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GENERALIZED RECURSION THEORY II

Recursion theory and set theory: a marriage of convenience [1/]

by

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1. Introduction. We expand here on a program which was initiated in [F1] and elaborated in one direction in [F2]. The aim of the program is to provide an abstract axiomatic framework to explain the success of various analogues to classical (set-theoretical) mathematics which have been formulated in operationally explicit terms. These analogue developments fall roughly into two groups: (a) recursive and/or constructive mathematics, and (b) hyperarithmetic and/or predicative mathematics.

The framework proposed in [F1] was given by two theories T_0 and T_1 with the following features:

(i) they are theories whose universe of discourse includes operations and classes as elements;

(ii) the notions in (i) are not interreducible, operations being given by rules of computation (in some sense or other) and classes by predicates (from a fairly rich language).

(iii) operations may be applied to any elements, including operations and classes;

(iv) the theories are non-extensional;

(v) T_1 is obtained from T_0 by adjunction of a single axiom for an operation e_N which gives quantification over N ;

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(vi) T_0 restricted to intuitionistic logic is constructively justified;

(vii) T_1 minus its theory of generalized inductive definitions is predicatively justified.

(viii) T_0 has a model in which the elements of $N \to N$ represent all the recursive functions;

(ix) T_1 has a model in which the elements of $N \to N$ represent all the hyperarithmetic functions.

(x) T_1 has a model in which the elements of $N \to N$ represent all set-functions of natural numbers.

The plan of the program is to explain cases in which analogues have been successful, e.g. in recursive mathematics as follows. Say one has a theorem $\phi^{(set)}$ of set-theoretical mathematics which has a positive recursive analogue $\phi^{(rec)}$. Then one tries to find a theorem ϕ of T_0 such that on the one hand ϕ specializes to $\phi^{(rec)}$ in the model (viii) and on the other hand to $\phi^{(set)}$ in the model (x). Similarly for the other analogues, using (vi)-(ix).

This plan was carried out in some detail for a portion of model theory in [F2], using an extension $T_1^{(\Omega)}$ of T_1; that theory had the same features as T_1, but also axioms for a class Ω of ordinals were adjoined. We explained thereby the success of Cutland's analogue development [C] in which: hyperarithmetic models $\sim$ countable models, and Π_1^1 chains of hyperarithmetic models $\sim$ models of cardinality $\leq \aleph_1$.

In this paper we expand the systems T_0, T_1 to new theories $T_0(S)$, $T_1(S)$ so as to increase their flexibility and range of applicability. S is a class which acts like the class of all sets in set-theory, and the new axioms (in § 2 below) provide strong, natural closure conditions on S. Otherwise the principal features of $T_0(S)$ and $T_1(S)$ are the same as for T_0 and T_1. These now constitute our proposed marriage of recursion theory and set theory for the "convenience" of achieving the program explained above. It seems that any such framework must give up some significant features or principles of ordinary set theory. *Our choice* is to *give up the identification of functions with graphs* and to *give up extensionality*. As to the latter, the principle of extensionality has no essential mathematical use; its standard purpose is to map an equivalence relation $\equiv_A$ in a class A onto the equality relation by passing to $A/\equiv_A$. Instead, one simply works with the structure $(A, \equiv_A)$ accompanied by the new "equality" $\equiv_A$. However, it is possible that *extensional, non-classical systems* can also be used for our purposes, (as has been suggested by H. Friedman). In any case, the choice of axioms should be based on pragmatic considerations (not necessarily in conflict with constructive principles) and, as such, is still subject to experimentation.

