1. Fix any $0 < R < 1$. Note that for any $|z| < R$, $|\zeta| = 1$.

\[
\frac{1}{\zeta - z} = \frac{1}{\zeta (1 - \frac{z}{\zeta})} = \frac{1}{\zeta} \sum_{n=0}^{\infty} \left(\frac{z}{\zeta}\right)^n
\]

so the convergence of the series is uniform in $\zeta$. Hence, $u(\zeta)(1 + \frac{z}{\zeta}) \sum_{n=0}^{\infty} \left(\frac{z}{\zeta}\right)^n$ converges uniformly to $u(\zeta)(1 + \frac{z}{\zeta})$. Hence

\[
\frac{1}{2\pi i} \int_\gamma \frac{u(\zeta)(1 + \frac{z}{\zeta})}{\zeta - z} \frac{\zeta}{\zeta^2} \frac{1}{2\pi i} \int_\gamma \frac{u(\zeta)}{\zeta} \frac{1}{2\pi i} \int_\gamma \frac{u(\zeta)}{\zeta^2}
\]

Since this is true for any $z$ with $|z| < R$, this must be the power expansion for $u$ so the desired result follows.

2. $e^{i\phi} + r e^{i\phi} = 1 + \frac{r e^{i\phi}}{(\phi - \theta)}$. Let us write $\phi - \theta = \alpha$.

Then

\[
e^{i\theta} - r e^{i\phi} = 1 - r e^{i(\phi - \theta)} = \frac{(1 + r \cos \alpha + ir \sin \alpha)(1 - r \cos \alpha - ir \sin \alpha)}{(1 - r \cos \alpha)^2 + (r \sin \alpha)^2} = \frac{1 - r^2 + 2ir \sin \alpha}{1 + r^2 - 2r \cos \alpha}, \text{ as required}
\]
Problem 3. We know that \( u_x^2 + u_y^2 = |J_f(x, y)| = |f'(z)|^2 \), so this question is similar to question 5 on homework 4.

Fix \( R < 1 \), and let \( D_R \) be the disk centered at 0 with radius \( R \). \( f(z) \) converges absolutely and uniformly on \( D_R \) and it is therefore sensible to compute

\[
|f'(z)|^2 = \left( \sum_{m=1}^{\infty} mc_m z^{m-1} \right) \left( \sum_{n=1}^{\infty} n \bar{c}_n z^{n-1} \right)
\]

\[
= \left( \sum_{n=1}^{\infty} n^2 |c_n|^2 |z|^{2(n-1)} \right) + \left( \sum_{m \neq n} \sum_{m=1}^{\infty} mnc_m \bar{c}_n z^{m-1} z^{n-1} \right)
\]

for all \( z \in D_R \). Moreover, because of uniform convergence we may interchange integration and summation to find

\[
\int_{D_R} |f'(z)|^2 dxdy = \left( \sum_{n=1}^{\infty} \int_{D_R} n^2 |c_n|^2 |z|^{2(n-1)} dxdy \right) + \left( \sum_{m \neq n} \int_{D_R} mnc_m \bar{c}_n z^{m-1} z^{n-1} dxdy \right).
\]

We can rewrite these integrals in polar coordinates. First of all, this gives

\[
\int_{D_R} mnc_m \bar{c}_n z^{m-1} z^{n-1} dxdy = \int_0^R \int_0^{2\pi} mnc_m \bar{c}_n r^{m+n-1} e^{i(m-n)\theta} r dr d\theta.
\]

This latter integral is 0 whenever \( m \neq n \) because \( \int_0^{2\pi} e^{ik\theta} d\theta = 0 \) whenever \( k \neq 0 \). Thus we obtain (again rewriting in polar coordinates)

\[
\int_{D_R} |f'(z)|^2 dxdy = \sum_{n=1}^{\infty} \int_0^R \int_0^{2\pi} n^2 |c_n|^2 r^{2n-1} dr d\theta.
\]

Each of the integrals in this sum may be computed simply, as

\[
\int_0^{2\pi} \int_0^1 n^2 |c_n|^2 r^{2n-1} dr d\theta = \frac{1}{2} \int_0^{2\pi} n |c_n|^2 R^{2n} d\theta = \pi n |c_n|^2 R^{2n}.
\]

Since \( \sum \pi n |c_n|^2 < \infty \), by Abel’s theorem, we obtain

\[
\lim_{R \to 1} \int_{D_R} |f'(z)|^2 dxdy = \sum \pi n |c_n|^2
\]

Finally since \( \int_{D} |f'(z)|^2 dxdy < \infty \), it follows by the monotone convergence theorem that
\[ \lim_{R \to 1} \int_{\mathbb{D}_R} |f'(z)|^2 \, dx \, dy = \int_{\mathbb{D}} |f'(z)|^2 \, dx \, dy \]

**Problem 4.** Define \( \psi(\zeta) = u(-i \frac{\zeta+1}{\zeta-i}) \) for all \( \zeta \neq 1 \) in \( \partial \mathbb{D} \). Since \( u \) is bounded, \( \psi \) is a piecewise continuous integrable function on \( \mathbb{D} \) (after extending it arbitrarily to 1). By the Poisson-Schwarz formula, the function

\[ P\psi(z) = \frac{i}{2\pi} \int_{\partial \mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2} \frac{d\zeta}{\zeta} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \psi(e^{i\theta}) \, d\theta \]

is harmonic on \( \mathbb{D} \), and extends continuously to \( \psi \) on \( \partial \mathbb{D} \setminus \{1\} \).

