1. First note that \( \cos: \mathbb{C} \to \mathbb{C} \) is surjective. Indeed \( \cos z = e^{iz} + e^{-iz} \). If we write \( e^{iz} = \alpha \), then given a complex number \( \omega \), it suffices to find \( \alpha \) such that \( \alpha + \frac{1}{\alpha} = \omega \). This is equivalent to \( \alpha^2 - 2\alpha \omega + 1 = 0 \). We can always find such \( \alpha \), and it will necessarily not be 0, so we can solve \( e^{iz} = \alpha \) to find \( z \) such that \( \cos z = \omega \).

Consider the map \( f: \mathbb{C} \to S \) given by \( f(z) = (\cos 2\pi z, \sin 2\pi z) \). Note that:

(i) \( f \) is surjective. Indeed if \( (z_1, z_2) \in S \), we can find \( z \) such that \( \cos 2\pi z = z_1 \). Then, since \( z_2^2 = 1 - z_1^2 \), \( z_2^2 = 1 - (\cos 2\pi z)^2 \) implies \( z_2^2 = \sin^2 (2\pi z) \) so \( (z_1, z_2) = (\cos 2\pi z, \sin 2\pi z) \) or \( (z_1, z_2) = (\cos (2\pi (z_1)), \sin (2\pi (-z))) \).

(ii) \( f(z + n) = f(z) \) \( \forall n \in \mathbb{Z} \) & \( f(x) = f(y) \) \( \Rightarrow x - y \in \mathbb{Z} \).

The first property follows directly from the fact that \( \cos \) & \( \sin \) are \( 2\pi \) periodic. If \( f(x) = f(y) \), then \( \cos 2\pi x = \cos 2\pi y \) & \( \sin 2\pi x = \sin 2\pi y \) \( \Rightarrow e^{2\pi i(x - y)} = 1 \) \( \Rightarrow x - y \in \mathbb{Z} \).
(iii) $f$ is holomorphic. Indeed, $f$ takes values from $\mathbb{C}$ to $\mathbb{C}$ composed with the coordinates on $S$ defined by Lemma 11.25 is just either $\cos^{-1}$ or $\sin^{-1}$ locally, both of which are holomorphic.

(i), (iii), & (iii) prove that there is a bijective holomorphic $f: \mathbb{C}/\mathbb{Z} \to S$. This must be a conformal equivalence since the inverse is automatically a holomorphic (the proof is the same as the proof in the case of open sets in $\mathbb{C}$).
(2) \( z(z) = \frac{1}{z} + \sum_{\lambda \in \mathbb{N} \setminus \{0\}} \left( \frac{1}{z-\lambda} + \frac{1}{\lambda + \frac{z}{\lambda^2}} \right) \)

Note that \( \left| \frac{1}{z-\lambda} + \frac{1}{\lambda + \frac{z}{\lambda^2}} \right| = \left| \frac{z}{(z-\lambda)\lambda + z^2} \right| \)

\( = |z| \left| \frac{\lambda^2 + (\lambda - z)\lambda}{(z-\lambda)\lambda^2} \right| \)

\( = \frac{|z|^2}{|z-\lambda| |\lambda|^2} \)

Thus, if we fix a radius \( R \), \( \sum_{\lambda \in \mathbb{N} \setminus \{0\}} \) then the series converges uniformly on \( U_{z,\epsilon} = \{ \lambda \in \mathbb{R} : d(z, \lambda) > \epsilon \} \), so \( z(z) \) is a holomorphic function on any such \( U_{z,\epsilon} \). Taking \( R \to \infty \) and \( \epsilon \to 0 \) gives that \( z(z) \) is meromorphic (at points in the lattice, there are simple poles because once we remove the term associated with a point \( \lambda \) from the series, the rest of the series defines a holomorphic function near \( \lambda \)).

(i) We know \( g(n) = \sum_{\lambda \in \mathbb{N} \setminus \{0\}} \left( \frac{1}{(z-\lambda)^n} - \frac{1}{\lambda^n} \right) \)

Since the series converges uniformly on compact subsets of \( \mathbb{C} \setminus \mathbb{N} \), we can differentiate the series term-wise. This gives the desired result.
(vi) \( \zeta(-z) = \frac{-1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z-n} - \frac{1}{\alpha^2} \right) \)

\[ \alpha = -\alpha \]

\[ = -\frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z-n} - \frac{1}{\alpha^2} \right) \]

\[ = -\frac{1}{z} + \left( \sum_{n=1}^{\infty} \frac{1}{z-n} + \frac{1}{\alpha^2} \right) \]

\[ = -\zeta(z) \]

(iii) Let \( F(z) = \zeta(z + \omega_1) - \zeta(z) \). Then \( F'(z) = -g(z + \omega_1) + g(z) = 0 \). Hence \( F(z) = \eta_1 \) for some constant \( \eta_1 \).

