CHAPTER 3

Multivalued Holonomic Approximation

This chapter is a continuation of ch.3. Here we prove that any section $F : V \to X^{(r)}$ can be approximated by a holonomic section $\widehat{F} : \widehat{V} \to X^{(r)}$ where $\widehat{V}$ is a non-Hausdorff covering space of V ("multifold").

3.1. Multifolds

A. Multifolded intervals $\widehat{I}$

Let $0 = a_0 < a_1 < \cdots < a_N < 1$ and $0 < b_1 < \cdots < b_N < b_{N+1} = 1$ be two sequences of points on the interval $[0, 1]$, such that $a_i < b_i$ for all $i = 1, \ldots, N$. Take disjoint copies of the intervals

$$[a_0, b_1],\ [a_1, b_1],\ [a_1, b_2],\ [a_2, b_2],\ \ldots\ [a_N, b_{N+1}],$$

and extend slightly these intervals, i.e. consider the open intervals $(a - \delta, b + \delta)$ instead of $[a, b]$. Here δ is a small positive number and we do not extend the ends $a_0 = 0$ and $b_{N+1} = 1$. Then identify the added twin (open) intervals as it depicted on the Fig. 3.1 and denote the produced non-Hausdorff multifolded interval by $\widehat{I}$. For simplicity we do not include the sequences a_i, b_i and the number δ in the notation $\widehat{I}$. Note that the non-Hausdorff interval $\widehat{I}$ arises together with the natural projection $\pi : \widehat{I} \to I$. We will always consider $\widehat{I}$ as a "non-Hausdorff ramification" of the interval I. In other words, we always consider $\widehat{I}$ together with the covering map $\pi : \widehat{I} \to I$.

B. Multifolded cubes $\widehat{I}^n$ and $\widehat{I}^n_h$

Let ρ be a small positive number and $I^k_{1-\rho} = [\rho, 1 - \rho]^k$. Now take the product $\widehat{I} \times I^{n-1}$ and for every $p \in I^{n-1} \setminus I^{n-1}_{1-\rho}$ stick together the non-Hausdorff interval $\widehat{I} \times p$ into the usual interval $I \times p$. Denote the constructed non-Hausdorff multifolded cube by $\widehat{I}^n$.

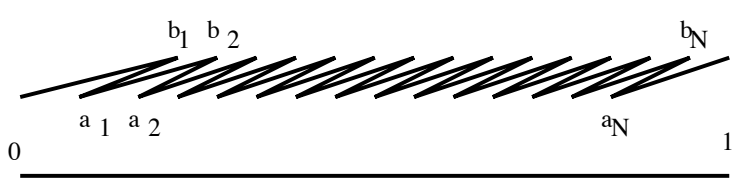

FIGURE 3.1. The non-Hausdorff manifold $\widehat{I}$.

The non-Hausdorff cube $\widehat{I}$ arises together with the natural projection $\pi : \widehat{I}^n \to I^n$. Note that $\mathcal{O}p\,(\partial\widehat{I}^n) = \mathcal{O}p\,\partial I^n$ and $\pi|_{\mathcal{O}p\,(\partial\widehat{I}^n)} = \mathrm{Id}\,\mathcal{O}p\,(\partial I^n)$. We will always consider $\widehat{I}^n$ as a "non-Hausdorff ramification" of the cube I^n. In other words, we always consider $\widehat{I}^n$ together with the covering map $\pi : \widehat{I}^n \to I$. Set $\mathcal{O}p\,\widehat{I}^n = (\mathcal{O}p\,I^n \setminus I^n) \cup \widehat{I}^n$.

Exercise. (a) Draw the ramification locus for the covering $\widehat{I}^2 \to I^2$.

[The answer: this is a ladder.]