§3 goes into some detail about how a variety of models of $T_0(S)$ and $T_1(S)$ can be constructed directly. There are two steps to be considered. First is the choice of an *applicative model*, either using familiar recursion theories or by generation. Examples of the former are denoted $\mathrm{Rec}(\omega)$ (ordinary recursion theory) and $\mathcal{E}^N\text{-}\mathrm{Rec}(\omega)$ (Π^1_1 recursion theory). Examples of the latter are given over any set-theoretical model $\mathfrak{M}$, resulting in three applicative structures $\mathrm{Rec}(\mathfrak{M})$, $\mathcal{E}^N\text{-}\mathrm{Rec}(\mathfrak{M})$ and $\text{Set-Fun}(\mathfrak{M})$; in the first two $\mathrm{Rec}(\omega)$ and $\mathcal{E}^N\text{-}\mathrm{Rec}(\omega)$ are lifted to $\mathfrak{M}$, and in the third all set functions of $\mathfrak{M}$ are fed into a generalized recursion theory. Next, given an applicative structure $\mathfrak{A}$ it is shown how to build a model $\mathfrak{A}^*$ of $T_0(S)$ in which any given collection of sets is represented. This finally leads to models such as $(\mathrm{Rec}(\omega))^*$, $\mathfrak{M}^*_{\mathrm{Rec}}$, $\mathcal{E}^N\text{-}\mathrm{Rec}(\omega))^*$, $\mathfrak{M}^*_{\mathcal{E}^N\text{-}\mathrm{Rec}}$ and $\mathfrak{M}^*_{\text{Set-Fun}}$; the last three of these are also models of $T_1(S)$.

§4 outlines how the *abstract constructive measure theory* of Bishop-Cheng [Bi, C] can be formalized in $T_0(S)$. That involved *prima-facie* use of a *power-class* operation which had been an obstacle in T_0 and other approaches. It is now handled essentially via $\mathcal{P}_S(X) = \{a \mid a \in S \wedge a \subseteq X\}$. A possible application of interest is given using the models $\mathrm{Rec}(\omega)^*$ or $\mathfrak{M}^*_{\mathrm{Rec}}$ of $T_0(S)$: if a recursive Borel set A is measurable in the sense of [Bi,C] and $\mu(A) > 0$ then A contains a recursive member (4.6). Some suggestions about how $T_0(S)$ might further be used to generalize classical and recursive mathematics are given in 4.7.

In §5 a theory of accessible ordinals $\mathcal{O}_S$ and (regular) number classes($\Omega_x^{(r)}$and) Ω_x for $x \in \mathcal{O}_S$ is developed within $T_0(S)$. In any model of $T_0(S)$ there are associated ordinals $|\mathcal{O}_S| = \sup\{|x| : x \in \mathcal{O}_S\}$ and $|\Omega_x|$ (defined similarly). Under the interpretation by $\mathfrak{M}^*_{\mathrm{Set\text{-}Fun}}$ we have $|\mathcal{O}_S|$ = least inaccessible ordinal and $|\Omega_x| = \omega_x$. On the other hand in both $\mathrm{Rec}(\omega)^*$ and $\mathfrak{H}^N\text{-}\mathrm{Rec}(\omega)^*$ we have $|\Omega_1| = \omega_1^c$ = least nonrecursive ordinal. It is conjectured that $|\mathcal{O}_S|$ = least recursively inaccessible ordinal and $\forall x \in \mathcal{O}_S[\,|x| = \alpha \Rightarrow |\Omega_x^{(r)}| = \omega_\alpha^c (= \tau_\alpha)]$ in these latter models. If so, this theory provides an approach to recursively accessible ordinals which is conceptually superior to that of Richter [R].

The paper concludes in 5.4 with a discussion of some further axioms which may be added to $T_1(S)$ and which are true in $\mathfrak{M}^*_{\mathfrak{H}^N\text{-}\mathrm{Rec}}$, such as the selection principle Sel_{Ω_1} for Ω_1. $T_1(S) + (\mathrm{Sel}_{\Omega_1})$ can be used for all the purposes in model theory which had been provided by $T_1^{(\Omega)}$ in [F2]. Now one can look for further applications in model theory by use of the development of higher number classes in $T_1(S)$. Another possible application is to "long" hierarchies of normal (critical) functions (originally due to Bachmann), which make use of higher number classes to define large countable ordinals. In certain specific cases these have been verified to be recursive by tedious calculations. The idea would be to obtain such results instead as a consequence of a treatment of these hierarchies within the framework of $T_1(S)$, using the fact that $|\Omega_1| = \omega_1^c$ in $\mathfrak{M}^*_{\mathfrak{H}^N\text{-}\mathrm{Rec}}$.

2. The theories $T_0(S)$ and $T_1(S)$. Knowledge of [F1], [F2] is not presumed here.

2.1 Syntax of the theories. The basic language is described as follows.

(Expansions of this syntax will consist simply in the adjunction of further constant symbols.)

