say that \( G \) is a subgroup of the group \( \text{Aut}(S) \) of conformal automorphisms of \( S \).

We say that \( G \) acts freely on \( S \) of any \( g \in G \), \( g \neq \text{Id} \) acts on \( S \) without fixed points, i.e. \( g(x) \neq x \) for any \( x \in S \), \( g \in G \), \( g \neq \text{Id} \).

Example 11.11. a). Consider the group \( G = \mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z} \) of pairs of integer numbers. Consider the action of \( G \) on \( \mathbb{C}^2 \) by translations:

\[
z = x + iy \mapsto g(z) = (x + m) + i(y + n), \quad \text{for} \quad g = (m, n) \in G.
\]

Then \( G \) is a discrete group acting freely on \( \mathbb{C} \).
b) Let \( G = \mathbb{Z}/p \), the finite group of \( p \) elements acting on \( \mathbb{C} \setminus 0 \) by the formula
\[
z \mapsto e^{\frac{2\pi i}{p} z}.
\]
Then this action is discrete and free. The same action on \( \mathbb{C} \) is discrete, but not free (why?).

Given a discrete group of transformations acting freely on a Riemann surface \( S \) we can define a new surface \( S/G \), called the quotient of \( S \) by \( G \) whose points are orbits \( Gx, x \in S \), of points of \( S \).

There is an obvious projection \( \pi : S \rightarrow S/G \) (which we will call tautological) which sends a point \( x \in S \) to its orbit \( Gx \). Any point \( x \in S \) has a neighborhood \( U_x \), such that for any \( g \in G, g \neq \text{Id} \), we have \( g(x) \notin U_x \). That means that the projection \( \pi|_{U_x} \) is injective and we call \( \pi(U_x) \) the coordinate neighborhood of the point \( X = \pi(x) \in S/X \).

**Example 11.12.** 1. Consider the group \( \mathbb{Z} \) acting on \( \mathbb{C} \) by the formula \( z \mapsto z + 2k\pi i \) for \( k \in \mathbb{Z} \).
Then the points of the quotient \( \mathbb{C}/\mathbb{Z} \) are complex numbers up to addition of a multiple of \( 2\pi i \).

**Lemma 11.13.** The quotient \( \mathbb{C}/\mathbb{Z} \) is conformally equivalent to \( \mathbb{C} \setminus 0 \).

Indeed, the exponential map \( \exp : z \mapsto e^z \) is the required biholomorphism. The inverse map \( \log : \mathbb{C} \setminus 0 \rightarrow \mathbb{C}/\mathbb{Z} \) is well defined (unlike the map to \( \mathbb{C} \) which is multiple valued).

2. Consider the same action restricted to the half-plane \( U = \{ \Re z > 0 \} \). Then \( U/\mathbb{Z} \) is conformally equivalent to the semi-infinite annulus \( A(1, \infty) = \{ 1 < |z| < \infty \} \). The holomorphic function \( \exp|_U \) induces a biholomorphism \( U/\mathbb{Z} \rightarrow A(1, \infty) \).

3. Consider another action of \( \mathbb{Z} \), this time on \( \mathbb{C} \setminus 0 \):
\[
z \mapsto 2^k z, \quad k \in \mathbb{Z}
\]
Then the quotient \( (\mathbb{C} \setminus 0)/\mathbb{Z} \) is a 2-torus \( T \).

4. Choose two complex numbers \( \omega_1, \omega_2 \in \mathbb{C} \setminus 0 \) such that \( \frac{\omega_1}{\omega_2} \notin \mathbb{R} \). Consider an action of \( \mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z} \) on \( \mathbb{C} \) given by the formula
\[
z \mapsto k\omega_1 + \ell\omega_2 \quad \text{for} \quad (k, \ell) \in \mathbb{Z} \oplus \mathbb{Z}.
\]
The quotient \( T(\omega_1, \omega_2) = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z} \) is again a torus. We will see that its conformal structure of \( T(\omega_1, \omega_2) \) depends on the choice of \( w_1, w_2 \). They are also called elliptic curves.
11.6 Covering maps

Given two Riemann surfaces $U, V$ a holomorphic map $p : U \to V$ is called a covering map if for any point $x \in V$ there exists a neighborhood $V_x \ni x$ such that the preimage $p^{-1}(V_x)$ can be presented as a union of finitely or countably many disjoint open sets:

$$p^{-1}(V_x) = U_1 \cup U_2 \cup \ldots,$$

such that $p|_{U_j} : U_j \to V_x$ is a biholomorphism for each $j = 1, 2, \ldots$. If $p : U \to V$ is a covering map than $U$ is called a cover of $V$.

**Exercise 11.14.** Prove that if $V$ is connected, then each point $x \in V$ has the same number (which can be infinite) of pre-images.

This number is called the order of the covering. Sometimes if $p : U \to V$ is a covering of order $k$, then one calls it a $k$-sheeted covering. A covering of order 1 is a biholomorphism.

**Example 11.15.** 1. The map $\exp : \mathbb{C} \to \mathbb{C} \setminus 0$, $\exp(z) = e^z$, is a covering map (of infinite order). Indeed, we have

$$\mathbb{C} \setminus 0 = V_+ \cup V_-, \quad V_+ := \mathbb{C} \setminus \{z; \text{Im} z = 0, \text{Re} z \leq 0\}, \quad V_- := \mathbb{C} \setminus \{z; \text{Im} z = 0, \text{Re} z \geq 0\}.$$

We have

$$\exp^{-1}(V_+) = \bigcup_{k=-\infty}^{\infty} U_j^+, \quad U_j^+ := \{z = x + iy; \ y \in ((2k-1)\pi, 2k\pi)\};$$

$$\exp^{-1}(V_-) = \bigcup_{k=-\infty}^{\infty} U_j^-, \quad U_j^- := \{z = x + iy; \ y \in (2k\pi, (2k+1)\pi)\}.$$  

The restrictions $\exp|_{U_k^\pm} : U_k^\pm \to V_\pm$ are biholomorphisms. The inverse maps are given by branches of the logarithm.

2. Similarly, the maps $\pi_k : \mathbb{C} \setminus 0 \to \mathbb{C} \setminus 0$ given by the formula

$$\pi_k(z) = z^k, \quad k = \pm 1, \pm 2, \ldots$$

---

2 Be aware that the word *cover* or *covering* is used in Mathematics also in a different sense, as a covering of a space by overlapping open sets.
are covering maps. In this case every point of \( \mathbb{C} \setminus 0 \) has exactly \( k \)-pre-images, i.e. \( \pi_k : \mathbb{C} \setminus 0 \to \mathbb{C} \setminus 0 \) is a \( k \)-sheeted covering.

### 11.7 Quotient construction and covering maps

The previous example can be generalized as follows. Let \( S \) be a Riemann surface and \( G \subset \text{Aut}(S) \) be a discrete group of its automorphisms. Consider the tautological projection \( \pi_G : S \to S/G \).

**Lemma 11.16.** The tautological projection \( \pi_G : S \to S/G \) is a covering map.

**Proof.** Take any \( x \in S \) and consider its orbit \( Gx = \{ g(x) ; \ g \in G \} \). Then by definition of a discrete group and its free action there exists a neighborhood \( U_x \ni x \) such that \( g(U_x) \cap \widetilde{g}(U_x) = \emptyset \) for \( g \neq \widetilde{g} \). The tautological projection \( \pi_G : X \to X/G \) sends \( U_x \) onto its trajectory \( \widetilde{U} := G(U_x) \). \( \widetilde{U} \) is by definition an open set containing the point \( Gx \in S/G \) and the preimage \( \pi_G^{-1}(\widetilde{U}_x) \) is the union

\[
\pi_G^{-1}(\widetilde{U}_x) = \bigcup_{g \in G} g(U_x)
\]

of disjoint open sets \( g(U_x) \), and the restriction of the projection \( \pi_G \) to each of these sets, \( \pi_G|_{g(U_x)} : g(U_x) \to \widetilde{U} \) is a biholomorphism. Hence, \( \pi_G : S \to S/G \) is a covering map. \( \blacksquare \)

It turns out that all covering maps come from the above quotient construction.

**Theorem 11.17.** Let \( p : U \to V \) be a covering map. Then there exists a discrete subgroup \( G \subset \text{Aut}(U) \) which acts freely on \( U \) and there exists a biholomorphism \( f : V \to U/G \) such that the following diagram commutes:

\[
\begin{array}{ccc}
U & \xrightarrow{f} & U/G \\
\downarrow{p} & \nearrow{\pi_G} & \\
V & \nearrow & \downarrow{
\pi_G}
\end{array}
\]

i.e. \( p = \pi_G \circ p \).

The group \( G \) is called the *deck transformation* group of the covering \( p : U \to V \). If time permits, we will discuss its construction later in the course.
11.8 Universal cover

Given a connected Riemann surface \( V \), its cover \( p : U \to V \) is called universal, if \( U \) is simply connected (simply connectedness assumes also connectedness). The following theorem explains the origin of the term universal.

**Theorem 11.18.**

1. For any Riemann surface \( V \) there exists a universal cover \( p : U \to V \).

2. The universal cover is unique in the following sense. Let \( p_1 : U_1 \to V \) and \( p_2 : U_2 \to V \) be two universal covers. Then there exists a biholomorphism \( f : U_1 \to U_2 \) such that the following diagram commutes:

\[
\begin{array}{ccc}
U_1 & \overset{f}{\to} & U_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
V & \overset{p_2 \circ F}{\to} & V
\end{array}
\]

i.e. \( p_1 = p_2 \circ F \).

3. Let \( V_1 \) and \( V_2 \) be two Riemann surfaces and \( p_1 : U_1 \to V_1 \) and \( p_2 : U_2 \to V_2 \) its universal covers. Then for any holomorphic map \( f : V_1 \to V_2 \) there exists a holomorphic map \( F : U_1 \to U_2 \) such that the following diagram commutes:

\[
\begin{array}{ccc}
U_1 & \overset{F}{\to} & U_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
V_1 & \overset{f}{\to} & V_2
\end{array}
\]

i.e. \( f \circ p_1 = p_2 \circ F \).

If \( U = \mathbb{C}P^1 \) then it is simply connected itself, and hence it serves as its own universal (one-sheeted) cover. The proof of this theorem is not difficult and its proof is related to the proof of Theorem 11.17. We may discuss it later in the course, but first we consider its applications.

**Theorem 11.19** (Uniformization theorem revisited). The universal cover of any Riemann surface is conformally equivalent to \( \mathbb{H}, \mathbb{C} \) or \( \mathbb{C}P^1 \). For any Riemann surface other than \( \mathbb{C}P^1 \) there exists a
discrete subgroup of $\text{Aut}(\mathbb{C})$, resp. $\text{Aut}(\mathbb{H})$, which freely acts on $\mathbb{C}$, resp. $\mathbb{H}$, such that $U$ is conformally equivalent to $\mathbb{C}/G$, resp. $\mathbb{H}/G$.

Thus Theorem 11.19 reduces the conformal classification of Riemann surfaces to the classification of discrete subgroups of the groups of automorphisms of $\mathbb{C}$ and $\mathbb{H}$ which acts without fixed points. The main variety of such subgroups is in $\text{Aut}(\mathbb{H})$. But the Riemann surfaces covered by $\mathbb{C}$ are also very important, They turned out to be only tori (elliptic curves) and $\mathbb{C}\setminus 0$. We will discuss them in Section 12.

**Proof.** Let $V$ be a Riemann surface and $U$ its universal cover. Then Theorem 11.17 yields a discrete subgroup of $U/G$ is conformally equivalent to $V$. But assumption, $V$, and hence $U$, is not $\mathbb{C}P^1$, and hence according to Theorem 11.18 $U$ is conformally equivalent to either $\mathbb{H}$ or $\mathbb{C}$.

11.9 Lattices

A **lattice** $\Lambda \subset \mathbb{C} = \mathbb{R}^2$ is any discrete subgroup. The **rank** of the lattice is the real dimension of a real subspace $\text{Span}_\mathbb{R}(\Lambda)$. Because $\dim \mathbb{R}^2 = 2$, any non-trivial (i.e. not consisting of only 0) lattice could be either of rank 1 or of rank 2.

