Math 116 Homework 8 Solutions

Saturday, November 23, 2019

**Problem 1.** As a map $\mathbb{CP}^1 \setminus \{\infty\} \to \mathbb{CP}^1$, $P$ is given by

$$P([z : 1]) = [z^4 + 2z^2 + 1 : 1].$$

Hence the extension to all of $\mathbb{CP}^1$ is given by

$$P([z : w]) = [z^4 + 2z^2 w^2 + w^4 : w^4].$$

Note also that since $P$ is a polynomial of degree 4 a general point in $\mathbb{C}$ has 4 pre-images, so the map has order 4. Now we determine the branching points. To do this, we need to determine where the differential of $P$ vanishes. For $z \in \mathbb{CP}^1 \setminus \{\infty\}$, we have $P'(z) = 0$ if and only if $4z^3 + 4z = 0$, i.e., if $z = 0$ or $z = \pm i$. Notice that $P(0) = 1, P(i) = P(-i) = 0$.

Thus 1 is a branching point of type $(1,1,2)$ if $P^{-1}(1) = \{\sqrt{-2}, -\sqrt{-2}, 0\}$, the differential only vanishes at 0, and the order of the map is 4, so this is the only possible type]. Similarly 0 is a branching point of type $(2,2)$ respectively if $P^{-1}(0) = \{\sqrt{-1}, -\sqrt{-1}\}$, the differential vanishes at both $i$ and $-i$, and the order of the map is 4, so this is the only possible type]. Finally $\infty$ is a branching point of type $(4)$ since $\infty$ has a unique pre-image (namely $\infty$).

**Problem 2.** (a) Theorem 11.23 in the notes shows that $f : S \to \mathbb{CP}^1$ is a branched cover, and in particular the proof shows that if $A$ and $B$ are as in the hint, then

$$f|_{S \setminus \tilde{A}} : S \setminus \tilde{A} \to \mathbb{CP}^1 \setminus B$$

is a covering map. Notice that $\mathbb{CP}^1 \setminus B$ is connected: if $B$ is empty, then the statement is that $\mathbb{CP}^1$ is connected, which we know to be true. If $B$ is
nonempty, then we may assume without loss of generality that $B$ contains $\infty$, in which case we are required to show that $\mathbb{C} \setminus \tilde{B}$ is connected for any finite set $\tilde{B}$. Let $z_0 \notin \tilde{B}$, and let $L_0$ be a line through $z_0$ not containing any point of $\tilde{B}$. (There are infinitely many lines through $z_0$, and they are pairwise disjoint, so such an $L_0$ exists because $\tilde{B}$ is finite). If $z$ is any other point of $\mathbb{C} \setminus \tilde{B}$, then every line through $z$ but one (namely, the parallel one) intersects $L$, and so by the same reasoning there is some line $L$ passing through $z$, intersecting $L_0$, and lying entirely within $\mathbb{C} \setminus \tilde{B}$. But then in fact $z_0$ and $z$ are connected by a piecewise linear path, so that the claim is proved.

Since $\mathbb{C}P^1 \setminus B$ is connected, all points have the same number $n$ of pre-images under $f|_{S \setminus \tilde{A}}$. A sketch of the proof is as follows: for each $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, let $W_k$ be the set of points of $V$ with exactly $k$ preimages under $p : U \to V$. The covering map property shows that each $W_k$ is open, and since $V$ is the disjoint union of the $W_k$, each is also closed. Since $V$ is connected, it follows that each $W_k$ is either empty or equal to $V$ itself, so that there is some $n$ such that every point of $V$ has exactly $n$ preimages.

Now suppose that $z$ is any point of $\mathbb{C}P^1$. Lemma 11.21 in the notes shows that $z$ has $n$ pre-images, when counted with multiplicity. The desired result follows.

(b) If $f$ has a unique simple pole, then (a) shows that every point of $\mathbb{C}P^1$ has exactly one preimage under $f$, so that $f$ is bijective. Since $f$ is holomorphic, this shows that $f$ is biholomorphic.

**Problem 3.** a) Assume first that 0 is neither a pole nor a zero of $f(z)$. Note that $\wp(z) - a = \infty$ has exactly two solutions (in any fundamental parallelogram), since $\wp(z)$ has a double pole at 0, so the previous question tells us that $\wp(z) - c$ has exactly two zeros (counted with multiplicity) for any $c \in \mathbb{C}$.

$\wp'(z)$ is a triple pole at 0, so it has 3 zeroes. The zeroes are $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, and $\frac{\omega_1 + \omega_2}{2}$, since, $\wp'(z)$ is odd means that, $-\wp'(\frac{\omega_1}{2}) = \wp'(-\frac{\omega_1}{2}) = \wp'(-\frac{\omega_1}{2} + \omega_1) = \wp'(\frac{\omega_1}{2})$. Thus, for $b \neq 0$, $\wp(z) - \wp(b)$ either has simple zeroes at $b$ and $-b$ or a double zero at $b$ if $b$ is one of $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, or $\frac{\omega_1 + \omega_2}{2}$.

