Math 116 Homework 5 Solutions

Saturday, November 2, 2019

Problem 1. First, if \( z \in D_+ \), then \( z = x + iy \) for \( y > 0 \) and \( x^2 + y^2 < 1 \). So

\[
\text{Im}\left( -\frac{1}{2} \left( z + \frac{1}{z} \right) \right) = -\frac{1}{2} \text{Im}\left( \frac{z(x^2 + y^2) + \overline{z}}{x^2 + y^2} \right) > 0
\]

since \( \text{Im}(z(x^2 + y^2) + \overline{z}) < 0 \). Now suppose that \( f(w) = f(z) \) for some \( z, w \in D_+ \), so that \( z + 1/z = w + 1/w \) and hence \( z^2 w + w = w^2 z + z \). Rearranging yields \( wz(z - w) = z - w \), which implies \( z = w \) since \( |wz| < 1 \). Hence \( f \) is injective.

Next let \( w \in \mathbb{H} \). If \( w = -\frac{1}{2} (z + \frac{1}{z}) \), then \( z^2 + 2wz + 1 = 0 \), and the quadratic formula shows that

\[
z = -\frac{2w \pm \sqrt{4w^2 - 4}}{2} = -w \pm \sqrt{w^2 - 1},
\]

where \( \sqrt{w^2 - 1} \) is any fixed choice of square root for \( w^2 - 1 \) (of which there are exactly two). If \( \alpha = -w + \sqrt{w^2 - 1} \) and \( \beta = -w - \sqrt{w^2 - 1} \), notice that

\[
\alpha \beta = w^2 - (w^2 - 1) = 1,
\]

so that \( \alpha = c\beta \) for some \( c \in \mathbb{R}_{>0} \). By interchanging \( \alpha \) and \( \beta \), we may assume \( c \geq 1 \). Note that in fact \( c > 1 \) since \( \alpha + \beta = w \not\in \mathbb{R} \), so that \( |\alpha| < 1 \). Also, since \( \text{Im}(\alpha + \beta) = \text{Im}(-2w) < 0 \), it follows that \( \alpha \in D_+ \). Hence \( f \) is surjective, and since it is a bijective holomorphic map, it is conformal.

Problem 2. Let \( g : \mathbb{D} \to \mathbb{D} \) be the biholomorphic map given by

\[
g(z) = \frac{z - z_1}{1 + \overline{z_1}z},
\]

so that \( g(z_1) = 0 \). Then let \( h = g \circ f \circ g^{-1} \), so that \( h \) is an automorphism of \( \mathbb{D} \) and \( h(0) = g(f(g^{-1}(0))) = g(f(z_1)) = g(z_1) = 0 \). Moreover, if \( w = g(z_2) \),
then $g^{-1}(w) = z_2$ and it follows easily that $h(w) = w$. Of course, since $h$ is biholomorphic and $z_1 \neq z_2$, it follows that $w \neq 0$. But then the Schwarz lemma implies that $h(z) = z$ for all $z \in \mathbb{D}$. But then $f = g^{-1} \circ h \circ g = g^{-1} \circ g$, which is the identity map on $\mathbb{D}$.

**Problem 3.** Suppose that $f(\beta) = 0$, and let $g : \mathbb{H} \to \mathbb{D}$ be given by

$$g(z) = \frac{z - \beta}{z - \overline{\beta}}.$$ 

It is not hard to see that this map is well-defined: $|z - \beta| < |z - \overline{\beta}|$ since $\beta, z \in \mathbb{H}$. Also, $g(\beta) = 0$. Consider now the automorphism $h = f \circ g^{-1}$ of $\mathbb{D}$. Then $h(0) = f(g^{-1}(0)) = f(\beta) = 0$. By the known classification of automorphisms of the disk, there is some $\theta \in \mathbb{R}$ such that $h(z) = e^{i\theta}z$ for all $z \in \mathbb{D}$. Then indeed

$$f(z) = f(g^{-1}(g(z))) = h(g(z)) = e^{i\theta} \frac{z - \beta}{z - \overline{\beta}},$$

as desired.
Note that all maps of the form \( f_{\alpha}(z) = \frac{z - \alpha}{1 - \alpha \overline{z}} \), \(|\alpha| < 1\). map circles to circles (Indeed all mobius transformations preserve the cross-ratio, & since \( \text{Re} m(f_{\alpha}) \subset \mathbb{D} \), circles cannot possibly get mapped to lines).

Suppose we find \( \alpha, r \) such that the circle \( |z - \frac{1}{2}| = \frac{1}{4} \) is mapped to the circle \( |z| = r \). Then \( f_{\alpha} \) would be a conformal equivalence between \( \mathbb{D} \triangle \left\{ |z - \frac{1}{2}| \leq \frac{1}{4} \right\} \) \( \cong \mathbb{D} \triangle \{ |z| < r \} \) since \( f_{\alpha} \) is injective must map the interior of the disk \( |z - \frac{1}{2}| = \frac{1}{4} \) to the disk \( |z| < r \) by the maximum modulus principle.

Furthermore, there is a unique \( r \) which satisfies the required property since annuli conformally equivalent precisely when the ratio of the radii is the same.

Suppose we find \( \alpha \) such that \( f_{\alpha}(\frac{1}{4}) = -f_{\alpha}(\frac{3}{4}) \) & \( \alpha \in \mathbb{R} \). Then, since \( f_{\alpha}(0) = \frac{1}{2} \) is the center of \( f_{\alpha}(\{ |z - \frac{1}{2}| \leq \frac{1}{4} \}) \) must be on the real line since \( f_{\alpha} \) maps real numbers to real numbers, and must be the origin since \( f_{\alpha}(\frac{1}{4}) \) & \( f_{\alpha}(\frac{3}{4}) \) are equidistant from the origin. A computation shows that \( \alpha = \frac{1}{16} (19 - \sqrt{105}) \) works, and that \( r = |f_{\alpha}(\frac{1}{4})| = \frac{1}{8} (13 - \sqrt{105}) \).