CHAPTER 3

Multivalued Holonomic Approximation

This chapter is a continuation of ch.3. Here we prove that any section $F : V \to X^{(r)}$ can be approximated by a holonomic section $\hat{F} : \hat{V} \to X^{(r)}$ where $\hat{V}$ is a non-Hausdorff covering space of $V$ ("multifold").

3.1. Multifolds

A. Multifolded intervals $\hat{I}$

Let $0 = a_0 < a_1 < \cdots < a_N < 1$ and $0 < b_1 < \cdots < b_N < b_{N+1} = 1$ be two sequences of points on the interval $[0,1]$, such that $a_i < b_i$ for all $i = 1, \ldots, N$. Take disjoint copies of the intervals $[a_0, b_1]$, $[a_1, b_1]$, $[a_1, b_2]$, $[a_2, b_2]$, $\ldots$, $[a_N, b_{N+1}]$, and extend slightly these intervals, i.e. consider the open intervals $(a_i - \delta, b_i + \delta)$ instead of $[a_i, b_i]$. Here $\delta$ is a small positive number and we do not extend the ends $a_0 = 0$ and $b_{N+1} = 1$. Then identify the added twin (open) intervals as it depicted on the Fig. 3.1 and denote the produced non-Hausdorff multifolded interval by $\hat{I}$. For simplicity we do not include the sequences $a_i$, $b_i$ and the number $\delta$ in the notation $\hat{I}$. Note that the non-Hausdorff interval $\hat{I}$ arises together with the natural projection $\pi : \hat{I} \to I$. We will always consider $\hat{I}$ as a "non-Hausdorff ramification" of the interval $I$. In other words, we always consider $\hat{I}$ together with the covering map $\pi : \hat{I} \to I$.

B. Multifolded cubes $\hat{I}^n$ and $\hat{I}_h^n$

Let $\rho$ be a small positive number and $I_{1-\rho}^k = [\rho, 1 - \rho]^k$. Now take the product $\hat{I} \times I_{1-\rho}^{n-1}$ and for every $p \in I_{1-\rho}^{n-1}$ stick together the non-Hausdorff interval $\hat{I} \times p$ into the usual interval $I \times p$. Denote the constructed non-Hausdorff multifolded cube by $\hat{I}^n$.

![Figure 3.1. The non-Hausdorff manifold $\hat{I}$.]
The non-Hausdorff cube $\hat{I}$ arises together with the natural projection $\pi : \hat{I}^n \to I^n$. Note that $\mathcal{O}_p(\partial \hat{I}^n) = \mathcal{O}_p \partial I^n$ and $\pi|_{\mathcal{O}_p(\partial \hat{I}^n)} = \text{Id} \mathcal{O}_p(\partial I^n)$. We will always consider $\hat{I}^n$ as a "non-Hausdorff ramification" of the cube $I^n$. In other words, we will always consider $\hat{I}^n$ together with the covering map $\pi : \hat{I}^n \to I^n$. Set $\mathcal{O}_p I^n = (\mathcal{O}_p I^n \setminus I^n) \cup \hat{I}^n$.

**Exercise.** (a) Draw the ramification locus for the covering $\hat{I}^2 \to I^2$.

[The answer: this is a ladder.]

Denote by $Diff(I^n, \mathcal{O}_p \partial I^n; I^{n-1})$ the space fibered over $I^{n-1}$ diffeomorphisms $h : I^n \to I^n$,

\[ I^{n-1} \times I \xrightarrow{h} I^{n-1} \times I \]

identical on $\mathcal{O}_p I^n$. In order to increase the reserve of multifolded cubes, we will consider, for any diffeomorphism $h \in Diff(I^n, \mathcal{O}_p \partial I^n; I^{n-1})$, the push-forward of the multifolded cube $\hat{I}^n$ by $h$. Indeed, for any multifolded cube $\hat{I}^n$ and any diffeomorphism $h \in Diff(I^n, \mathcal{O}_p \partial I^n; I^{n-1})$ (in fact, for any diffeomorphism $h \in Diff(I^n, \mathcal{O}_p \partial I^n)$) there exists a unique non-Hausdorff ramification $\pi : \hat{I}^n \to I^n$ and a unique diffeomorphism $\hat{h} : \hat{I}^n \to \hat{I}^n_h$, such that the diagram

\[ \hat{I}^n \xrightarrow{\hat{h}} \hat{I}^n_h \]

commutes. The ramification ladder for $\hat{I}^n_h$ will have curved rungs.