Moreover, if $f$ is any even elliptic function, the same argument shows that the order of the pole or zero at $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, or $\frac{\omega_1 + \omega_2}{2}$ must be even because all ordered derivatives are odd functions, so they vanish $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, and $\frac{\omega_1 + \omega_2}{2}$.

Thus we can take $a_1, ..., a_r$ and $b_1, ..., b_r$ such that each $a_i$ is a zero of $f$, each $b_i$ is a pole of $f$, and if $z$ is a zero or pole of $f$ then exactly one of $z$ and $-z$ is represented in either list. (We list each several times if it occurs with
multiplicity). Then the function

\[ f(z) \prod_{i=1}^{r} \frac{\phi(z) - \phi(b_i)}{\phi(z) - \phi(a_i)} \]

is an elliptic function which has no poles (since the factors in the numerator helped cancel the poles of \( f \), the factors in the denominator helped cancel the poles the factors of the numerator had added, and no poles are added by the denominator since the denominator is only 0 when \( f \) is zero) and hence is constant, proving the claim.

\( b \) If 0 is a pole or zero of \( f(z) \), then there is some \( k \in \mathbb{Z} \) such that \( f(z)\phi(z)^k \) has no pole or zero at 0 (since \( f \) is even, all terms in the Laurent series of \( f \) are even, so the order of the pole or zero must also be even) and then the previous reasoning applies.

**Problem 4.** Let \( P \) be the parallelogram spanned by \( \omega_1, \omega_2 \), translated so that there are no zeros or poles on \( \partial P \), and oriented in the usual way. By the generalized argument principle (whose proof is identical to the usual argument principle), we have

\[
\frac{1}{2\pi i} \int_{\partial P} \frac{zf'(z)}{f(z)} dz = \left( \sum_{j=1}^{r} a_j \right) - \left( \sum_{j=1}^{r} b_j \right) = \sum_{j=1}^{r} (a_j - b_j).
\]

On the other hand, the integral of this function along opposite sides of \( \partial P \) will cancel except for a constant times an \( f'(z)/f(z) \) term, by periodicity of \( f'(z)/f(z) \). Thus, if \( \gamma_1 \) is the side from \( \omega_1 \) to \( \omega_1 + \omega_2 \) and \( \gamma_2 \) is the side from \( \omega_1 + \omega_2 \) to \( \omega_2 \) (or appropriately translated as above), then we have

\[
\int_{\partial P} \frac{zf'(z)}{f(z)} dz = \omega_1 \int_{\gamma_1} \frac{f'(z)}{f(z)} dz + \omega_2 \int_{\gamma_2} \frac{f'(z)}{f(z)} dz,
\]

so to complete the problem we need only show that the two integrals in this latter sum evaluate to integers. Indeed, we will only consider the first integral, the other being completely similar.

We know that the path \( \gamma_1 \) is given by \( \gamma_1(t) = \omega_1 + \omega_2 t \). Our goal is to transfer everything in sight to the unit circle, where we can apply the argument principle. First, compute

\[
\int_{\gamma_1} \frac{f'(z)}{f(z)} dz = \int_{0}^{1} \frac{f'(\gamma_1(t))}{f(\gamma_1(t))} \gamma_1'(t) dt.
\]
Now let \( h(z) = \exp(2\pi i(z - \omega_1)/\omega_2) \), and let \( \tilde{\gamma}_1 = h \circ \gamma_1 \), so that \( \tilde{\gamma}_1(t) = \exp(2\pi i t) \). Since \( f \) is \( \omega_2 \)-periodic, we can define \( \tilde{f} \) by \( \tilde{f} \circ h = f \). Explicitly,

\[
\tilde{f}(z) = f\left( \frac{\omega_2 \log(z)}{2\pi i} + \omega_1 \right).
\]

Compute

\[
\tilde{f}'(z) = f'(h^{-1}(z)) \frac{\omega_2}{2\pi i z}
\]

and

\[
\tilde{\gamma}_1'(t) = \frac{2\pi i}{\omega_2} \tilde{\gamma}_1(t)\gamma_1'(t)
\]

Since \( \tilde{f}(\tilde{\gamma}_1(t)) = \gamma_1(t) \), this implies

\[
\int_0^1 \frac{f'(\gamma_1(t))}{f(\gamma_1(t))} \gamma_1'(t) \, dt = \int_0^1 \frac{\tilde{f}'(\tilde{\gamma}_1(t))}{\tilde{f}(\tilde{\gamma}_1(t))} \tilde{\gamma}_1'(t) \, dt = \int \frac{\tilde{f}'(z)}{\tilde{f}(z)} \, dz.
\]

This latter integral is the integral of a holomorphic function around the unit circle, so by the argument principle it is an integer multiple of \( 2\pi i \). By the above, this establishes the required result.