**Lemma 11.20.** Any rank 1 lattice has the form $\Lambda = \{k\omega\}$ for $\omega \in \mathbb{C}, \omega \neq 0, k \in \mathbb{Z}$. Any rank two lattice has the form $\Lambda = \{k\omega_1 + \ell\omega_2\}$ where $k, \ell \in \mathbb{Z}, \omega_1, \omega_2 \in \mathbb{C}, \omega_1, \omega_2 \neq 0$ and $\frac{\omega_2}{\omega_1} \notin \mathbb{R}$.

**Proof.** We consider here only the case of a rank 2 lattice and leave the rank 1 case as an exercise to the reader. The discreteness of the lattice guarantees that there exists $\omega_1 \in \Lambda \setminus 0$ such that $|\omega_1| \leq |\omega|$ for all $\omega \in \Lambda \setminus 0$. Denote $L = \text{Span}_\mathbb{R}(\omega_1)$. We claim that there exists $\omega_2 \in \Lambda \setminus L$ such that $\text{dist}(\omega_2, L) \leq \text{dist}(\omega, L)$ for any $\omega \in \Lambda \setminus L$. Indeed suppose there exists a sequence $\omega_j \in \Lambda \setminus L$ such that $\text{dist}(\omega_j, L) < \text{dist}(\omega_{j+1}, L)$, for all $j = 2, 3, \ldots$. that for any $\omega \in \Lambda$ and any $k \in \mathbb{Z}$ we have

$$\text{dist}(\omega, L) = \text{dist}(\omega + k\omega_1, L).$$

(Recall that $\omega_1 \in L$). Denote $d := \text{dist}(\omega_2, L)$. Then by adding to $\omega_j$ an appropriate multiple of $k\omega_1$ of $\omega_1$ we can arrange that $|\omega - k\omega_1| \leq \sqrt{|\omega_1|^2 + d^2}$ but then we get an infinite set of points of $\Lambda$ in
a bounded domain which contradicts the fact that \( \Lambda \) is discrete. Take now the lattice

\[
\tilde{\Lambda} := \{k\omega_1 + \ell\omega_2; \ k, \ell \in \mathbb{Z}\}
\]

generated by \( \omega_1 \) and \( \omega_2 \). Clearly, \( \tilde{\Lambda} \subset \Lambda \). We claim the \( \tilde{\Lambda} = \Lambda \). Indeed, if there exists \( om \in \Lambda \setminus \tilde{\Lambda} \), then by adding to \( \omega \) a multiple of \( \omega_2 \) we can bring it to a distance \( < d \) to \( L \):

\[
\text{dist}(\omega + k\omega_2, L) < d
\]

, but by the choice of \( \omega_2 \) it means that

\[
\text{dist}(\omega + k\omega_2, L) = 0,
\]

i.e. \( \omega + k\omega_2 \in L \). But then, adding an appropriate multiple of \( \omega_1 \) we can arrange that \( |\omega + k\omega_2 + \ell\omega_1| < |\omega_1| \), but then \( \omega + k\omega_2 + \ell\omega_1 = 0 \), i.e. \( \omega \in \tilde{\Lambda} \).

A lattice \( \Lambda \) can be interpreted as a subgroup of \( \text{Aut}(\mathbb{C}) \) acting on \( \mathbb{C} \) by translations. It turns out that converse is also true:

**Lemma 11.21.** Any discrete subgroup \( G \) of \( \text{Aut}(\mathbb{C}) \) which acts on \( \mathbb{C} \) freely is a lattice.

**Proof.** Any conformal automorphism \( g : \mathbb{C} \to \mathbb{C} \) has the form \( g(z) = az + b \). We claim that if \( a \neq 1 \) \( g \) has a fixed point. Indeed in that case the equation \( az + b = z \) has a solution \( z_0 = \frac{b}{1-a} \). Hence, \( g(z) = z + b \), i.e. \( G \) is a subgroup of the subgroup \( \mathbb{C} \subset \text{Aut}(\mathbb{C}) \) which consists of translations. Therefore it is a lattice.

Hence we can form a quotient torus \( T(\omega_1, \omega_2) = \mathbb{C}/\Lambda \). Clearly, some of these tori are conformally equivalent, which allows us to restrict the class of lattices we want to study. Here are some of the operations which allows us to do that.

1. **Action** \( z \mapsto cz, \ c \in \mathbb{C} \). The tori \( T(\omega_1, \omega_2) \) and \( T(c\omega_1, c\omega_2) \) for any complex number \( c \in \mathbb{C} \) \( \setminus \) \( 0 \) are conformally equivalent. Indeed, the linear map \( z \mapsto cz \) sends the lattice \( \Lambda(\omega_1, \omega_2) \) generated by \( \omega_1, \omega_2 \) to the lattice \( \Lambda(c\omega_1, c\omega_2) \) generated by \( c\omega_1, c\omega_2 \). Hence, orbits of the former lattice are sent by this map to the orbits of the latter one, and this map is clearly invertible.

Thus \( T(\omega_1, \omega_2) \cong T(1, \tau = \frac{\omega_2}{\omega_1}) \),

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2. Automorphisms of a lattice. The lattice $\Lambda(\omega_1, \omega_2)$ is preserved by interchanging $\omega_1$ and $\omega_2$. Hence, we can always assume that $\omega_1$ and $\omega_2$ define the standard orientation of $\mathbb{R}^2 = \mathbb{C}$ (i.e. the same orientation as defined by 1 and $i$. This is equivalent to the requirement that $\tau = \frac{\omega_2}{\omega_1} \in \mathbb{H}$.

In particular any lattice is conformally equivalent to a lattice $\Lambda(1, \tau)$ where $\tau \in \mathbb{H}$ via an automorphism $\mathbb{C} \to \mathbb{C}$, and hence every torus $T(\omega_1, \omega_2)$ is conformally equivalent to the torus $T(1, \tau)$ with $\tau \in \mathbb{H}$.

Let us denote by $PSL(2, \mathbb{Z})$ the subgroup of $Aut(\mathbb{H}) = PSL(2, \mathbb{R})$ which consists of matrices with integer entries (and determinant 1). It is called the modular group.

**Theorem 11.22** (Conformal classification of tori). Tori $T(\omega_1, \omega_2)$ and $T(\tilde{\omega}_1, \tilde{\omega}_2)$ are conformally equivalent if and only if $\tau = \frac{\omega_2}{\omega_1}$ and $\tilde{\tau} = \frac{\tilde{\omega}_2}{\tilde{\omega}_1}$ are related by the action of an element of the modular group, i.e. there exists a matrix
\[
\begin{pmatrix}
  n & m \\
  \ell & k
\end{pmatrix}
\]
with integer values and $\det = kn - m\ell = 1$ such that
\[
\tilde{\tau} = \frac{n\tau + m}{\ell\tau + k}.
\]

**Proof.** Denote $\Lambda := \Lambda(\omega_1, \omega_2), \tilde{\Lambda} = \Lambda(\tilde{\omega}_1, \tilde{\omega}_2)$. Let $f : C/\Lambda \to C/\tilde{\Lambda}$ be a conformal equivalence. Then according to the uniqueness of universal cover theorem [11.18] there exists a biholomorphism $F : C \to C$ such that the following diagram commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C \\
\downarrow{\pi_\Lambda} & & \downarrow{\pi_\Lambda} \\
C/\Lambda & \xrightarrow{f} & C/\tilde{\Lambda}
\end{array}
\]

i.e. $f \circ \pi_\Lambda = \pi_\tilde{\Lambda} \circ F$, where $\pi_\Lambda : C \to C/\Lambda$ and $\pi_\tilde{\Lambda} : C \to C/\tilde{\Lambda}$ are tautological projections. We have $F(z + \Lambda) = F(z) + \tilde{\Lambda}$, and hence $F(z + 1) = F(z) + c_1, F(z + \tau) = F(z) + c_2$, where $c_1, c_2 \in \tilde{\Lambda}$. Hence, $F'(z)$ is a bi-periodic function with periods 1 and $\tau$, and hence a constant, but then $F(z) = A + Bz$, where $A, B \in \mathbb{C}$. and the two lattices $\Lambda$ and $\tilde{\Lambda}$ differ by a rotation, scaling and translation. After we used the transformation $F(z) = A + Bz$ to identify the lattices $\Lambda$ and $\tilde{\Lambda}$ we see that $(\omega_1, \omega_2)$
and \((\bar{\omega}_1, \bar{\omega}_2)\) are two different bases of the same lattice \(\Lambda\), which in addition by assumption define the same orientation of \(\mathbb{C}\). Hence, there exists and integer valued matrix \(A = \begin{pmatrix} k & \ell \\ m & n \end{pmatrix}\) such that \(\bar{\omega}_1 = k\omega_1 + \ell\omega_2, \bar{\omega}_2 = m\omega_1 + n\omega_2\). The matrix is invertible (as an integer valued matrix) and it is orientation preserving. Hence, \(\det A = kn - ml = 1\).

\[ \bar{\tau} = \frac{\bar{\omega}_2}{\bar{\omega}_1} = \frac{n\tau + m}{\ell\tau + k}. \]

The action of the modular group on \(\mathbb{H}\) is discrete but not free. Hence, the quotient \(\mathbb{H}/PSL(2, \mathbb{Z})\) is a what is called singular Riemann surfaces.

**Theorem 11.23.** Denote

\[ U := \left\{ \tau \in \mathbb{H}; |z| \geq 1, -\frac{1}{2} \leq \text{Re} \tau \leq \frac{1}{2} \right\} \]

Every torus \(T(\omega_1, \omega_2)\) is conformally equivalent to a torus \(T(1, \tau)\) with \(\tau \in U\).

**Proof.** As it is explained in the proof of Theorem 11.22 given a lattice \(\Lambda = \Lambda(\omega_1, \omega_2)\) automorphisms of this lattice are real linear transformation of \(\mathbb{C} = \mathbb{R}^2\) which map a basis \(\omega_1, \omega_2\) of \(\Lambda\) to another basis \(\bar{\omega}_1, \bar{\omega}_2\) of \(\Lambda\), i.e. \(\bar{\omega}_1 = k\omega_1 + \ell\omega_2, \bar{\omega}_2 = m\omega_1 + n\omega_2\), where \(k, \ell, m, n\) are integers. The condition that this is an automorphism means that the matrix is invertible, and the inverse is also integer valued, which means that \(\det = kn - ml = \pm 1\), and taking into account that we consider only orientation preserving automorphisms, we should have \(kn - ml = 1\). On the other hand, as we discussed above, by a complex linear isomorphism \(\mathbb{C} \to \mathbb{C}\) given by \(z \mapsto \frac{1}{\omega_1}z\), we identify the lattices \(\Lambda(\omega_1, \omega_2)\) and \(\Lambda(1, \tau = \frac{\omega_2}{\omega_1})\), where \(\tau \in \mathbb{H}\) (due to our orientation convention). The change of a basis \((\omega_1, \omega_2) \mapsto (\bar{\omega}_1, \bar{\omega}_2)\) amounts to changing

\[ \tau \mapsto \bar{\tau} = \frac{\bar{\omega}_2}{\bar{\omega}_1} = \frac{n\tau + m}{\ell\tau + k}. \]

In other words, the action of \(SL(2, \mathbb{Z})\) on \(\mathbb{H}\) via fractional linear transformations is equivalent to the action of \(SL(2, \mathbb{Z})\) as the automorphism group of the lattice \(\Lambda = \Lambda(\omega_1, \omega_2)\).
Hence, our problem can be interpreted as the problem of finding an appropriate basis \((\tilde{\omega}_1, \tilde{\omega}_2)\) of the lattice \(\Lambda(\omega_1, \omega_2)\). First, let us interpret the conditions on \(\tilde{\tau}\) in terms of \(\tilde{\omega}_1, \tilde{\omega}_2\). The condition \(|\tilde{\tau}| \geq 1\) means \(|\tilde{\omega}_2| > |\tilde{\omega}_1|\). To clarify geometric meaning of the condition \(|\text{Re} \tilde{\tau}| < \frac{1}{2}\) let us write \(\tilde{\tau} = x + iy\). Then
\[
\tilde{\omega}_2 = \tilde{\tau}\tilde{\omega}_1 = (x + iy)\tilde{\omega}_1 = x\tilde{\omega}_1 + y(i\tilde{\omega}_1)
\]
The vectors \(\tilde{\omega}_1\) and \(i\tilde{\omega}_1\) are orthogonal; and have the same length. So the condition \(|\text{Re} \tilde{\omega}_2 = |x| \leq 12\) means that the orthogonal projection of \(\tilde{\omega}_2\) onto the line \(L := \text{Span}(\tilde{\omega}_1)\) lies in the interval \([-\tilde{\omega}_1, \tilde{\omega}_1]\) \(\subset L\).