Denote by $Diff\,(I^n, \mathcal{O}p\,\partial I^n; I^{n-1})$ the space of fibered over I^{n-1} diffeomorphisms $h : I^n \to I^n$,

$$\begin{tikzcd} I^{n-1} \times I \arrow[rr, "h"] \arrow[dr, "\pi"'] & & I^{n-1} \times I \arrow[dl, "\pi"] \\ & I^{n-1} \times 0 & \end{tikzcd} ,$$

identical on $\mathcal{O}p\,I^n$. In order to increase the reserve of multifolded cubes, we will consider, for any diffeomorphism $h \in Diff\,(I^n, \mathcal{O}p\,\partial I^n; I^{n-1})$, the push-forward of the multifolded cube $\widehat{I}^n$ by h. Indeed, for any multifolded cube $\widehat{I}^n$ and any diffeomorphism $h \in Diff\,(I^n, \mathcal{O}p\,\partial I^n; I^{n-1})$ (in fact, for any diffeomorphism $h \in Diff\,(I^n, \mathcal{O}p\,\partial I^n)$) there exists a unique non-Hausdorff ramification $\pi : \widehat{I}^n_h \to I^n$ and a unique diffeomorphism $\hat{h} : \widehat{I}^n \to \widehat{I}^n_h$, such that the diagram

$$\begin{tikzcd} \widehat{I}^n \arrow[r, "\widehat{h}"] \arrow[d, "\pi"'] & \widehat{I}^n_h \arrow[d, "\pi"] \\ I^n \arrow[r, "h"] & I^n \end{tikzcd}$$

commutes. The ramification ladder for $\widehat{I}^n_h$ will have curved rungs.

C. Multifolds and multivalued sections

A *multifold* over an n-dimensional manifold V is a non-Hausdorff manifold $\widehat{V}$ together with a projection $\pi : \widehat{V} \to V$ such that

- π is a diffeomorphism over a neighborhood of the $(n-1)$-skeleton of an appropriate triangulation of V;
- for any simplex Δ^n of the triangulation there exist a multifolded cube $\widehat{I}$ and a commutative diagram

$$\begin{tikzcd} \pi^{-1}(\mathcal{O}p\,\Delta^n) \arrow[r, "\widehat{\varphi}"] \arrow[d, "\pi"'] & \mathcal{O}p\,\widehat{I}^n \arrow[d, "\pi"] \\ \mathcal{O}p\,\Delta^n \arrow[r, "\varphi"] & \mathcal{O}p\,I^n \end{tikzcd}$$

  where φ and $\widehat{\varphi}$ are diffeomorphisms.

A *multivalued section* (=*multisection*) of a fibration $p : X \to V$ is a map $\widehat{f} : \widehat{V} \to X$, where $\widehat{V}$ is a multifold over V, such that the diagram

$$\begin{tikzcd} & X \arrow[d, "p"] \\ \widehat{V} \arrow[ur, "\widehat{f}"] \arrow[r, "\pi"] & V \end{tikzcd}$$

commutes.

A *fibered* (over P) multisection of a fibration $id \times p : P \times X \to P \times V$ is a fibered map $\{\widehat{f}_p\} : P \times \widehat{V} \to P \times X$, $p \in P$, such that the diagram

$$\begin{tikzcd} & P \times X \arrow[d, "id \times p"] \\ P \times \widehat{V} \arrow[ur, "\{\widehat{f}_p\}"] \arrow[r, "id \times \pi"] & P \times V \end{tikzcd}$$

commutes. In particular, two multisections $\widehat{f}_0, \widehat{f}_1 : \widehat{V} \to X$ are *homotopic*, if there exists a fibered multisection $\{\widehat{f}_t\} : I \times \widehat{V} \to I \times X$.

A multisection $\widehat{f} : \widehat{V} \to X$ is called *regular* if there exists a section $f : V \to X$ such that the diagram

$$\begin{tikzcd} & X \\ \widehat{V} \arrow[ur, "\widehat{f}"] \arrow[r, "\pi"] & V \arrow[u, "f"'] \end{tikzcd}$$

commutes. We will write $\widehat{f} = f$ if $\widehat{f}$ is regular and $\widehat{f} = f \circ \pi$. For any multifold $\widehat{V}$ over V any section $f : V \to X$ can be uniquely covered by the regular multisection $\widehat{f} = f \circ \pi : \widehat{V} \to V \to X$, and hence one can define the homotopy between sections and multisections. A section $f_0 : V \to X$ and a multisection $\widehat{f}_1 : \widehat{V} \to X$ are called C^0-*close*, if the multisections $\widehat{f}_0, \widehat{f}_1 : \widehat{V} \to X$ are C^0-close.