Let now \( U(z) = P\psi(\frac{z-i}{z+i}) \). Integrate to find

\[ U(z) = \frac{i}{2\pi} \int_{\partial \mathbb{D}} \frac{1 - \frac{|z-i|^2}{|z+i|^2}}{|\zeta - \frac{z-i}{z+i}|^2} \psi(\zeta) \frac{d\zeta}{\zeta} \]

\[ = \frac{i}{2\pi} \int_{\partial \mathbb{D}} \frac{|z + i|^2 - |z - i|^2}{|\zeta(z + i) - (z - i)|^2} \psi(\zeta) \frac{d\zeta}{\zeta} \]

\[ = \frac{i}{2\pi} \int_{\partial \mathbb{D}} \frac{4y}{|\zeta(z + i) - (z - i)|^2} u \left( -i \frac{\zeta + 1}{\zeta - 1} \right) \frac{d\zeta}{\zeta} \]

\[ = \frac{i}{2\pi} \int_0^{2\pi} \frac{4y}{|e^{i\theta} - (z - i)|^2} \left( -i \frac{e^{i\theta} + 1}{e^{i\theta} - 1} \right) \, d\theta. \]

Make the substitution \( \theta = -i \log(\frac{z+i}{z+i}) \), so that \( e^{i\theta} = \frac{z+i}{z+i} \) and \( d\theta = \frac{2}{t+i} \, dt \).

Note that \( t = -i \frac{e^{i\theta}+1}{e^{i\theta}-1} \), so that as \( e^{i\theta} \) ranges around the circle, \( t \) ranges over \( \mathbb{R} \). So we obtain

\[ U(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{4y|t+i|^2}{|(t-i)(z+i) - (t+i)(z-i)|^2} \frac{u(t)}{t^2 + 1} \, dt \]

\[ = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{4y}{|2i(t-z)|^2} u(t) \, dt \]

\[ = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} u(t) \, dt, \]

as desired. (Note that we used the fact that \( |t+i|^2 = t^2 + 1 \) for all \( t \in \mathbb{R} \).)
(Let \( U = \{ z \in \mathbb{H} : |z| \geq 1, -\frac{1}{2} \leq \text{Re} z \leq \frac{1}{2} \} \))

(5) Suppose by way of contradiction that some orbit intersects \( U \) twice i.e. we find \( g_1, g_2, z \) such that

\[ g_1 \cdot z = g_2 \cdot z = \xi, \quad g_1 \cdot z, g_2 \cdot z \in U. \]

Then, \( g_2^{-1} g_1^{-1} (z) \in U \)

\[ \Rightarrow \exists z_1, z_2 \in U \text{ with } z_1 \neq z_2 \text{ and } g_1 \in G \text{ with } g_1 \cdot z_1 = z_2. \]

Without loss of generality let us assume \( \text{Im}(z_1) \leq \text{Im}(\xi) \)

and write \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, g \neq I \)

Note that \( \text{Im}(g \cdot z_1) = \text{Im}(\frac{a z_1 + b}{c z_1 + d}) = \frac{\text{Im}(z_1)}{1 |c z_1 + d|^2} \)

Hence \( |c z_1 + d|^2 \leq 1 \). Let us write \( z_1 = x + i y. \) We know \( |x| \leq \frac{1}{2} \).

Since \( |c z_1 + d|^2 \leq 1 \),

\[ (c x + d)^2 + (c y)^2 \leq 1 \]

\[ \Rightarrow c^2 (x^2 + y^2) + d^2 + 2 d c x \leq 1 \]

\[ \Rightarrow c^2 + d^2 - 1 |d c| \leq 1 \]

(3) Since \( x^2 + y^2 \geq 1 \) & \( 2 d c x \geq -1 |d c| \)

\[ \Rightarrow |d c| \leq 1. \]

(3) Since \( c^2 + d^2 \geq 2 |d c| \).

Note that \( |d c| = 1 \Rightarrow 2 d c x > -1 |d c| \Rightarrow c^2 + d^2 - 1 |d c| < 1 \)

\[ \Rightarrow |d c| < 1, \text{ which is a contradiction. Hence} \]

\( |d c| = 0. \) Hence, \( c = d = \pm 1 \) or \( d = 0 \) \& \( b = -c = \pm 1 \).

If \( c = 0 \), then in fact \( g \cdot z_1 = z_1 \pm i b. \) Since \( b \in \mathbb{Z} \),

\( b = 0, \text{ contradiction.} \)

If \( d = 0 \), the in fact \( g \cdot z_1 = \frac{a z_1 - 1}{z_1} \)

\[ \Rightarrow g \cdot z_1 = \frac{a - 1}{z_1}. \text{ Now } \text{Re}(g \cdot z_1) = a - \text{Re}(\frac{1}{x + i y}) = a - \frac{x}{x^2 + y^2} \]

\[ |x| < \frac{1}{2}, |x^2 + y^2| > 1 \Rightarrow |\frac{x}{x^2 + y^2}| < \frac{1}{2} \Rightarrow a = 0 \Rightarrow |z_1| < 1, \text{ contradiction.} \]