Hence, the strategy for finding the appropriate basis \((\tilde{\omega}_1, \tilde{\omega}_2)\) should be the following (comp. the proof of Lemma 11.20). Choose as \(\tilde{\omega}_1\) the shortest vector of \(\Lambda\). This guarantees that whatever vector we choose as \(\tilde{\omega}_2\) the condition \(\tilde{\tau} = \frac{\tilde{\omega}_2}{\tilde{\omega}_1} \geq 1\) would be automatically satisfied. Note that all points \(\Lambda\) are located on affine lines \(L + \lambda, \lambda \in \Lambda\), parallel to \(L\). Let \(L_1\) be the line \(L + \lambda\) closest to \(L\) and chosen on the appropriate side of \(L\), i.e. to ensure that \(\tilde{\omega}_1, \lambda\) define the complex (i.e. counter-clockwise) orientation of \(\mathbb{C}\). Points of \(\Lambda\) on \(L_1\) are spaced at a distance \(|\tilde{\omega}_1|\). Hence, every closed interval of length \(|\tilde{\omega}_1|\) contains at least one point of \(\Lambda\), and therefore there exists \(\lambda \in L\) such that the orthogonal projection of \(\lambda\) to \(L\) lies in the interval \([-\tilde{\omega}_1, \tilde{\omega}_1]\) \(\subset L\). Set \(\omega_2 := \lambda\). Then \((\omega_1, \omega_2)\) form a basis of \(\Lambda\) with the required properties.

\[\blacksquare\]

**Remark 11.24.** Theorem 11.23 can be strengthened as follows:

*Every torus \(T(\omega_1, \omega_2)\) is conformally equivalent to a unique torus \(T(1, \tau)\) with
\[
\tau \in \text{Int } U \cup \left\{ \tau = e^{i\theta}, \ \theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right] \right\}.
\]*

**11.10 Branched covers**

We begin with a lemma describing the behavior of a holomorphic function near its critical point.
Lemma 11.25. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a domain $\Omega \ni 0$. Suppose $f(0) = f'(0) = \cdots = f^{k-1}(0) = 0$ and $f^k(0) \neq 0$ for some $k > 1$. Then there exists a local coordinate $u$ near 0 such that the map $f$ is given by $u \mapsto u^k$. In other words, there exists a biholomorphism $h : U \to \tilde{U} \subset \mathbb{C}$ defined on a neighborhood $U \ni 0$, $U \subset \Omega$ such that $h(0) = 0$ and the following diagram commutes:

$$
\begin{array}{ccc}
\tilde{U} & \xrightarrow{u \mapsto u^k} & \mathbb{C} \\
\downarrow{h} & & \downarrow{f} \\
U & & \\
\end{array}
$$

Proof. We can write $f(z) = z^k g(z)$ where $g(z) = a \neq 0$. Let $\log z$ be a branch of the logarithm which is defined in a neighborhood $V \ni a = g(0) \neq 0$, and let $U$ be a neighborhood of 0, so small that $g(U) \subset V$. Define a function $h(z)$ on $U$ by the formula

$$h(z) = ze^{\frac{\log g(z)}{k}}.$$

We have $h(0) = 0$ and $h'(0) \neq 0$. Hence, decreasing, if necessary, the neighborhood $U$ we can assume that $h$ is a biholomorphism of $U$ onto its image $\tilde{U} = h(U) \ni 0$. But we have $f(z) = z^k g(z) = h^k(z)$. □

A holomorphic map $f : S_1 \to S_2$ is called a branched cover if there is a discrete set $B \subset S_2$ such that

- the restriction

$$f|_{S_1 \setminus f^{-1}(B)} : S_1 \setminus f^{-1}(B) \to S_1 \setminus B$$

is a covering map;

- each point $b \in B$ has a neighborhood $U$ such that $f^{-1}(U)$ can be presented as a disjoint union $U_1 \cup U_2 \cup \ldots$, and there exist local coordinates $w$ on $U$ and $z_j$ on $U_j$, $j = 1, 2, \ldots$ such that $f|_{U_j} : U_j \to U$ can be written in these coordinates as $w = z^{k_j}$, where $k_j$ are positive integer numbers.

Remark 11.26. If all $k_j$ are equal to 1 then the branched cover is just the usual cover. Hence, it is usually assumed that for each branching point, at least for one of the exponents we have $k_j > 1$. 

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The set $B$ is called the *branching locus* and the points $b \in B$ are called *branching points* of $f$. Note that if $S_2$ is connected then $S_2 \setminus B$ is connected as well. Hence the order of the covering $f|_{S_1 \setminus f^{-1}(B)} : S_1 \setminus f^{-1}(B) \to S_1 \setminus B$ is well defined. This number is usually referred to as the order of the branched covering $f$. Thus for a branched cover $f : S_1 \to S_2$ when $S_2$ is connected, the number of pre-images of any point $z \setminus B$ is the same, and it is equal to the order of the branched cover.

**Lemma 11.27.** Let $f : S_2 \to S_1$ be a branched map of order $d$. Suppose that a branching point $b \in B$ has $p$ pre-images and $(k_1, \ldots, k_p)$ are exponents of its pre-images. Then $\sum_{j=1}^{p} k_j = d$. In other words, a branching point has $d$ pre-images if one counts them with multiplicities given by the exponents $k_j$.

**Proof.** By definition the point $b \in B$ has a neighborhood $U$ such that $f^{-1}(U)$ can be presented as a disjoint union $U_1 \cup U_2 \cup \ldots$, and there exist local coordinates $w$ on $U$ and $z_j$ on $U_j$, $j = 1, 2, \ldots$ such that $f|_{U_j} : U_j \to U$ can be written in these coordinates as $w = z_j^{k_j}$. Hence, for a point $b' \in U \setminus b$ the preimage $f^{-1}(b')$ has exactly $k_j$ pre-images in $U_j$, so the total number of pre-images, which is by definition is the order $d$ of the branched cover $f$, is equal to $\sum_{j=1}^{p} k_j$.  

The set of exponents $(k_1, \ldots, k_p)$ is called the *type* of the branching point $b$. Hence, comparing the number of preimages $\#f^{-1}(b)$ of a branching point $b$ and a number of preimages $\#f^{-1}(b')$ of a nearby point $b'$ differ by $\sum_{j=1}^{p} (k_j - 1) = d - p$. Sometimes one refers to exponents $k_j$ (when $k_j > 1$ as the *order* of the corresponding pre-images of the branching points.

**Example 11.28.** The map $z \to z^p$ is a branch cover $\mathbb{C} \to \mathbb{C}$ of order $p$ with the unique branching point $0$. Viewed as a holomorphic map $\mathbb{C}P^1 \to \mathbb{C}P^1$ the map $z \mapsto z^p$ is a branched cover of order $p$ with branching points $0, \infty \in \mathbb{C}P^1$.

It turns out that any non-constant holomorphic map between two compact Riemann surfaces is a branch cover.

**Theorem 11.29.** Let $S_1, S_2$ be compact Riemann surfaces and $f : S_1 \to S_2$ a non-constant holomorphic map. Suppose $S_2$ is compact. Then $f : S_1 \to S_2$ is a branched cover.
Proof. First we observe that the map \( f \) is surjective, i.e. \( f(S_1) = S_2 \). Indeed, by the open image theorem \( f(S_1) \) is an open set. On the other hand, the image of a compact set is compact, and hence \( f(S_1) \) is also a closed subset of \( S_2 \). Hence, the connectedness of \( S_2 \) implies that \( f(S_1) = S_2 \). As \( f \) is not constant there are only finite set \( C \) of its critical points, i.e. points \( c \) where the differential \( df \) vanishes. Denote \( B := f(C) \subset S_2 \). We claim that \( f \) is a branched cover with the branching locus \( B \). Take a point \( b \in B \) denote by \( a_1, \ldots, a_p \) its pre-images. For a sufficiently small neighborhood \( U \ni b \) in \( S_2 \) its pre-image \( f^{-1}(U_b) \) can be presented as a disjoint union of neighborhoods \( V_j \ni a_j \). By making \( U_b \) smaller we can arrange that there exist a holomorphic coordinate \( u \) in \( U_b \) and a holomorphic coordinate \( v_j \) in \( V_j \) for each \( j = 1, \ldots, p \). We can assume, in addition that the coordinate \( u \) is centered at \( b \), and \( v_j \) is centered at \( a_j \), i.e. \( v_j(a_j) = 0, u(b) = 0 \). Applying Lemma 11.25 (and possibly choosing \( U \) even smaller and further adjusting coordinates \( v_j \)) we can arrange that the map \( f\big|_{V_j} : V_j \to U \) can be written as

\[
  u = f(v_j) = v_j^{k_j}, \quad j = 1, \ldots, p, \tag{11.10.1}
\]

Here \( k_j \) is the order of the first non-zero derivative at the point \( v_j \).

Consider now a point \( z \in S_2 \setminus B \). Then for each point \( a \in f^{-1}(z) \) we have \( d_au \neq 0 \). Hence, there exists a neighborhood \( V_a \ni a \) such that \( f\big|_{V_a} \) is a biholomorphism of \( V_a \) onto a neighborhood of \( z \). Hence, \( f^{-1}(z) \) is a discrete set, and hence due to compactness of \( S_1 \) consists of finitely many points. Choosing a neighborhood \( U \ni a \) in \( \bigcup_{a \in f^{-1}(z)} f(V_a) \) we conclude that

\[
  f^{-1}(U) = \bigcup_{a \in f^{-1}(z)} (V_a = V_a \cap f^{-1}(U))
\]

and

\[
  f\big|_{V_a} : V_a \to U
\]

is a biholomorphism. This proves that

\[
  f\big|_{S_1 \setminus f^{-1}(B)} : S_1 \setminus f^{-1}(B) \to S_2 \setminus B
\]

is a covering map, and together with (11.10.1) this implies that \( f : S_1 \to S_2 \) is a branched cover. ■
Corollary 11.30. For any compact Riemann surface $S$ and any non-constant meromorphic function

$$f : S \to \mathbb{C}P^1$$

is a branched cover.

The branching point of $f$ are critical values of $f$ (i.e. $f(a) \in \mathbb{C}P^1$ if $d_a f = 0$) and possibly $\infty$ in the case when $f$ has non-simple poles.
Part II

Famous meromorphic functions
In this part we introduce and study some famous meromorphic functions. First, in Chapter 12 we discuss the theory of elliptic functions, which are doubly periodic meromorphic functions, or equivalently meromorphic functions on tori. The main role in this story plays the Weierstrass function $\wp(z)$. In Chapter 13 we introduce and discuss properties of the Euler Gamma function $\Gamma(z)$, and finally in Chapter 14 we define and study some elementary properties of the Riemman zeta-function $\zeta(z)$ which plays a major role in Number Theory. Of course, we cannot discuss the number-theoretic applications in the framework of this course.
Chapter 12

Elliptic functions

12.1 Elliptic integrals

12.1.1 Motivation

Several geometric and physical problems lead to computations of integrals of the form

\[ \int_a^x \frac{dt}{\sqrt{P(t)}}, \quad \text{or more generally} \quad \int_0^x R(t, \sqrt{P(t)}) \, dt, \]

where \( P(t) \) is a polynomial and \( R(t, u) \) is a rational function of 2 variables. When \( \deg P \leq 2 \) this integral is not hard to compute in terms of elementary functions. However, if \( \deg P \geq 3 \) then in general this integral cannot be expressed through elementary functions. The integrals of this kind when \( \deg P = 3 \) or 4 are called elliptic. The term is motivated by Example 12.1.1 below.

Example 12.1. 1. Computing the arc length of an arc of an ellipse.