> Individual (general) variables: a, b, c, ..., f, g, h, ..., x, y, z
>
> Class variables: A, B, C, ..., X, Y, Z
>
> Individual constants: $\underline{0}, \underline{k}, \underline{s}, \underline{d}, \underline{p}, \underline{p}_1, \underline{p}_2, \underline{j}, \underline{i}_n (n < \omega)$
>
> Class constant: S

Basic terms: variables or constants of either sort.
Individual terms are denoted $t, t_1, t_2, \ldots$
Class terms are denoted $T, T_1, T_2, \ldots$

Atomic formulas:

(i) Equations between terms of either sort

(ii) $App(t_1, t_2, t_3)$, also written $t_1 t_2 \simeq t_3$

(iii) $t \in T$

Formulas are generated by $\neg, \wedge, \vee, \Rightarrow$, and the quantifiers $\exists$ and $\forall$ applied to either sort of variable.

$\phi, \psi, \theta, \ldots$ range over formulas. We may write ϕ with a distinguished free variable as $\phi(x, \ldots)$ or $\phi(x)$. Then $\phi(t, \ldots)$, $\phi(t)$ resp., denotes $Sub(t/x)\phi$; similarly for class variables. The Gödel-number of a formula ϕ is denoted $\ulcorner\phi\urcorner$.

We write $C\ell(a)$ for $\exists A(a = A)$ and $x \in a$ for $\exists A(a = A \wedge x \in A)$.

2.2 Stratified and elementary formulas. By a stratified formula we mean one which contains equations only between individual terms. If $\phi(X)$ is stratified and $\psi(x)$ is any formula then $\phi(\hat{x}\,\psi(x))$ or $\phi(\hat{\psi})$ is defined to be the result of substituting $\psi(t)$ for each occurrence of $(t \in X)$ in ϕ. This is assumed to avoid collision of variables. Also for stratified formulas it makes sense to write $\phi(X^+)$ for a formula with only positive occurrences of subformulas $(t \in X)$.

By an <u>elementary formula</u> is meant a stratified formula without bound class variables and without the constant S.

Note that the formulas $C\ell(a)$, $x \in a$ are not stratified.

2.3 <u>Application terms</u>. These are generated in an extension of the basic language as follows:

(i) every basic term of either sort is an application term;

(ii) if τ_1, τ_2 are application terms so also is $\tau_1\tau_2$.

In the following, $\tau, \tau_1, \tau_2, \ldots$ range over application terms. $\tau_1\tau_2 \ldots \tau_n$ is written for $(\ldots(\tau_1\tau_2)\ldots)\tau_n$ (association to the left). Certain formulas involving application terms are translated into the basic language as follows:

$\tau \simeq x$ is $\tau = x$ when τ is a basic term

$\tau_1\tau_2 \simeq x$ is $\exists y_1 \exists y_2 [\tau_1 \simeq y_1 \wedge \tau_2 \simeq y_2 \wedge y_1 y_2 \simeq x]$

$\tau_1 \simeq \tau_2$ is $\forall x[\tau_1 \simeq x \Leftrightarrow \tau_2 \simeq x]$

$\tau\downarrow$ is $\exists x(\tau \simeq x)$ ("τ is defined")

$\phi(\tau)$ is $\exists x[\tau \simeq x \wedge \phi(x)]$.

$\tau_1 = \tau_2$ is written for $\tau_1 \simeq \tau_2$ when $\tau_1\downarrow$ and $\tau_2\downarrow$ is known or assumed. $\tau_1 \neq \tau_2$ is written for $\neg(\tau_1 \simeq \tau_2)$ under the same conditions.

The constant $\underline{p}$ will act as a pairing operator. We write

$$(\tau_1, \tau_2) = \underline{p}\,\tau_1\tau_2 \quad \text{and}$$

$$(\tau_1, \ldots, \tau_n, \tau_{n+1}) = ((\tau_1, \ldots, \tau_n), \tau_{n+1})$$

Tuples are indicated by bars: $\bar{\tau} = (\tau_1, \ldots, \tau_n)$.

2.4 <u>Class terms</u>. Consider any stratified formula $\phi(x, X^+; \bar{y}; \bar{A})$, for which we also write $\phi(x,X)$. We write $Clos_\phi(X)$ for $\forall x[\phi(x,X) \Rightarrow x \in X]$. We write $\phi_c(x, - ; \bar{y}; \bar{A})$ or $\phi_c(x, -)$ for $[Clos_\phi(X) \Rightarrow x \in X]$.