**C. Multifolds and multivalued sections**

A *multifold* over an $n$-dimensional manifold $V$ is a non-Hausdorff manifold $\hat{V}$ together with a projection $\pi : \hat{V} \to V$ such that

- $\pi$ is a diffeomorphism over a neighborhood of the $(n-1)$-skeleton of an appropriate triangulation of $V$;
- for any simplex $\Delta^n$ of the triangulation there exist a multifolded cube $\hat{I}$ and a commutative diagram

\[ \pi^{-1}(\mathcal{O}_p \Delta^n) \xrightarrow{\varphi} \mathcal{O}_p \hat{I}^n \]

\[ \pi \]

\[ \mathcal{O}_p \Delta^n \xrightarrow{\varphi} \mathcal{O}_p I^n \]

where $\varphi$ and $\hat{\varphi}$ are diffeomorphisms.
A multivalued section (=multisection) of a fibration $p : X \to V$ is a map $\hat{f} : \hat{V} \to X$, where $\hat{V}$ is a multifold over $V$, such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\hat{f}} & \hat{V} \\
p \downarrow & & \downarrow \pi \\
V & \xrightarrow{=} & V
\end{array}
$$

commutes. A fibered (over $P$) multisection of a fibration $id \times p : P \times X \to P \times V$ is a fibered map $\{\hat{f}_p\} : P \times \hat{V} \to P \times X$, $p \in P$, such that the diagram

$$
\begin{array}{ccc}
P \times X & \xrightarrow{(f_p)} & P \times \hat{V} \\
\downarrow_{id \times p} & & \downarrow_{id \times \pi} \\
P \times V & \xrightarrow{=} & P \times V
\end{array}
$$

commutes. In particular, two multisections $f_0, \hat{f}_1 : \hat{V} \to X$ are homotopic, if there exists a fibered multisection $\{\hat{f}_t\} : I \times \hat{V} \to I \times X$. A multisection $\hat{f} : \hat{V} \to X$ is called regular if there exists a section $f : V \to X$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & V \\
\hat{V} & \xrightarrow{\hat{f}} & \downarrow \pi \\
& & V
\end{array}
$$

commutes. We will write $\hat{f} = f$ if $\hat{f}$ is regular and $\hat{f} = f \circ \pi$. For any multifold $\hat{V}$ over $V$ any section $f : V \to X$ can be uniquely covered by the regular multisection $\hat{f} = f \circ \pi : \hat{V} \to V \to X$, and hence one can define the homotopy between sections and multisections. A section $f_0 : V \to X$ and a multisection $\hat{f}_1 : \hat{V} \to X$ are called $C^0$-close, if the multisections $f_0, \hat{f}_1 : \hat{V} \to X$ are $C^0$-close.

### 3.2. Holonomic approximation by multivalued sections

Let $p : X \to V$ be a fibration. A multisection $\hat{F} : \hat{V} \to X^{(v)}$ is called holonomic, if there exists a multisection $f : \hat{V} \to X$ such that $F = f|_A$.

**Theorem 3.1 (Multivalued Holonomic Approximation Theorem).** Any section $F : V \to X^{(v)}$ can be $C^0$-approximated by a holonomic multivalued section $\hat{F} : \hat{V} \to X$. If $F$ is already holonomic over $A \subset V$, then one can approximate $F$ such that $\hat{F} = F$ over $\mathcal{O}p A$.

**Theorem 3.2 (Parametric Multivalued Holonomic Approximation Theorem).** Any holonomic over $\mathcal{O}p \partial I^m$ fibered section $F : I^m \times V \to I^m \times X^{(v)}$ can be $C^0$-approximated by a fibered holonomic multivalued section $\hat{F} : I^m \times \hat{V} \to I^m \times X$ such that $\hat{F} = F$ over $\mathcal{O}p \partial I^m$. If, in addition, $F$ is already holonomic over $I^m \times A \subset I^m \times V$, then one can approximate $F$ such that $\hat{F} = F$ over $\mathcal{O}p A \times I^m$.

We will prove the theorem 3.1. For parametric version 3.2 the proof is quite analogous. Using the Holonomic Approximation Theorem?? we can proceed inductively
over the skeleton of a triangulation and reduce the Theorem 3.1 to its special case for the pair \((V, A) = (\mathcal{O}p I^n, \partial I^n)\):

**Theorem 3.3 (Multivalued Holonomic Approximation over a cube).** Any section \(F : I^n \rightarrow X^{(r)}, \ V \simeq I^n, \) holonomic over \(\partial I^n, \) can be \(C^0\)-approximated by a holonomic multivalued section \(\hat{F} : \hat{I}_h^n \rightarrow X\) such that \(\hat{F} = F\) over \(\partial I^n.\)

**Proof.** Using the Theorem ?? we can apply the holonomic approximation parametrically to the family of cubes \(I^{n-1} \times t \subset \mathcal{O}p (I^{n-1} \times t), \ t \in I.\) We can choose a \(t\)-family, \(t \in I,\) of fibered over \(I^{n-1}\) diffeotopies

\[ h^\tau : \mathcal{O}p (I^{n-1} \times t) \rightarrow \mathcal{O}p (I^{n-1} \times t), \ \tau \in [0, 1], \]

such that the formula \(h^\tau (x, t) = h^\tau_t (x, t)\) defines a fibered over \(I^{n-1}\) diffeomorphism \(h^\tau : I^n \rightarrow I^n\) for all \(\tau \in [0, 1].\) See Fig.3.2 for \(h^1.\) Finally, we can construct the required holonomic multisection by choosing and sticking together, after an appropriate deformation, the partial holonomic sections over \(\mathcal{O}p (h^1_t (I^{n-1} \times t)),\) as it depicted on Fig.3.3. Note, that the obtained multifolded cube will be the push-forward of a standard multifolded cube \(\hat{I}_h^n\) by \(h^1\) (see 3.1B), i.e. \(\hat{I}_h^n.\) \(\square\)