Consider an ellipse

\[ E(a, b) = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}, \]

or parametrically given by

\[ x = a \cos t, \]
\[ y = b \sin t. \]
Integrating the length of the vector \( \mathbf{v} = (\dot{x}, \dot{y}) = (-a \sin t, b \cos t) \) we compute the arc length \( s(\tau) \) between the points \((x(0), y(0)) = (a, 0) \) and \((a \cos \tau, b \sin \tau), \tau \in (0, \frac{\pi}{2})\), by the formula

\[
s(\tau) = \int_{0}^{\tau} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \, dt.
\]

Changing the variable \( \sin t = u \) we get

\[
s(\tau) = \int_{\arcsin u}^{\arcsin t} \frac{\sqrt{(a^2 - b^2)u^2 + b^2} \, du}{\sqrt{1 - u^2}} = \int_{\arcsin u}^{\arcsin t} \frac{\sqrt{(a^2 - b^2)u^2 + b^2}(1 - u^2) \, du}{1 - u^2}.
\]

2. **Pendulum in the constant gravity field.** The next example comes from Mechanics. In appropriate units its equation of motion can be written as

\[
\ddot{\theta} = -\sin \theta
\]

and the law of conservation of energy gives

\[
\frac{1}{2} (\dot{\theta})^2 - \cos \theta = E.
\]

Hence,

\[
\frac{d\theta}{dt} = \sqrt{2(E + \cos \theta)}, \quad (12.1.1)
\]

and separating variables we get

\[
t - t_0 = \int_{\theta_0}^{\theta} \frac{d\tau}{\sqrt{2(E + \cos \tau)}}.
\]

Changing the variable \( u = \cos \tau \) we get

\[
t - t_0 = \int_{\arccos \theta_0}^{\arccos \theta} \frac{du}{\sqrt{2(E + u)(1 - u^2)}}.
\]
12.1.2 Elliptic curves

To understand the proper meaning and geometry related to elliptic integrals we take a closer look at functions $\sqrt{P(u)}$ when $P$ is polynomial of degree 3. In fact, for determinacy we will consider just the polynomial $P(u) = u^3 - 1$. It will be crucially important to allow the variable $u$ to take complex values, and hence to visualize the graph of this function we consider the algebraic curve

$$S = \{z_2^2 = P(z_1)\}, \text{ where } P(z_1) = z_1^3 - 1.$$

To verify that $S$ is a smooth algebraic curve, and hence a Riemann surface we just need to check, according to Lemma 11.3 that both partial derivatives never vanish at the same point of $S$. Indeed, if $\frac{\partial F}{\partial z_1} = \frac{\partial F}{\partial z_2} = 0$ then $z_2 = P(z_1) = 0$. But this implies that $z_1 = \zeta_k$ and hence $\frac{\partial F}{\partial z_1} = -P'(\zeta_k) = -3\zeta_k^2 \neq 0$ for $\zeta_k = 0, 1, 2$.

Consider a holomorphic differential 1-form $\frac{dz_1}{z_2}$ on $\mathbb{C}^2 \setminus \{z_2 = 0\}$. The restriction $\alpha$ of $\frac{dz_1}{z_2}$ to the surface $S$ is defined everywhere except points $(\zeta_k, 0) \in S$.

**Lemma 12.2.** The form $\alpha$ extends holomorphically to the whole $S$ as a nowhere vanishing holomorphic 1-form.

**Proof.** To extend $\alpha$ to points $(\zeta_k, 0) \in S$ let us observe that the equation $z_2^2 = P(z_1)$ which holds on $S$ implies that $2z_2dz_2 = P'(z_1)dz_1$, i.e. for $z_2 \neq 0$ we have

$$\frac{dz_1}{z_2} = 2 \frac{dz_2}{P'(z_1)}. \tag{12.1.2}$$

But $P'(\zeta_k) \neq 0$, and hence the form $\frac{dz_1}{z_2}$ can be defined as a holomorphic 1-form on a neighborhood of $(\zeta_k, 0)$ as equal to $2 \frac{dz_2}{P'(\zeta_k)} \neq 0$.

Given a path $\gamma$ connecting in $S$ two points $(a_0, b_0), (a_1, b_1) \in S$ the integral $\int_\gamma \alpha$ depends only on the homotopy class of the pass $\gamma$, i.e. it remains unchanged if we continuously deform $\gamma$ keeping its end-points fixed. This integral gives the exact meaning to the integral

$$\int_{a_0}^{a_1} \frac{du}{\sqrt{u^3 - 1}}$$

and we will study it below in Section 12.6.
12.1.3 Projectivization

The Riemann surface $S$ is not compact and we would like to compactify it in a way similar to our compactification of $\mathbb{C}$ into the Riemann sphere $\mathbb{C}P^1$.

This is done by a general procedure, called projectivization which we already discussed in Section 11.3.2 above.

To projectivize the affine algebraic curve

$$S = \{z_2^2 - z_1^3 - 1 = 0\}$$

we complete each monomial to degree 3 by multiplying by an appropriate power $z_3^k$, i.e. consider the projective curve

$$\overline{S} = \{z_3 z_2^2 - z_1^3 + z_3^3 = 0\}. \quad (12.1.3)$$

Let us rewrite this equation in affine coordinates in the charts $\hat{U}_1, \hat{U}_2, \hat{U}_3$. These equations are obtained by dividing (12.1.3) by $z_1^3, z_2^3$ and $z_3^3$, respectively.

(i) $\overline{S} \cap \hat{U}_1 = \{z_3 z_2^2 - z_1^3 + z_3^3 = 0\}$,
(ii) $\overline{S} \cap \hat{U}_2 = \{z_3^2 - z_1^3 + z_3^3 = 0\}$,
(iii) $\overline{S} \cap \hat{U}_3 = \{z_2^3 - z_1^3 + 1 = 0\}$.

In equation (iii) we recognize the equation for the Riemann surface $S \subset \mathbb{C}^2$, written in coordinates $(z_{13}, z_{23})$ instead of $(z_1, z_2)$.

**Lemma 12.3.** $\overline{S} \subset \mathbb{C}P^2$ is a compact Riemann surface. It differs from $S \subset \hat{U}_2$ by a unique point with projective coordinates $p = (0, 1, 0)$.

**Proof.** Note that $\overline{S} \setminus \hat{U}_3 \subset \hat{U}_2$. Indeed, if $z \in \overline{S} \setminus (\hat{U}_3 \cup \hat{U}_2)$ then $z_2 = z_3 = 0$ which in view of equation (12.1.3) then implies that $z_1 = 0$, which is impossible. On the other hand, $\overline{S} \setminus \hat{U}_3 = \{p = (0, 1, 0)\}$.

In the coordinate chart $\hat{U}_2$ about $p$ the equation takes the form (ii):

$$G(z_{32}, z_{12}) = z_{32} - z_{12}^3 + z_{32}^3 = 0.$$
We note $\frac{\partial^2 G}{\partial z_{32}^2}(p) = 1 \neq 0$, and hence the implicit function theorem implies that $\overline{S}$ is a compact Riemann surface, which is called the *projectivization* of the Riemann surface $S$, see Lemma 11.7.

Let $\alpha$ be the differential 1-form which we defined on $S \subset \overline{U}_3$ in Section 12.1.2 above.

**Lemma 12.4.** The form $\alpha$ extends to $\overline{S}$ as a holomorphic non-vanishing form.

**Proof.** Recall that the form $\alpha$ has two equivalent expressions (see (12.1.2),

$$\alpha = \frac{dz_{13}}{z_{23}} = \frac{2dz_{23}}{P'(z_{13})} = \frac{2dz_{23}}{3z_{13}^2}.$$  

Both expression holds where $P(z_{13}) = z_{13}^3 - 1 \neq 0$ and $P'(z_{13}) = 3z_{13}^2 \neq 0$. Both inequalities hold when $|z_{13}| > 1$. Rewriting the latter expression for $\alpha$ in coordinates $z_{12}$ and $z_{32}$ (taking into account that $z_{13} = \frac{z_{32}}{z_{32}}$, $z_{23} = \frac{1}{z_{23}}$) we get

$$\alpha = -\frac{2}{3} \frac{dz_{32}}{z_{12}^2}. \quad (12.1.4)$$

Recall that on $\overline{S}$ we have

$$z_{32}^3 + z_{32}^3 = z_{12}^3.$$  

Differentiating we get

$$(1 + 3z_{32}^2)dz_{32} = 3z_{32}^2dz_{12}.$$  

Plugging this expression into (12.1.4) we get

$$\alpha = -\frac{2}{3} \frac{dz_{12}}{1 + 3z_{32}^3}. \quad (12.1.5)$$

This expression extends $\alpha$ to the point $p$ with coordinates $z_{12} = z_{32} = 0$ as a non-vanishing holomorphic form. ■

**Lemma 12.5.** The affine coordinate functions $z_{13}|_{S \subset \overline{S} \cap \overline{U}_3} : S \rightarrow \mathbb{C}$ and $z_{23}|_{S} : S \rightarrow \mathbb{C}$ extend as meromorphic functions to $\overline{S} \rightarrow \mathbb{C}P^1$, denoted $z$ and $w$, respectively. These functions are related by the equation

$$w^2 = P(z) = z^3 - z.$$  

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The holomorphic differential form $\alpha$ is equal to

$$\alpha := \frac{dz}{w}.$$ 

**Proof.** This follows from the removal of singularities theorem. 

12.1.4 From a cubic curve to a torus $T(\omega_1, \omega_2)$

**Proposition 12.6.** Let $\mathcal{S}$ be a compact Riemann surfaces which admits a non-vanishing holomorphic 1-form $\alpha$. Then there exists a lattice $\Lambda = \Lambda(\omega_1, \omega_2) \subset \mathbb{C}$ and a biholomorphism

$$h : \mathcal{S} \to T(\omega_1, \omega_2) = \mathbb{C}/\Lambda(\omega_1, \omega_2).$$

Moreover, the map $(h \circ \pi_{\Lambda}) : \mathbb{C} \to \mathcal{S}$ pulls back the form $\alpha$ to $du$, where $u$ is the coordinate in $\mathbb{C}$:

$$(h \circ \pi_{\Lambda})^* \alpha = du.$$ 

Here $\pi|_{\Lambda} : \mathbb{C} \to \mathbb{C}/\Lambda$ is the tautological projection.

**Proof.** Let $p : \mathcal{S} \to \mathcal{S}$ be the universal cover of $\mathcal{S}$. One can pull-back the form $\alpha$ to $\mathcal{S}$. The holomorphic differential 1-form $\beta := f^* \alpha$ is defined by the formula

$$\beta_z(T) = \alpha_{p(z)}(d_z p(T)),$$

where $z \in U$, $T$ is a tangent vector to $U$ at the point $z$ and $dp_{\alpha}$ is the differential of $p$ at the point $z$. Similar to the form $\alpha$ the form $\beta$ is not vanishing (because the differential $d_z p$ is not zero for every $z$ by the definition of a covering map. Being a holomorphic, the differential 1-form $\beta$ is closed in $\mathcal{S}$, but then it is exact because $U$ is simply connected. Therefore there exists a holomorphic function $f : \mathcal{S} \to \mathbb{C}$ such that $\beta = df$.

To complete the proof we need the following

**Lemma 12.7.** $f : \mathcal{S} \to \mathbb{C}$ is a biholomorphism.
Postponing the proof of the lemma we finish first the proof of Proposition 12.6. According to Lemma 12.7

\[ p \circ f^{-1} : \mathbb{C} \to \mathbb{S} \]

is a covering map. But then according to Theorem 11.17 and Lemma 11.21 there exists a lattice \( \Lambda \subset \mathbb{C} \) and a biholomorphism \( h : \mathbb{S} \to \mathbb{C}/\Lambda \) such that \( p \circ f^{-1} = h \circ \pi_{\Lambda} \). The rank of the lattice has to be equal 2, because if it is equal to 1 then the quotient \( \mathbb{C}/\Lambda \) would be a cylinder which is not compact. Hence, \( \Lambda = \Lambda(\omega_1, \omega_2) \) and therefore \( \mathbb{S} \) is conformally equivalent to the torus \( T(\omega_1, \omega_2) \).

Recall that by construction \( df = p^*\alpha \). On the other hand we have \( df = f^*du \), where \( u \) is the coordinate in \( \mathbb{C} \). Hence,

\[ du = (f^{-1})^*(p^*\alpha) = (p \circ f^{-1})^*\alpha = (h \circ \pi_{\Lambda})^*\alpha. \]

\[ \blacksquare \]

**Proof of Lemma 12.7**

We split the proof into several steps.