Similarly \( \zeta(z + \omega_2) - \zeta(z) = \eta_2 \).

b) Clearly \( \oint \zeta(z) \, dz = 2\pi i \) since \( \oint \frac{1}{z} \, dz = 2\pi i \).

\( \oint \zeta(z) - \frac{1}{z} \, dz = 0 \) since \( \zeta(z) - \frac{1}{z} \) is a holomorphic inside \( P \). On the other hand,

\( \oint_{\partial P} \zeta(z) \, dz = \oint_{w_2-w_1} \zeta(z) \, dz - \oint_{w_1-w_2} \zeta(z) \, dz \)

\[ + \oint_{w_2+w_1} \zeta(z) \, dz - \oint_{w_1+w_2} \zeta(z) \, dz \]

\[ - \oint_{w_1-w_2} \zeta(z) \, dz - \oint_{w_2-w_1} \zeta(z) \, dz \]

\[ - \oint_{\frac{w_2-w_1}{2}} \zeta(z) \, dz - \oint_{\frac{w_1-w_2}{2}} \zeta(z) \, dz \]

\[ + \oint_{\frac{w_2+w_1}{2}} \zeta(z+w_1) \, dz - \oint_{\frac{w_1+w_2}{2}} \zeta(z) \, dz \]

\[ + \oint_{\frac{w_1-w_2}{2}} \zeta(z+w_2) - \oint_{\frac{w_2+w_1}{2}} \zeta(z) \, dz \]
which gives \( \eta_1 w_2 - \eta_2 w_1 = 2\pi i \).

[The question as stated is incorrect, and in fact \( \eta_1 w_2 - \eta_2 w_1 \neq 2\pi i \), we can see this by interchanging our labels for \( w_1 \) & \( w_2 \).]

3. a) Note that
\[
(1 + h^{2n-1} e^{\frac{\varphi}{z}})(1 + h^{2n-1} e^{-\frac{\varphi}{z}}) = 1 + \left[ h^{2n-1} (e^{\frac{\varphi}{z}} + e^{-\frac{\varphi}{z}}) + h^{4n-2} \right].
\]

For \(|h| < 1\),
\[
\sum |h^{2n-1} (e^{\varphi/z} + e^{-\varphi/z})| + h^{4n-2} \leq \sum |h^{2n-1} (e^{\varphi/z} + e^{-\varphi/z})| + |h|h^{4n-2} \leq |e^{\varphi/z} + e^{-\varphi/z}| 2 \sum_{k=2} |h|h^{4k-2}.
\]

Hence, by lemma 13.3, for any fixed \( \varphi \), the product converges absolutely. Moreover, if \( |\varphi| \leq R \), then \( |e^{\varphi/z} + e^{-\varphi/z}| \leq C_R \) for some constant \( C_R \), and examining the proof of theorem 13.2 & lemma 13.3 gives that the product converges uniformly in \( \varphi \) when \( |\varphi| \leq R \).

So the product defines a holomorphic function on all of \( \mathbb{C} \).

b) \( \Theta (z + 2 \log h) = \prod_{1}^{\infty} \left( 1 + h^{2n+1} e^{\frac{\varphi}{z}} \right) \left( 1 + h^{2n+1} e^{-\frac{\varphi}{z}} \right) \)

Note that
\[
\frac{\prod_{1}^{\infty} \left( 1 + h^{2n+1} e^{\frac{\varphi}{z}} \right) \left( 1 + h^{2n+1} e^{-\frac{\varphi}{z}} \right)}{(1 + h e^{\varphi/z})} = (1 + h^{-1} e^{-\varphi/z}) \prod_{1}^{\infty} \left( 1 + h^{2n+1} e^{\frac{\varphi}{z}} \right) \left( 1 + h^{2n+1} e^{-\varphi/z} \right).
\]
Taking the limit as $n \to \infty$ on both sides of the equality yields

$$
\theta(z + 2\log h) = \frac{1 + h^{-1}e^{-z}}{1 + h e^{-z}} \theta(z).
$$

Let

$$
\left(1 + \frac{x}{h} = x \right).
$$

4) Define $g$ as the limit of the functions

$$
g_n(z) = \prod_{i=1}^{n} \left(1 - \frac{z}{a_i} \right). \quad \text{For any}
$$

fixed radius $R$, $g_n(z)$ converge uniformly to $g$ (by the proof of Theorem 13.2, Lemma 13.3), so $g$ is holomorphic

For any $a_i$, $g_n(a_i) = 0$ for sufficiently large $a_i$, so $g(a_i) = 0$. If $z \neq a_i$, then none of the terms in the product is zero. So in fact by the product converges to a non-zero number by Theorem 13.2.