3.2. Holonomic approximation by multivalued sections

Let $p : X \to V$ be a fibration. A multisection $\widehat{F} : \widehat{V} \to X^{(r)}$ is called *holonomic*, if there exists a multisection $f : \widehat{V} \to X$ such that $F = J^r_f$.

THEOREM 3.1 (**Multivalued Holonomic Approximation Theorem**). *Any section $F : V \to X^{(r)}$ can be C^0-approximated by a holonomic multivalued section $\widehat{F} : \widehat{V} \to X$. If F is already holonomic over $A \subset V$, then one can approximate F such that $\widehat{F} = F$ over $\mathcal{O}p\, A$.*

THEOREM 3.2 (**Parametric Multivalued Holonomic Approximation Theorem**). *Any holonomic over $\mathcal{O}p\, \partial I^m$ fibered section $F : I^m \times V \to I^m \times X^{(r)}$ can be C^0-approximated by a fibered holonomic multivalued section $\widehat{F} : I^m \times \widehat{V} \to I^m \times X$ such that $\widehat{F} = F$ over $\mathcal{O}p\, \partial I^m$. If, in addition, F is already holonomic over $I^m \times A \subset I^m \times V$, then one can approximate F such that $\widehat{F} = F$ over $\mathcal{O}p\, A \times I^m$.*

We will prove the theorem 3.1. For parametric version 3.2 the proof is quite analogous. Using the Holonomic Approximation Theorem **??** we can proceed inductively

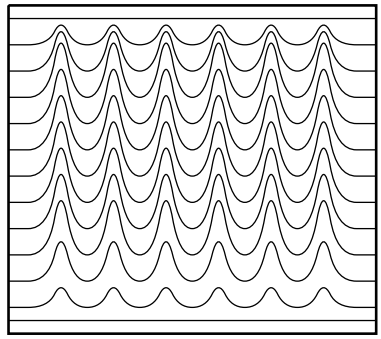

FIGURE 3.2. The foliation $h^1(I^{n-1} \times t)$, $t \in I$.

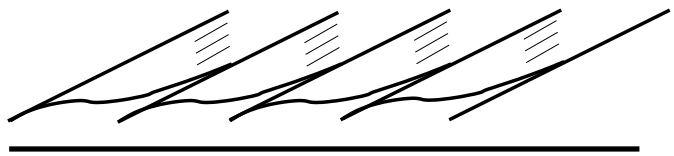

FIGURE 3.3. From partial sections to a multisection.

over the skeleton of a triangulation and reduce the Theorem 3.1 to its special case for the pair $(V, A) = (\mathcal{O}p\, I^n, \partial I^n)$:

THEOREM 3.3 (**Multivalued Holonomic Approximation over a cube**). *Any section $F : I^n \to X^{(r)}$, $V \simeq I^n$, holonomic over ∂I^n, can be C^0-approximated by a holonomic multivalued section $\widehat{F} : \widehat{I}^n_h \to X$ such that $\widehat{F} = F$ over ∂I^n.*

PROOF. Using the Theorem **??** we can apply the holonomic approximation parametrically to the family of cubes $I^{n-1} \times t \subset \mathcal{O}p\,(I^{n-1} \times t)$, $t \in I$. We can choose a t-family, $t \in I$, of fibered over I^{n-1} diffeotopies

$$h^\tau_t : \mathcal{O}p\,(I^{n-1} \times t) \to \mathcal{O}p\,(I^{n-1} \times t)\,, \quad \tau \in [0, 1],$$

such that the formula $h^\tau(x, t) = h^\tau_t(x, t)$ defines a fibered over I^{n-1} diffeomorphism $h^\tau : I^n \to I^n$ for all $\tau \in [0, 1]$. See Fig. 3.2 for h^1. Finally, we can construct the required holonomic multisection by choosing and sticking together, after an appropriate deformation, the partial holonomic sections over $\mathcal{O}p\,(h^1_t(I^{n-1} \times t))$, as it depicted on Fig. 3.3. Note, that the obtained multifolded cube will be the push-forward of a standard multifolded cube $\widehat{I}^n$ by h^1 (see 3.1B), i.e. $\widehat{I}^n_{h^1}$. $\square$