**Lemma 12.8 (Step 1).** There exists \( r > 0 \) such that every point \( a \in \mathbb{S} \) has a neighborhood \( \Delta_a \) such that \( f(\Delta_a) = D_r(f(a)) \) and \( f|_{\Delta_a} : \Delta_a \to D_r(f(a)) \) is a biholomorphism. If for \( a, b \in \mathbb{S} \), \( a \neq b \) we have \( f(a) = f(b) \) then \( \Delta_a \cap \Delta_b = \emptyset \).

**Proof.** Take a point \( a \in U \) and denote \( \bar{a} = p(a) \in \mathbb{S} \). By the definition of a covering there exists a neighborhood \( U \ni \bar{a} \) such that its pre-image \( p^{-1}(U, a) \) can be presented as the union \( U_1 \cup U_2 \cup \ldots \) such that

- \( U_i \cap U_j = \emptyset \) if \( i \neq j \), and
- \( p|_{U_j} : U_j \to U \) is a biholomorphism for \( j = 1, \ldots \).

In particular, there exists \( j \) such that \( U_j \ni a \). We denote \( \bar{U} := U_j \).
Let $\zeta$ be a local holomorphic coordinate in $U$ (it exists if $U$ is chosen small enough). Then $\overline{\zeta} := \zeta \circ p$ can be chosen as a local coordinate on each $\overline{U}$. The non-vanishing holomorphic form $\alpha|_{U(\alpha)}$ can be written as $\alpha = g(\zeta)d\zeta$, where $g(\zeta) \neq 0$. Hence the form $\beta = p^*\alpha$ is equal $g(\overline{\zeta})d\overline{\zeta}$ on each $\overline{U}$. The form $\beta = p^*\alpha$ is exact, $\beta = df = f'(\overline{\zeta})d\zeta$. Hence, on $f'(\overline{\zeta}) = g(\overline{\zeta}) \neq 0$. Hence the map $f|_{\overline{U}} : \overline{U} \to \mathbb{C}$ is injective if the neighborhood $U$ is chosen small enough. The image $V := f(\overline{U})$ is open. Hence there exists $r > 0$ such that $D_r(f(\overline{a})) \subset V$. Denote

$$\Delta_a = f^{-1}(D_2(f(\overline{a}))) \cap \overline{U}, \quad \overline{\Delta}_a := f^{-1}(D_r(f(\overline{a}))) \cap \overline{U}.$$ 

Note that the choice of $r$ depends only on the the point $\overline{a} \in \overline{S}$, and not on $a$. Hence, in view of compactness of $\overline{S}$ one can choose $r > 0$ which works for all points $\overline{a} \in \overline{S}$.

Suppose now that there exist $a, b \in \overline{S}$ such that $a \neq b$ but $f(a) = f(b)$. We claim that $\Delta_a \neq \Delta_b$. Indeed, suppose there exists $c \in \Delta_a \cap \Delta_b$. Then $b \in \Delta_c$ and we have

$$\left( f(\Delta_c) = D_2(f(\overline{c})) \right) \cap \left( f(\Delta_a) = D_2(f(\overline{a})) \right).$$

Therefore, $D_2(f(\overline{c})) \subset D_2(f(\overline{a}))$. But this implies that

$$\Delta_c \subset \overline{\Delta}_a := f^{-1}(D_r(f(\overline{a}))) \cap \overline{U}.$$ 

Hence, $b \in \overline{\Delta}_a$, but this contradicts to the fact that $f|_{\overline{\Delta}_a}$ is injective.

Lemma 12.9 (Step 2). The map $f : \overline{S} \to \mathbb{C}$ is a covering map.

Proof. It remains to show that $f$ is surjective. Indeed, in by Lemma 12.8 this would imply that for each $z \in \mathbb{C}$ the pre-image $f^{-1}(D_2(z))$ is a disjoint union of neighborhoods which are biholomorphically mapped by the map $f$ onto $D_2(z)$, which is the definition of a covering map. By the open image theorem we know that $f(U)$ is open. If $f(U) \neq \mathbb{C}$ then there exists a point $z \in \mathbb{C} \setminus f(U)$ which is a boundary point of $f(U)$. Hence, there exists $a \in \overline{S}$ such that $|f(a) - z| < \frac{r}{2}$. But then $z \in D_2(f(\overline{a})) = f(\Delta_a) \subset f(U)$, which is a contradiction.

Lemma 12.10 (Step 3). The map $f : \overline{S} \to \mathbb{C}$ is a biholomorphism.
**Proof.** This follows from simply connectedness of \( \mathbb{C} \) and Theorem 11.18. Because \( \mathbb{C} \) is simply connected, the identity map \( \text{Id} : \mathbb{C} \to \mathbb{C} \) is also a universal cover, but then according to Theorem 11.18.2 the universal cover is unique, i.e there exists a biholomorphism \( F : \mathbb{C} \to \hat{S} \) such that \( f \circ F = \text{Id} \). But then \( f = F^{-1} \) is a biholomorphism as well.

This concludes the proof of Lemma 12.7.

### 12.1.5 Summary of the construction

Let us summarize what we achieved in this section.

1. For a degree 3 polynomial \( P(u) = u^3 + a_1u^2 + a_2u + a_3 \) without multiple roots we associated first a Riemann surface \( S = \{w^2 = P(z)\} \) in \( \mathbb{C}^2 \) and then compactified it to a Riemann surface \( \overline{S} \) in \( \mathbb{C}P^2 \supset \mathbb{C}^2 \), called *projectivization* of \( S \).

2. We showed that the coordinate functions \( z|_S, w|_S \) extend to \( \overline{S} \) as meromorphic functions satisfying the equation \( w^2 = P(z) \), and the differential form \( \alpha = \frac{dz}{w} \) is defined on the whole \( \overline{S} \) as a non-vanishing holomorphic form.

3. We found a lattice \( \Lambda = \Lambda(\omega_1, \omega_2) \subset \mathbb{C} \) and a biholomorphism

\[
f : T(\omega_1, \omega_2) = \mathbb{C}/\Lambda(\omega_1, \omega_2) \to \overline{S}
\]

such that

\[
(\pi_\Lambda \circ f)^* \alpha = du
\]

where \( u \) is the coordinate in \( \mathbb{C} \) and \( \pi_\Lambda : \mathbb{C} \to \mathbb{C}/\Lambda \) is the tautological projection. In other words, if we denote

\[
\overline{z}(u) = z(f(\pi_\Lambda(u))), \overline{w}(u) = w(f(\pi_\Lambda(u)))
\]

then

\[
\frac{d\overline{z}(u)}{\overline{w}(u)} = du, \quad \text{or} \quad \frac{d\overline{z}(u)}{du} = \overline{w}(u).
\]
Hence,
\[
\left( \frac{d\overline{z}(u)}{du} \right)^2 = \overline{w}(u)^2 = P(u),
\]
i.e. the function \( \overline{z}(u) \) is a solution of the differential equation
\[
\left( \frac{d\overline{z}(u)}{du} \right)^2 = P(u).
\] (12.1.6)

In the next section we will find this solution explicitly.

### 12.2 The Weierstrass \( \wp \)-function

A holomorphic or meromorphic function on \( T(\omega_1, \omega_2) \) is exactly the same as a doubly periodic function on \( \mathbb{C} \):
\[
f(z + \omega_1) = f(u), \quad f(u + \omega_2) = f(z).
\]

There is no interesting holomorphic functions with this properties. Indeed any such function is bounded, and by Liouville’s theorem has to be constant. Hence, we will be studying meromorphic doubly periodic functions, i.e. holomorphic maps
\[
T(\omega_1, \omega_2) \rightarrow \mathbb{C}P^1.
\]

Such functions are called elliptic. We begin with the famous example of an elliptic function, called Weierstrass \( \wp \)-function. Suppose we are given a lattice \( \Lambda = \Lambda(\omega_1, \omega_2) \). The Weierstrass \( \wp \)-function is defined by the formula
\[
\wp(u) = \sum_{\lambda \in \Lambda \setminus \{0\}} \left( \frac{1}{(u - \lambda)^2} - \frac{1}{\lambda^2} \right) + \frac{1}{u^2}.
\] (12.2.1)

**Lemma 12.11.** The series in formula (12.2.1) absolutely uniformly converges on compact sets in \( \mathbb{C} \setminus \Lambda \). The function is doubly periodic with periods \( \omega_1, \omega_2 \) and hence define a meromorphic function \( T(\omega_1, \omega_2) \rightarrow \mathbb{C}P^1 \) with a unique pole of order 2.

**Proof.** Denote \( \varepsilon = \{\inf |u - \lambda|; \lambda \in \Lambda\} \). then we have
\[
\left| \frac{1}{(u - \lambda)^2} - \frac{1}{\lambda^2} \right| = \frac{2\lambda |u| + |u|^2}{|\lambda|^2 |u - \lambda|^2} \leq \frac{C}{|\lambda|^3},
\]
where $C$ depends on $\epsilon$ and $R := |u|$. Hence,

$$\sum_{\lambda \in \Lambda \setminus 0} \left| \frac{1}{(u - \lambda)^2} - \frac{1}{|\lambda|^2} \right| \leq C \sum_{\lambda \in \Lambda \setminus 0} \frac{1}{|\lambda|^3}.$$  

But the series in the right hand side converges, as it is clear by comparison with the converging integral

$$\int_{\mathbb{C} \setminus \{|u| \leq 1\}} \frac{dxdy}{|u|^3} = \int_1^\infty \int_0^{2\pi} \frac{drd\phi}{r^2} = 2\pi.$$  

This proves that the function $\varphi(u)$ is meromorphic on $\mathbb{C}$. To check that it is doubly periodic, let us compute the derivative by term-wise differentiation:

$$\varphi'(u) = -\sum_{\lambda \in \Lambda} \frac{1}{(u - \lambda)^3}.$$  

Clearly $\varphi'(u)$ is doubly periodic with periods $\omega_1$ and $\omega_2$, because addition to $\lambda$ multiples of $\omega_1$ and $\omega_2$ leaves the sum unchanged. Hence

$$\frac{d}{du} \left( \varphi(u + \omega_j) - \varphi(u) \right) = \varphi'(u + \omega_j) - \varphi'(u) = 0, \quad j = 1, 1, 2$$

Hence, $\varphi(u + \omega_j) - \varphi(u) = c_j, \quad j = 1, 2$ for some constants $c_1$ and $c_2$. But $\varphi(u)$ is an even function: $\varphi(-u) = \varphi(u)$. Hence

$$c_j = \varphi \left( \frac{\omega_j}{2} \right) - \varphi \left( -\frac{\omega_j}{2} \right) = 0,$$

and therefore $\varphi(u)$ is doubly periodic and thus defines a meromorphic function $T(\omega_1, \omega_2) \to \mathbb{C}P^1$ with a unique pole of order to in the image of the origin 0 under the tautological projection $\mathbb{C} \to T(\omega_1, \omega_2)$.

\section*{12.2.1 Differential equation for $\varphi(u)$}

Consider the Laurent expansion of $\varphi(u)$ about 0. We have

$$\varphi(u) = \frac{1}{u^2} + au^2 + bu^4 + \ldots.$$  

The absence of odd terms follows from the fact that \( \varphi \) is even (why?). The vanishing of the constant term is clear from the fact that

\[
\varphi(u) - \frac{1}{u^2} = \sum_{\lambda \in \Lambda} \frac{1}{(u - \lambda)^2} - \frac{1}{\lambda^2}
\]

vanishes at 0.

Differentiating twice we get

\[
\varphi''(u) = \frac{6}{u^4} + 2a + \ldots
\]

Therefore

\[
\varphi''(u) - 6\varphi^2(u) = -10a + \ldots
\]

is a holomorphic doubly periodic function, and hence constant, i.e.

\[
\varphi''(u) = 6\varphi^2(u) - 10a.
\]

Thus we have

\[
\frac{d}{du} \left( \varphi'(u)^2 \right) = 2\varphi'(u)\varphi''(u) = 12\varphi'(u)\varphi^2(u) - 20a\varphi'(u) = \frac{d}{du} \left( 4\varphi^3(u) - 20a\varphi(u) \right),
\]

i.e.

\[
(\varphi'(u))^2 = 4\varphi^3(u) - 20a\varphi(u) + b,
\]

where \( a, b \) are some constants which depend on the lattice \( \Lambda \).

