FOUNDATIONS OF UNLIMITED CATEGORY THEORY:

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The Problem of the Foundations of Category Theory (FCT)

- Eilenberg and Mac Lane introduced Category Theory (CT) in 1945.
- The problem of its foundations was there from the beginning.
- For example, unlimited categories such as *Grp*, *Top*, *Cat*, of all groups, topological spaces, categories, etc., are problematic: they are objects in *Cat*.
- Even more, the category of all functors between any two such categories *A* and *B*, is problematic.

Mac Lane led the way in FCT

- Pursuit through various systems of set theory (1961, 1969, etc.) ending with "one-universe" theory. Taken as basis of *Categories for the Working Mathematician* (1971).
- Systematic use of "small" and "locally small" categories.
- Now universally accepted that such distinctions are essential for statements of many theorems.
- Prime example: Freyd's AFT ("not baroque").

Why are there still claims for Autonomous Categorical Foundations (CF)?

- Despite such detailed evidence "on the ground" to the contrary, case for autonomous CF still being made for philosophical and/or ideological reasons.
- One argument: ZFC can be replaced by Lawvere's ETCS (Mac Lane 1986, 1998, McLarty 2004, etc.)
- My view: Deceptive ideological shell game
- Moreover, what this misses about set membership: e.g. the main step in the definition of L is not retrieved in category theory.

Set-theoretical Foundations of CT

- Granted some such (NB!) foundations are necessary, let us review the approaches to set-theoretical foundations of Mac Lane, Grothendieck, et al., with respect to two basic requirements (R1) and (R2) for unlimited category theory.

The Requirements for FCT

- Within an axiomatic system S as basic framework:
- (R1) Should be able to form the category of all structures of a given kind, e.g. *Grp*, *Top*, *Cat*
- (R2) Should be able to form the category of all functors from *A* to *B*, for any categories *A*, *B*
- (R3) Should be able to prove existence of basic mathematical structures and carry out usual set-theoretical operations
- (R4) Finally, consistency of S should be established relative to some currently accepted system of set theory.

How do standard approaches fare?

- 1. S = BGC (Eilenberg-Mac Lane '45, Mac Lane '61) small = set, large = proper class. (R1) and (R2) not met (as in the following standard and semi-standard approaches). (R3) in full. (R4) BGC is conservative over ZFC.

- 2. S = ZFC + "there exists a universe" (Mac Lane '69, '71). For a given universe U, small = set in U, large = proper subset of U, super-large = set in V. (R4) S equivalent in strength to ZFC + "there exists a strongly inaccessible cardinal"

Standard to semi-standard approaches

- 3. S = ZFC + "there exist arbitrarily large universes" (Grothendieck c. '69). Relativize 'small', 'large', 'super-large' to any universe U.
  (R4) strength: ZFC + "there exist arbitrarily large strong inaccessibles"

- 4. S = ZFC/s (Fef '69). s is an added symbol with axioms for $(s, \in) \preccurlyeq (V, \in)$. small = set in s, etc.
  (R4) Conservative over ZFC (Montague, Vaught).

Standard and semi-standard approaches (cont'd)

- 5. S = ZFC/s + "s is a universe" (Fef '69). (R4) Requires Mahlo hierarchy of strongly inaccessible cardinals for consistency proof
- M. A. Shulman, "Set theory for category theory" (arXiv 2008) gives a useful survey of 1-5 and advantages/disadvantages in applications

A Non-standard approach

- S = S* (Feferman 1974, 2006, 2011, and t.a.)
  S* is an extension of MKC theory of sets and classes by NFU+P , an enriched *stratified* theory of classes.
- (R4) S* is consistent relative to ZFC + "there exist two strongly inaccessible cardinals".
- Relation to Engeler and Röhrl (1969).

Stratified systems
Background: NF and NFU

- NF (Quine 1937) -- Variables A, B, C,...,X, Y, Z, basic relations =, $\in$. A formula is stratified if it comes from a formula of Simple Type Theory by suppressing types.
- NF axioms: Extensionality (Ext) and Stratified Comprehension Axiom scheme (SCA), i.e. $(\exists A)(\forall X)[X \in A \leftrightarrow \varphi]$ for all stratified φ (no 'A').
- Consistency of NF is a long open problem.

NF and NFU (Cont'd)

- NFU replaces (Ext) by (Ext)′, Weakened Extensionality, allowing Urelements
- Jensen (1969) proved consistency of NFU (+infinity, choice) rel. to PA (Z, ZC), using methods of Specker and Ehrenfeucht/ Mostowski; also stronger extensions.
- NFU proves closure under unordered pair, unions, intersections, power class, complement. Also have a universal class V. So, V∈V.

The problem of Pairing; NFU + P

- Usual definition of pairing, (X,Y) = {{X},{X,Y}} works but at the cost of going 2 up in type levels.
- Solution: NFU + P adds a binary operation symbol P with Pairing Axiom,
  $P(X,Y) = P(Z,W) \rightarrow X=Z \wedge Y=W$.
- For SCA in NFU+P, expand definition of stratification so "type level" of P(s,t) same as for both s,t.

Relations, Structures and CT in NFU+P

- NFU+P proved consistent by a simple modification of Jensen's methods
- Tuples obtained by iterated pairing, then relations and functions as usual. Can prove closure under $X \times Y$ and $X \rightarrow Y$ for any X, Y.
- First-order structures are tuples (A,...,R,...,F...). Use SCA to form the classes *Grp*, *Top*, *Cat*, etc., of all groups, topological spaces, categories, etc.

(R1) and (R2) for NFU+P

- (R1) is met in full. In particular, can form the structure *Cat* = (Cat, Funct,...), "the category of all categories" and prove *Cat* $\in$ Cat.
- (R2) is met in full. Can prove that if *A*, *B* are in Cat then ($A \rightarrow B$) is in Cat. More generally, can also prove that the 2-category (Cat, Funct, Nat,...) is in Cat.

(R3) for NFU + P

- (R3) has two specific problems in NFU+P:
- (Prob 1) to go from an equivalence relation to the class of equivalence classes
- (Prob 2) to form unrestricted Cartesian products.

Adding Universal Choice to take care of Problem 1

- Form the extension NFU+(P, C) obtained by adding a constant symbol C with axiom
  (UC) $\exists X(X \in A) \rightarrow C(A) \in A$.
- The stratification condition for SCA is now that C(t) is always assigned type level one less than that for t.
- UC allows us to define equivalence classes--given (A, E)--as $X/E = C(\{Y | (X,Y) \in E\})$. Then the map $F(X)=X/E$ from A to A/E exists by extended SCA.
- Consistency (R4) still holds; cf. below.

A partial solution to Problem 2

- Don't know of any way to get full Cartesian prods in an enriched stratified theory of classes.
- But can form $\Pi F(x)[x \in I]$ for any class I of sets and F from I to classes, in the system S* of sets and classes of Fef ('74, '06); sets may be assigned any type level in the stratification conditions.
- Let S† be the extension of S* by C with UC and its stratification conditions. Consistency of S† is proved by a modification of that for S* in Fef('74);sets are the constructibles up to a strongly inaccessible κ .

What remains to be done?

- The advantages and disadvantages of working in S† need to be tested by working with specific cases (e.g.,Freyd AFT, Yoneda Lemma, Kan Extension Theorem, etc.)
- The ecumenical point of view about the appropriate framework for FCT in Fef('77) was that that should be carried out via some sort of theory of operations and collections. One should still pursue alternatives.
- My candidates: systems of Explicit Mathematics and Operational Set Theory.

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