The case is simple because once we remove the term $(1 - \frac{z}{a_i})$, the remaining argument shows that the remaining product is no longer zero at $a_i$. 
b). According to the hint, we need to choose $Y_n$ so that
\[
\sum_{i=1}^{\infty} g(z) e^{\delta_n (z-a_n)} A_n \text{ converges if it does, it's straightforward to verify } f(a_n) = A_n \text{, only one term in the sum contributed.}
\]

It suffices to find $Y_n$ such that the series converges uniformly on each disk of radius $R$.

In particular, it suffices to find $Y_n$ so that
\[
(*) \quad \left| e^{\delta_n (z-a_n)} A_n g_n(z) \right| \leq \frac{1}{2^n} \text{uniformly for } \|z\| \leq R
\]

$\forall n \geq N_R$, where $N_R$ is a constant depending on $R$, and we choose $Y_n$ in such a way that $\|e^{\delta_n (z-a_n)} g_n(z)\| \leq \frac{1}{2^n}$.

Note that if $|a_n| \geq 2R$, we can achieve $Y_n = e^{\delta_n (z-a_n)} g_n(z)$ the same magnitude argument as $a_n$, then the argument of $e^{\delta_n (z-a_n)}$ is between $\pi \pm \pi/6$.

Now just choose the magnitude of $Y_n$ so large so that (*) holds for $R = \frac{n}{2}$. Then (*) holds for any fixed $R$ whenever $n \geq 2R$, so the desired result follows.
By Theorem 13.2, the convergence of the product \( \prod_{n=1}^{\infty} (1+a_n) \) is equivalent to the convergence of the series \( \sum_{n=1}^{\infty} \log(1+a_n) \). But the convergence of series \( \sum_{n=1}^{\infty} \log(1+a_n) \) is equivalent to the convergence of the series \( \sum_{n=1}^{\infty} a_n! \).

Indeed, since \( \sum_{n=1}^{\infty} |a_n|^2 < \infty \), for sufficiently large \( n \), \( |a_n| < \frac{1}{2} \).

Then, \( \log(1+a_n) - a_n \) can be expressed as:

\[
\log(1+a_n) - a_n = \log(1+a_n) \approx a_n^2 + \frac{a_n^3}{3} + \cdots = a_n^2 \left[ 1 + a_n + a_n^2 + \cdots \right]
\]

Thus, \( \log(1+a_n) - a_n \) can be bounded by:

\[
\frac{a_n^2}{2} \left[ 1 + a_n + a_n^2 + \cdots \right] = \frac{a_n^2}{2(1-a_n)} \leq \frac{a_n^2}{2}
\]

Hence, \( \sum_{n=1}^{\infty} |\log(1+a_n) - a_n| < \infty \), so the desired result follows.
Let \( a_n(\theta) = \frac{1}{2^n} \sum_{k=1}^{n} \cos \frac{\theta}{2^k} \). Note that
\[
= \frac{1}{2^n} \sum_{k=1}^{n} \cos \frac{\theta}{2^k} \cdot \cos \frac{\theta}{2^k} \cdot \sin \left( \frac{\theta}{2^n} \right) = a_{n-1}(\theta) \cdot \sin \left( \frac{\theta}{2^{n-1}} \right)
\]

Thus, by induction, \( a_n(\theta) \cdot \sin \left( \frac{\theta}{2^n} \right) = \frac{\sin \theta}{2^n} \). Hence if
\[
\theta \neq 0, \quad a_n(\theta) \cdot \sin \left( \frac{\theta}{2^n} \right) = \frac{\sin \theta}{2^n}
\]

For sufficiently large \( n \), \( \sin \left( \frac{\theta}{2^n} \right) \neq 0 \), so
\[
a_n(\theta) = \frac{\sin \theta}{\theta} \cdot \frac{\sin \left( \frac{\theta}{2^n} \right)}{\sin \left( \frac{\theta}{2^n} \right)} = \frac{\sin \theta}{\theta} \cdot \frac{\sin \left( \frac{\theta}{2^n} \right)}{\sin \left( \frac{\theta}{2^n} \right)}
\]

Hence, \( \lim_{n \to \infty} a_n(\theta) = \frac{\sin \theta}{\theta} \cdot \frac{\sin \left( \frac{\theta}{2^n} \right)}{\sin \left( \frac{\theta}{2^n} \right)} = \frac{\sin \theta}{\theta} \cdot \frac{\sin \left( \frac{\theta}{2^n} \right)}{\sin \left( \frac{\theta}{2^n} \right)}
\]

The result is also true for \( \theta = 0 \) when we interpret \( \frac{\sin \theta}{\theta} = \frac{1}{w} \) as \( \lim_{w \to 0} \frac{\sin w}{w} \).