In fact, traditionally one uses different choice of constants and write this differential equation in the form

\[
(\varphi'(u))^2 = 4\varphi^3(u) - g_2\varphi(u) - g_3. \tag{12.2.2}
\]

To explain a somewhat strange notation \( g_2 \) and \( g_3 \) for the coefficients we need a deeper theory of elliptic functions which we cannot discuss in this course.
12.2.2 Identifying Weierstrass $\wp$-function with the solution of the pendulum equation

We observe that equation (12.2.2) is similar to equation (12.1.6) for an appropriate cubic polynomial $P$.

In fact, we will see that the solution of this equation is the Weierstrass function for an appropriate lattice.

Lemma 12.12. Let $S = \{z_2^3 = P(z_1)\}$ where $P$ is a cubic polynomial and $\overline{S} = S \cup \{p\}$ is the projectivization of the curve $S$. Let $z : \overline{S} \to \mathbb{CP}^1$ be the meromorphic extension of the coordinate $z_1|_S$ (see Lemma 12.5). The function $z$ has a single pole of multiplicity 2 at the point $p$.

Proof. We continue for determinacy to work with the polynomial $P(z_1) = z_1^3 - z_1$. As we had seen in Section 12.1.3 near the point $p$ the curve $\overline{S}$ is defined in coordinates $z_{12}, z_{13}$ by the equation

$$z_{32} - z_{12}^3 + z_{32}^3 = 0,$$

and $z_{12}$ can be chosen as a local coordinate near $p$. Expressing $z_{32}$ from this equation we get

$$z_{32} = z_{12}^3 + o(z_{12}^3).$$

At the same time the function $z$ in these coordinate chart is equal to

$$\frac{z_{12}}{z_{32}} = \frac{1}{z_{12}^2} + o(z_{12})^3,$$

i.e. it has a pole of order 2. ■

Lemma 12.13. Any meromorphic function $f(z)$ on $T(\omega_1, \omega_2) = \mathbb{C}/\Lambda$ with a unique pole of order 2 is equal to $A\wp(u + u_0) + B$, where $\wp(u) = \wp_\Lambda(u)$ is the Weierstrass function for the lattice $\Lambda$, and $A, B \in \mathbb{C}$ are constants.

Proof. We can view $f$ is a bi-periodic meromorphic function on $\mathbb{C}$ with periods $\omega_1, \omega_2$. By shifting the origin $u \mapsto u + u_0$ we can assume that the poles of $f$ are in the vertices of $\Lambda$. By scaling with an appropriate constant $C$ we can assume that near 0 the Laurent expansion of $f$ has the form

$$f(u) = \frac{1}{u^2} + \frac{a}{u} + g(u).$$
where $g$ is a holomorphic near 0. Consider the difference $g(u) = f(u) - \wp(u)$. This is either a holomorphic bi-periodic function (and hence is a constant), or a meromorphic function with order 1 poles at vertices of $\Lambda$. Hence, the following lemma completes the proof.

**Lemma 12.14.** A meromorphic function on $T(\omega_1, \omega_2)$ cannot have a unique simple pole.

**Proof.** Let $g : \mathbb{C}/\Lambda(\omega_1, \omega_2) \to \mathbb{C}P^1$ be a holomorphic function with a unique pole. We lift $g$ as a bi-periodic function on $\mathbb{C}$. By combining it with a translation, if necessary, we can assume that the poles are at vertices of $\Lambda$. Consider a parallelogram $P$ with vertices

$$A_{-} = -\frac{\omega_1 - \omega_2}{2}, \ A_{+} = -\frac{\omega_1 + \omega_2}{2}, \ A_{++} = \frac{\omega_1 - \omega_2}{2}, \ A_{++} = \frac{\omega_1 + \omega_2}{2}.$$ 

There is only 1 simple pole of $g$ inside $P$. Hence, the residue theorem says

$$\int_{\partial P} g(u) du = 2\pi i \text{Res } g \neq 0.$$ 

On the other hand the integral is 0. Indeed, the integrals over the opposite sides of the parallelogram cancels, because the integrand are equal due to the periodicity but the direction of integration is opposite. This contradiction finishes off the proof of Lemma 12.14 and with it of Lemma 12.13 as well.

Hence, the solution of equation (12.1.6) for a pendulum is given by the formula

$$\theta = \arccos(A + B\wp(t + t_0))$$ 

for an appropriate lattice $\Lambda$. We leave to the reader to fix the constants $A$ and $B$.

Equation (12.2.2) also implies that the meromorphic $\mathbb{C}/\Lambda \to \mathbb{C}^2$ given by the formula

$$u \mapsto (\wp(u), \wp'(u))$$ 

maps the torus $T(\omega_1, \omega_2) \setminus 0$ (where 0 is the pole of $\wp(u)$ and $\wp'(u)$ onto the cubic surface

$$S = \{z_2^2 = 4z_1^3 - g_2z - g_3\}.$$ 

This maps extends as a holomorphic map of the torus $T(\omega_1, \omega_2)$ onto the compactification $\overline{S}$ of $S$. This map is a biholomorphism and the inverse map $\overline{S} \to T(\omega_1, \omega_2)$ is given by integration of the holomorphic form $\alpha$ constructed above in Lemma 12.7.
12.2.3 More about geometry of the Weierstrass $\wp$-function

Given a holomorphic map $f : S_1 \to S_2$ a point $a \in S_1$ is called a branching point of order $k > 1$, if in some local coordinates $z$ near $a$ and $w$ near $f(z)$ the map $f$ can be written as

$$w = f(z) = z^k g(z),$$

where $g(a) \neq 0$. The branching point is characterized by the property $f'(a) = f''(a) = \cdots - f^{(k-1)}(a) = 0$ and $f^{(k)}(a) \neq 0$. A pole of order > 1 of a meromorphic function $\overline{S} \to \mathbb{C}P^1$ is also a branching if viewed in appropriate coordinates.

As we see from (12.2.2) the Weierstrass function $\wp$ has 3 branching points $u_1, u_2, u_3$ where the derivative $\wp'(z)$ vanishes. These points are roots of the cubic polynomial $P(z) = 4z^3 - g_2 z - g_3$. The second derivatives do not vanish if the polynomial $P$ has no multiple roots. The function $\wp$ also has a unique pole of order 2. Has the holomorphic map $\wp : T(\omega_1, \omega_2) \to \mathbb{C}P^1$ has 4 branching points of order 2. The following picture (called ”C.S. Peirce quincuncial projection”) illustrates this map
(viewed as a doubly periodic map from $\mathbb{C}$.
Chapter 13

The Gamma function

13.1 Product development

The infinite product of complex numbers $\prod_{j=1}^{\infty} p_j$ is called *convergent* if there exists $\lim_{n \to \infty} \Pi_n$, where $\Pi_n = \prod_{j=1}^{n} p_j$, and this limit is not 0. Sometimes one slightly relaxes the latter condition by allowing a finite number of terms in the products to be equal to 0, while the rest of the product is required to converge to a non-zero number.

**Lemma 13.1.** The common term $p_n$ of a convergent product tends to 1 when $n \to \infty$.

Indeed, $p_n = \frac{\Pi_n}{\Pi_{n-1}} \to 1$.

In view of this lemma we can define (the principal branch of) $\log p_n$ for all but finite number of terms of the product.

**Theorem 13.2.** The product $\prod_{j=1}^{\infty} p_j$ converges if and only if the series $\sum_{j=1}^{\infty} \log p_j$ converges.

**Proof.** It is clear that if $\sum_{j=1}^{\infty} \log p_j$ converges then $\prod_{j=1}^{\infty} p_j$ converges. Indeed,

$$e^{\sum_{j=1}^{n} \log p_j} = \Pi_n = \prod_{j=1}^{n} p_j,$$
so the convergence of partial sums $S_n := \sum_{j=1}^{n} \log p_j$ implies the convergence of partial products to a non-zero number.

The converse is slightly more tricky. Indeed, for the main branch of $\log$ its not true in general that $\log(ab) = \log a + \log b$. Suppose there exists a non-zero limit $\lim_{n} \Pi_n = \Pi$. Then $\frac{\Pi_n}{\Pi} \to 1$ and $p_n \to 1$. Hence, for large $n$ we can write $p_n = r_n e^{i\phi_n}$, where $|\phi_n| < \pi$.

Let $\Pi_n = R_n e^{i\Theta_n}$, $\Pi = R e^{i\Theta}$, where $\Theta_n = \sum_{k=1}^{n} \phi_k$. Then

$$R_n \to R, \quad \text{and} \quad \ln R_n \to \ln R,$$

and there exists a sequence of integer numbers $k_n$ such that

$$\Theta_n - \Theta - 2\pi k_n = \theta_n \to 0.$$ 

Choose $n$ large enough, so that $|\theta_n| < \frac{\pi}{2}$ and $|\phi_n| < \pi$. Then

$$\phi_n = \Theta_n - \Theta_{n-1} = 2\pi(k_n - k_{n-1}) + \theta_n - \theta_{n-1},$$

i.e.

$$\phi_n - \theta_n + \theta_{n-1} = 2\pi(k_n - k_{n-1}).$$

On the other hand,

$$|\phi_n - \theta_n + \theta_{n-1}| < 2\pi.$$

Therefore, the sequence $k_n$ stabilizes, i.e.

$$k_n = k_{n-1} = K \quad \text{if} \quad n \quad \text{is large enough}.$$

Thus, $\Theta_n = \sum_{j=1}^{n} \phi_j \to \Theta + 2K\pi$. Thus

$$\sum_{j=1}^{n} \log p_j = \sum_{j=1}^{n} \ln r_j + i \sum_{j=1}^{n} \phi_j \to \log R + i\Theta + 2\pi K i.$$  

The product $\prod_{j=1}^{\infty} p_j$ is called *absolutely convergent* if the series $\sum_{j=1}^{\infty} \log p_j$ absolutely converges.  

---

1Warning: the absolutely convergence implies but not equivalent to the convergence of the product $\prod_{j=1}^{\infty} |p_j|$ (why?)
Lemma 13.3. Denote \( p_n = 1 + a_n \). The product \( \prod_{n=1}^{\infty} p_n \) absolutely converges if and only if the series \( \sum_{n=1}^{\infty} |a_n| \) converges.

**Proof.** We have \( \frac{|\log (1 + a_n)|}{|a_n|} \to 1 \) if \( |a_n| \to 0 \). But both (if and only if) assumptions imply that \( a_n \to 0 \).

Hence, for large \( n \) we have

\[
\frac{|a_n|}{2} < |\log(1 + a_n)| < 2|a_n|,
\]

and hence both series are simultaneously converge or not.  \( \blacksquare \)

Exercise 13.4. 1. Prove that if \(|z| < 1\) then

\[
(1 + z)(1 + z^2)(1 + z^4)(1 + z^8) \cdots = \frac{1}{1 - z}.
\]

*Hint:* First verify using Lemma 13.3 that the product absolutely converges, and then use the fact that every integer has a unique binary presentation.

2. Show that the product

\[
\prod_{\text{prime } p} \left( 1 - \frac{1}{p} \right)^{-1} = \prod_{\text{prime } p} \left( \frac{1}{p} + \frac{1}{p^2} + \ldots \right)
\]

diverges.

*Hint:* Any integer can be uniquely factored as a product of primes and the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

3. Prove that the series

\[
\sum_{\text{prime } p} \frac{1}{p}
\]

diverges.

4. Let \( \pi(n) \) denote the number of primes \( \leq n \). Use Exercise 3 to show that there is no \( C, \epsilon > 0 \) such that

\[
\pi(n) < Cn^{1-\epsilon}.
\]
Meromorphic functions with prescribed poles and zeroes

Let us begin with the following simple lemma.

**Lemma 13.5.** Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function without zeroes. Then there exists an entire function \( g : \mathbb{C} \to \mathbb{C} \) such that \( f(z) = e^{g(z)} \).

**Proof.** The function \( \frac{f'(z)}{f(z)} \) is holomorphic. Hence, the differential form
\[
\alpha := \frac{f'(z)dz}{f(z)}
\]
is closed on \( \mathbb{C} \), and hence exact. Therefore, there exists a holomorphic function \( g(z) \) such that
\[
dg = g'(z)dz = \alpha = \frac{f'(z)dz}{f(z)}.
\]
Thus \( g'(z) = \frac{f'(z)}{f(z)} \) and
\[
\frac{d}{dz} \left( f(z)e^{-g(z)} \right) = \left( f'(z) - f(z)g'(z) \right)e^{-g(z)} = f(z) \left( \frac{f'(z)}{f(z)} - g'(z) \right)e^{-g(z)} = 0.
\]
Therefore,
\[
f(z) = Ce^{g(z)} = e^{g(z) + \log C}, \quad C \neq 0.
\]

**Corollary 13.6.** Suppose an entire function \( f : \mathbb{C} \to \mathbb{C} \) has finitely many zeroes. Denote zeroes not at the origin by \( a_1, \ldots, a_n \) (where multiple zeroes are repeated), and let \( m \) be the multiplicity of the zero at the origin (possibly, \( m = 0 \)). Then there exists an entire function \( g : \mathbb{C} \to \mathbb{C} \) such that
\[
f(z) = z^m e^{g(z)} \prod_{j=1}^{n} \left( 1 - \frac{z}{a_j} \right).
\]

**Proof.** Apply Lemma 13.5 to the entire function
\[
\frac{f(z)}{z^m \prod_{j=1}^{n} \left( 1 - \frac{z}{a_j} \right)}
\]
which has no zeroes.

This corollary can be generalized to the case of infinitely many zeroes. But we will do it only in some special cases.
13.3 Some product and series developments for trigonometric functions

Lemma 13.7.

\[ \frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}, \quad z \notin \mathbb{Z}. \]

Proof. We first note that the series in the right hand side is uniformly converging on compact sets in \( \mathbb{C} \setminus \mathbb{Z} \), and hence the sum is a meromorphic function with double poles at points of \( \mathbb{Z} \). But so is the left-hand side. The coefficients with \( \frac{1}{(z-n)^2} \) in the Laurent expansion about \( n \) in both sides are the same, so the difference

\[ h(z) = \frac{\pi^2}{\sin^2 \pi z} - \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \]

is an entire holomorphic function. Note that

\[ |\sin(x+iy)| = \frac{1}{2}|e^{ix}e^{-y} - e^{-ix}e^{y}| \leq e^{-|y|} \rightarrow 0, \quad |y| \rightarrow \infty, \]

and hence

\[ \frac{\pi^2}{\sin^2 \pi(x+iy)} \rightarrow 0. \quad (13.3.1) \]

We also note

\[ \sum_{n=-\infty}^{\infty} \frac{1}{|x+iy-n|^2} = \sum_{-\infty}^{\infty} \frac{1}{(x-n)^2 + y^2} \]

is monotone decreasing in \( |y| \). On the other hand, \( h(x+iy) \) is \( 1 \)-periodic in \( x \). Hence, \( |h(z)| \) is bounded, and therefore, in view of Liouville’s theorem,

\[ h(z) = \text{const.} \quad (13.3.2) \]

Moreover, for \( x = \frac{1}{2} \) we have

\[ \frac{(\frac{1}{2} - n)^2}{(\frac{1}{2} - n)^2 + y^2} = \frac{(\frac{1}{2} - n)^2}{1 + \frac{y^2}{(\frac{1}{2} - n)^2}} \leq \frac{(\frac{1}{2} - n)^2}{1 + 4y^2}. \]
Hence,

\[
\left| \sum_{-\infty}^{\infty} \frac{1}{(\frac{1}{2} + iy - n)^2} \right| \leq \frac{1}{1 + 4y^2} \sum_{-\infty}^{\infty} \frac{1}{(\frac{1}{2} - n)^2} \to 0. \tag{13.3.3}
\]

Combining (13.3.1) and (13.3.3) we conclude that

\[
h\left(\frac{1}{2} + iy\right) \to 0, \quad |y| \to \infty,
\]
and therefore, in view of (13.3.2) we get \(h(z) = 0\).

\[\square\]

**Lemma 13.8.**

\[
\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z - n} + \frac{1}{n} \right).
\]

**Proof.** First, note that the series in the right-hand side uniformly converges on compact sets in \(\mathbb{C} \setminus \mathbb{Z}\), and hence its sum is a meromorphic function with simple poles at points of \(\mathbb{Z} \subset \mathbb{C}\). Hence, we can differentiate the series term-wise to get

\[
\frac{d}{dz} \left( \frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z - n} + \frac{1}{n} \right) \right) = -\sum_{-\infty}^{\infty} \frac{1}{(z - n)^2}.
\]

On the other hand, we have

\[
(\pi \cot \pi z)' = -\frac{\pi^2}{(\sin \pi z)^2} = -\sum_{-\infty}^{\infty} \frac{1}{(z - n)^2}.
\]

Thus,

\[
\pi \cot \pi z = C + \frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z - n} - \frac{1}{n} \right).
\]

But both functions, \(\pi \cot \pi z\) and \(\frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z - n} - \frac{1}{n} \right)\) are odd (i.e. changing signs when \(z \mapsto -z\)), and therefore, \(C = 0\).

\[\square\]

**Lemma 13.9.**

\[
\sin \pi z = \pi z \prod_{n \neq 0} \left( 1 - \frac{z}{n} \right) e^{\frac{z^2}{2n^2}} = \pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right).
\]

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Proof. First note that the product \( \pi \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}} \) absolutely converges because the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) absolutely converges, see Lemma 13.3. Moreover, it converges uniformly on every compact set in \( \mathbb{C} \). The function \( \pi \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}} \) has simple zeroes at all real integer points \( 0, \pm 1, \pm 2, \ldots \). But so does \( \sin \pi z \). Hence, the entire function

\[
h(z) = \frac{\sin \pi z}{\pi \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}}
\]

has no zeroes, and hence by applying Lemma 13.5 we get:

\[
\sin \pi z = e^{g(z) \pi \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}}.
\] (13.3.4)

Let us prove that \( g(z) \) is a constant. To do this we compute the logarithmic derivative (i.e. \( \frac{f'}{f} \)) of both parts of the equation (13.3.4). We get

\[
\pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n \neq 0} \left( \frac{1}{z - n} + \frac{1}{n} \right).
\]

Comparing with the expression for \( \pi \cot \pi z \) from Lemma 13.8 we conclude that \( g'(z) = 0 \), and hence \( g(z) = C \) is a constant. Thus, from (13.3.4) we get

\[
\frac{\sin \pi z}{z} = e^{C} \pi \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}.
\]

Passing to the limit when \( z \to 0 \) we get \( 1 = e^{C} \), and hence we obtain the required formula

\[
\sin \pi z = \pi \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}.
\] (13.3.5)

Combining terms with \( n \) and \( -n \) we get the second expression for the right-hand side:

\[
\sin \pi z = \pi \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).
\] (13.3.6)
13.4 The Gamma function: definition and some properties

Denote

\[ G(z) := \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-\frac{1}{n}}. \]

Then (13.3.5) can be rewritten as

\[ \frac{\sin \pi z}{\pi} = zG(z)G(-z). \quad (13.4.1) \]

Note that we have \( G(0) = 1 \).

The value

\[ \gamma = -\log G(1). \]

is called Euler’s constant. Thus,

\[ e^{-\gamma} = \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right) e^{-\frac{1}{n}}. \quad (13.4.2) \]

Note that the partial product \( \Pi_n = \prod_{k=1}^{n} \left( 1 + \frac{1}{k} \right) e^{-\frac{1}{k}} \) is equal to

\[ \Pi_n = (n + 1)e^{-\sum_{1}^{n} \frac{1}{k}}. \quad (13.4.3) \]

Hence,

\[ \gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) = 0.57722\ldots \quad (13.4.4) \]

We define now the Gamma function by the formula

\[ \Gamma(z) = \frac{e^{-\gamma z}}{zG(z)} = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right)^{-1} e^{\frac{z}{n}} = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{ne^{\frac{z}{n}}}{z + n}. \quad (13.4.5) \]

Thus, \( \Gamma(z) \) is a meromorphic function without zeroes on \( \mathbb{C} \) with poles at \( 0, -1, -2, \ldots \).

Lemma 13.10 (Euler’s formula).

\[ \Gamma(z) = \lim_{n \to \infty} \frac{n!n^z}{z(z + 1)(z + 2)\ldots(z + n)} \quad (n^z = e^{z\ln n}). \quad (13.4.6) \]
Proof. By definition,

\[
\Gamma(z) = \lim_{n \to \infty} \left( \frac{n!e^{-\gamma z}}{z} \prod_{j=1}^{n} \frac{e^{\frac{j}{z}}}{z+j} \right) = \lim_{n \to \infty} \frac{n!e^{-\gamma z}e^{(1+\ldots+rac{1}{z})}}{z(z+1)(z+2)\ldots(z+n)}
\]

But according to (13.4.4) we have

\[
e^{-\gamma z} = \lim_{n \to \infty} (n)e^{-\sum_{k=1}^{n} \frac{1}{k}}
\]

and hence we get the required formula (13.4.6). ■


1. \(\Gamma(z+1) = z\Gamma(z); \) in particular, \(\Gamma(n+1) = n!;\)
2. \(\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}; \) in particular, \(\Gamma(\frac{1}{2}) = \sqrt{\pi};\)
3. \(\frac{d}{dz} \left( \frac{\Gamma'(z)}{\Gamma(z)} \right) = \sum_{0}^{\infty} \frac{1}{(z+n)^2}.\)

Proof.

1. Using Euler’s formula (13.4.6) we get

\[
\Gamma(z+1) = \lim_{n \to \infty} \frac{n!n^{z+1}}{(z+1)(z+2)\ldots(z+n)} = z \left( \lim_{n \to \infty} \frac{n}{n+1} \right) \left( \lim_{n \to \infty} \frac{(n+1)!n^z}{z(z+1)(z+2)\ldots(z+n)} \right) = z\Gamma(z).
\]

Part 2 follows immediately from (13.4.1), and Part 3 can be proven by a direct computation. ■

We will need below the following estimate for \(\Gamma(z).\)

Lemma 13.12. Let \(z = x + iy, x > 0.\) Then

\[
|\Gamma(x+iy)| \geq \frac{x\Gamma(x)}{\sqrt{x^2 + y^2}} \frac{\sqrt{\pi|y|}}{\sqrt{\sinh \pi|y|}}.
\]

In particular, for \(x \in [1,2]\) we have

\[
|\Gamma(x+iy)| \geq Ce^{-\frac{|y|}{2}}.
\]

Proof. We have

\[
|\Gamma(x+iy)| = \frac{e^{-\gamma x}}{\sqrt{x^2 + y^2}} \prod_{1}^{\infty} \frac{ne^n}{\sqrt{(n+x)^2 + y^2}} = \left( \frac{e^{-\gamma x}}{x} \prod_{1}^{\infty} \frac{ne^n}{(n+x)^2} \right) \left( \frac{x}{\sqrt{x^2 + y^2}} \prod_{1}^{\infty} \frac{1}{\sqrt{1 + \frac{y^2}{(n+x)^2}}} \right)
\]

\[
\geq \frac{x\Gamma(x)}{\sqrt{x^2 + y^2}} \prod_{1}^{\infty} \frac{1}{\sqrt{1 + \frac{y^2}{n^2}}} = \frac{x\Gamma(x)}{\sqrt{x^2 + y^2}} \frac{\sqrt{\pi|y|}}{\sqrt{\sinh \pi|y|}}.
\]

\[\blacksquare\]
13.5 Integral representation of the Gamma function

Lemma 13.13. The integral
\[ \hat{\Gamma}(z) := \int_{0}^{\infty} e^{-t} t^{z-1} \, dt \] (13.5.1)
absolutely converges if \( \text{Re} \, z > 0 \). It satisfies the equation
\[ \hat{\Gamma}(z + 1) = z \hat{\Gamma}(z). \] (13.5.2)

Proof. Let \( z = x + iy, x > 0 \). We have
\[ |t^{z-1}| = |e^{(x-1+iy)\ln t}| = t^{x-1}. \]
The integral \( \int_{0}^{1} \) converges because the function \( t^{x-1} \) is integrable when \( x > 0 \), and integral \( \int_{1}^{\infty} \) converges because the factor \( e^{-t} \) decays faster than any power of \( t \). Equation (13.5.2) follows from the integration by part formula. \( \blacksquare \)

Hence, the integral \( \int_{0}^{\infty} e^{-t} t^{z-1} \, dt \) defines a holomorphic function on the half-plane \( \{ \text{Re} \, z > 0 \} \) which satisfy the same relation (13.5.2) as \( \Gamma(z) \). Moreover, \( \hat{\Gamma}(1) = \Gamma(1) = 1 \). It turns out that

Theorem 13.14. \[ \hat{\Gamma}(z) = \Gamma(z). \]

In particular, \( \hat{\Gamma}(z) \) admits a meromorphic extension to the whole \( \mathbb{C} \).

Proof. The ratio
\[ h(z) = \frac{\hat{\Gamma}(z)}{\Gamma(z)} \]
is a holomorphic function on the half-plane \( \{ \text{Re} \, z > 0 \} \). Moreover, it is 1-periodic: \( h(z + 1) = h(z) \), and hence can be extended by periodicity to a 1-periodic holomorphic function on the whole plane \( \mathbb{C} \). Let us study the behavior of \( h(x + iy) \) when \( |y| \to \infty \) in the strip \( U := \{ x \in [1, 2] \} \) Note the the map \( g(z) = e^{-2\pi i z} = e^{-2\pi y} e^{2\pi iy} \) defines a biholomorphism of the strip \( U \) onto \( \mathbb{C} \setminus \{ \text{Im} \, z = 0, \text{Re} \, z > 0 \} \). Moreover, the function \( f := h \circ g^{-1} : \mathbb{C} \setminus \{ \text{Im} \, z = 0, \text{Re} \, z > 0 \} \to \mathbb{C} \) extends holomorphically to \( \mathbb{C} \setminus 0 \). Indeed, both functions, \( g(z) \) end \( e^{-2\pi z} \) are 1-periodic.
Let us analyze the behavior of $f$ near 0 and at $\infty$.

In view of the equality $|e^{-t^2}| = |e^{-t^{2-1}}| \leq e^{-t}$ we have $\hat{\Gamma}(z) \leq 1$, and thus $|\hat{\Gamma}(g^{-1}(z))| \leq 1$. Let us now estimate below $|\hat{\Gamma}(g^{-1}(z))|$. We have

$$g^{-1}(re^{i\phi}) = 1 + \frac{\phi}{2\pi} - i\frac{\ln r}{2\pi}.$$ 

Using Lemma 13.12 and plugging $z = 1 + \frac{\phi}{2\pi} - i\frac{\ln r}{2\pi}$ we get

$$|\hat{\Gamma}\left(1 + \frac{\phi}{2\pi} - i\frac{\ln r}{2\pi}\right)| \geq Ce^{-\frac{\ln r}{4}} \geq \begin{cases} Cr^{-\frac{1}{4}}, & r \geq 1; \\ Cr^\frac{3}{4}, & r < 1. \end{cases}$$

Thus

$$|f(z)| = \frac{\hat{\Gamma}(z)}{|\Gamma(z)|} = \begin{cases} O(|z|^\frac{1}{2}), & |z| \to \infty, \\ O(|z|^{-\frac{1}{4}}), & |z| \to 0. \end{cases}$$

According to the removable singularity Theorem 7.1 this implies that $f$ has removable singularities at 0 and $\infty$, and hence $f(z) = \text{const}$. But then $h(z) = \text{const}$, and taking into account that $\hat{\Gamma}(1) = \Gamma(1) = 1$ we get $h(z) = 1$, and hence $\hat{\Gamma}(z) = \Gamma(z)$.

**Exercise 13.15.** Prove the following identities:

1. Legendre’s duplication formula

   $$\Gamma(2z) = \frac{n^{z-\frac{1}{2}}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right).$$

2. Gauss’ formula.

   $$(2\pi)^{n-\frac{1}{2}} \Gamma(z) = n^{z-\frac{1}{2}} \Gamma\left(\frac{z}{n}\right) \Gamma\left(\frac{z + 1}{n}\right) \ldots \Gamma\left(\frac{z + n - 1}{n}\right).$$
Chapter 14

The Riemann $\zeta$-function

The series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

absolutely converges if $\text{Re} \, s > 1$. Moreover it converges uniformly on $\{\text{Re} \, s > 1 + \epsilon\}$ for any $\epsilon > 0$. In particular, its sum is a holomorphic function on $\{\text{Re} \, s > 1\}$. As we will show below, it extends to $\mathbb{C}$ as a meromorphic function with the unique simple pole at the point $s = 1$. This function was introduced by B. Riemann and it is now called the Riemann $\zeta$-function. This is one of the most famous functions due to its huge importance in number theory and Mathematics in general.\(^1\)

14.1 Product development for $\zeta(s)$

Theorem 14.1.

$$\frac{1}{\zeta(s)} = \prod_{p \text{ prime}} (1 - p^{-s}), \quad \sigma = \text{Re} \, s > 1.$$  

We assume that primes in the product are ordered in the increasing order.

Proof. First we note the the product in the right-hand side absolutely converges because the series

\(^1\)It is customary to denote the argument of the $\zeta$-function by the letter $s$ rather than $z$.  

\[ \sum_{p \text{ prime}} p^{-s} \text{ is converging, see Lemma 13.3} \]

Then

\[ \zeta(s)(1 - 2^{-s}) = \sum_{n=1}^{\infty} n^{-s} - \sum_{n=1}^{\infty} (2n)^{-s} = \sum_{m \text{ odd}} m^{-s}. \]

Similarly,

\[ \zeta(s)(1 - 2^{-s})(1 - 3^{-s}) = \sum m^{-s}, \]

where the sum is taken over all integer \( m \) not divisible by 2 and 3. Continuing we get

\[ \zeta(s)(1 - 2^{-s})(1 - 3^{-s}) \cdots (1 - p_n^{-s}) = \sum m^{-s}, \]

where the sum is taken over all integer not divisible by first \( n \) prime numbers: 2, 3, \ldots, \( p_n \). Note that \( \sum m^{-s} = 1 + T_n \), where \( T_n = N_n^{-s} + \ldots \), where \( N_n > p_n \). Hence, as \( \lim_{n \to \infty} T_n = 0 \) and thus

\[ \zeta(s)(1 - 2^{-s})(1 - 3^{-s}) \cdots (1 - p_n^{-s}) \to 1 \]

, i.e.

\[ \zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s}). \]

\[ \blacksquare \]

### 14.2 Meromorphic extension of \( \zeta(s) \)

**Theorem 14.2.** \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \) extends meromorphically from \( \{ \Re s > 1 \} \) to \( \mathbb{C} \) with a unique simple pole at \( s = 1 \) with \( \Res_1 \zeta = 1 \).

**Lemma 14.3.**

\[ \zeta(s) \Gamma(s) = \int_{0}^{\infty} \frac{x^{s-1}}{e^x - 1} \, dx, \quad \Re s > 1. \]

**Proof.** First we note \( x^{s-1} = e^{(s-1) \ln x} \) and that the integral in the right-hand-side converges at 0 and at infinity.

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Recall that

\[ \Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx \]

when \( \text{Re} \, s > 0 \), see Lemma 13.13. Therefore, by the variable change \( x \mapsto nx \) we get

\[ n^{-s} \Gamma(s) = \int_0^\infty x^{s-1} e^{-nx} \, dx, \]

and summing up over \( n = 1, \ldots \) we get

\[ \left( \sum_{n=1}^\infty n^{-s} \right) \Gamma(s) = \int_0^\infty x^{s-1} \left( \sum_{n=1}^\infty e^{-nx} \right) \, dx, \]

or

\[ \zeta(s) \Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx. \]

We note that all the involved series converge absolutely when \( \text{Re} \, s > 1 \), and hence all the series manipulations are justified.

Denote by \( U_\epsilon \) an infinite domain in \( \mathbb{C} \) given by

\[ U = \{|z| < \epsilon\} \cup \{|\text{Re} \, z > 0, |\text{Im} \, z| < \epsilon\}, \quad 0 < \epsilon < 2\pi. \]

**Lemma 14.4.**

\[ \zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\partial U_\epsilon} \frac{(-z)^{s-1}}{e^z - 1} \, dz. \]

Here \( (-z)^{s-1} = e^{(s-1)\log(-z)} \), where \( \log z \) is the principal branch of \( \log \) defined on the complement of the negative real semi-axis. We also note that by Cauchy theorem the integral is independent of \( \epsilon \) as far as \( \epsilon < 2\pi \).

**Proof.** The contour \( \partial U_\epsilon \) consists of an arc \( A \) of a circle \( \{|z| = \epsilon\} \) and two rays \( B_- \) and \( B_+ \). It is straightforward to see that \( \int_A \frac{(-z)^{s-1}}{e^z - 1} \, dz \) tends to 0 when \( \epsilon \to 0 \), so that the integral over \( \partial U \) converges
\[
\int_{\partial U_\epsilon} (-z)^{s-1} e^z - 1 \, dz = - \int_0^\infty x^{s-1} e^{(s-1)\pi} e^x - 1 \, dx + \int_0^\infty x^{s-1} e^{(s-1)\pi} e^x - 1 \, dx
\]

\[
= 2i \sin(s - 1)\pi \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx.
\]

But in view of Lemma [4.3] we have

\[
\int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx = \zeta(s)\Gamma(s),
\]

and hence

\[
\int_{\partial U_\epsilon} (-z)^{s-1} e^z - 1 \, dz = 2i \sin(s - 1)\pi \zeta(s)\Gamma(s) = -2i \sin \pi s \zeta(s)\Gamma(s).
\]

According to Theorem [13.11.2] we have

\[
(sin \pi s)\Gamma(s) = \frac{\pi}{\Gamma(1 - s)}.
\]

Hence,

\[
-2i (sin \pi s)\zeta(s)\Gamma(s) = - \frac{2\pi i \zeta(s)}{\Gamma(1 - s)}.
\]

Therefore,

\[
\zeta(s) = - \frac{\Gamma(1 - s)}{2\pi i} \int_{\partial U_\epsilon} (-z)^{s-1} e^z - 1 \, dz.
\]

**Proof of Theorem [14.4]** The contour integral

\[
\int_{\partial U_\epsilon} (-z)^{s-1} e^z - 1 \, dz
\]

in Lemma [4.4] converges for all \( s \in \mathbb{C} \). It converges over the circular part \( A \) of the contour because the circle is compact and the function is continuous, and over each ray \( B_\epsilon \), the exponential term in the denominator \( e^z - 1 \) grows faster than any power of \( z \). On the other hand \( \Gamma(1 - s) \) is meromorphic
with poles at $1, 2, \ldots$. But $\zeta(s)$ is holomorphic in $\{\text{Re } s > 1\}$. Hence, these poles are cancelled by the zeroes of the integral. On the other hand, the pole at 1 has residue 1, while $\int_{\partial U_\epsilon} \frac{dz}{\epsilon^z - 1} = 1$ in view of the residue theorem. Hence, $\zeta(s)$ has a single pole at 1 with residue 1.

**14.3 Zeroes of the $\zeta$-function**

Recall that Bernoulli numbers $B_n$ can be defined as coefficients in the Laurent expansion of the function $\frac{1}{e^z - 1}$:

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_n}{2n!} z^{2n-1}. \quad (14.3.1)$$

**Proposition 14.5.**

$$\zeta(-n) = \begin{cases} 
-\frac{1}{2}, & n = 0, \\
0, & n = 2k, \\
(-1)^k \frac{B_k}{2k}, & n = 2k - 1.
\end{cases}$$

**Proof.** According to Lemma [14.4] we have

$$\zeta(-n) = \frac{\Gamma(1 + n)}{2\pi i} \int_{\partial U_\epsilon} \frac{(-z)^{-(n+1)}}{e^z - 1} dz.$$ 

Hence, by the residue theorem the values $\zeta(-n)$ are the coefficients of the expansion given by formula (14.3.1) multiplied by $(-1)^n n!$.

The values $-2k, k > 0$, are called trivial zeroes of $\zeta(s)$. It is clear from Theorem [14.1] that $\zeta(s)$ has no zeroes in the half-plane $\{\text{Re } s > 1\}$. One can also show that there are no non-trivial zeroes in the half-plane $\{\text{Re } s < 0\}$ and on the lines $\{\text{Re } s = 0, 1\}$. Riemann conjectured that all non-trivial zeroes lie on the line $\{\text{Re } s = \frac{1}{2}\}$. This conjecture, known as the Riemann hypothesis, is still open and became one of the most famous mathematical problems. It is known that many fundamental results in number theory could be deduced from the Riemann hypothesis.