Then we shall use $\underset{=}{i}_{\ulcorner\phi\urcorner}(\bar{y},\bar{A})$ to denote the smallest class X satisfying $Clos_{\phi}$, i.e. the class inductively defined by ϕ. We thus write

$$\{x|\phi_c(x, -; \bar{y}; \bar{A})\} \text{ or } \cap X[Clos_{\phi}(X)] \text{ for } \underset{=}{i}_{\ulcorner\phi\urcorner}(\bar{y},\bar{A}).$$

Note that this is given as an operation $\underset{=}{i}_k$ applied to the tuples of individual and class parameters of ϕ (for $k = \ulcorner\phi\urcorner$). As a special case of this, given $\phi(x;\bar{y};\bar{A})$ which does not contain X we write $\{x|\phi(x;\bar{y};\bar{A})\}$ or $\{x|\phi(x)\}$ for $\{x|\phi_c(x;\bar{y};\bar{A})\}$. As another special case we write

$$\{x|x \in B \wedge \phi_c(x, -; \bar{y}, \bar{A})\} \text{ for } \{x|\phi^*_c(x, -; \bar{y}; \bar{A}, B)\}$$

where $\phi^*(x,X;\bar{y},\bar{A},B)$ is $x \in B \wedge \phi(x, X; \bar{y}, \bar{A})$. The axioms will guarantee that all these operations lead to classes.

We write

$$A \subseteq B \text{ for } \forall x(x \in A \Rightarrow x \in B), \text{ and}$$

$$A \equiv B \text{ for } A \subseteq B \wedge B \subseteq A .$$

Further we write

$$f: A \to B \text{ for } \forall x \in A \exists y \in B(fx \simeq y) \text{ (or } \forall x \in A(fx \in B)) .$$

2.5 <u>The axioms of</u> $T_0(S)$.

I. <u>Applicative axioms</u>

(i) (Unicity) $xy \simeq z_1 \wedge xy \simeq z_2 \Rightarrow z_1 = z_2$

(ii) (Constants) $(\underset{=}{k}\,xy\!\downarrow) \wedge \underset{=}{k}\,xy = x$

(iii) (Substitution) $(\underset{=}{s}\,xy\!\downarrow) \wedge \underset{=}{s}\,xyz \simeq xz(yz)$

(iv) (Definition by cases) $(\underset{=}{d}\,abxy\!\downarrow) \wedge (x = y \Rightarrow \underset{=}{d}\,abxy = a)$

$\wedge\ (x \neq y \Rightarrow \underset{=}{d}\,abxy = b)$

(v) (pairing, projections) $(\underline{p}x_1x_2\downarrow) \wedge (\underline{p}_i z\downarrow) \wedge \underline{p}_i(\underline{p}x_1x_2) = x_i$

(vi) (zero) $\underline{p}\,xy \neq 0$

II. Special axioms

(i) (Classes are elements) $\forall X\, \exists x(X = x)$

(ii) (Totality of class operations on elements) $(\underline{i}_n z\downarrow) \wedge (\underline{j}\, z\downarrow)$.

III. Elementary inductive definitions. For each elementary $\phi(x, X^+)$ and any $\psi(x)$,

$$\exists C\{x|\phi_c(x,-)\} \simeq C \wedge \mathrm{Clos}_\phi(C) \wedge [\mathrm{Clos}_\phi(\psi) \Rightarrow \forall x \in C.\psi(x)]\}.$$

IV. Join

$$\forall x \in A.C\ell(fx) \Rightarrow \exists C\{j(A,f) \simeq C \wedge$$
$$\forall z[z \in C \Leftrightarrow \exists x \in A\, \exists y\, (y \in fx \wedge z = (x,y))]\}.$$

V. S-axioms.

These will be explained after drawing consequences of I-IV, and introducing more notation. (Not all of that will be needed to state V, but serves later purposes as well).

Remarks. (1) The axioms I-IV are slightly stronger than the system T_o introduced in [F1]. The axioms of elementary comprehension and inductive generation in T_o are subsumed under the present IV. Also the logic is not restricted to be intuitionistic as it was in [F1]. (2) By II(i), operations applied to classes are special cases of operations applied to elements. II(ii) is taken for convenience. The operation $\underline{j}$ applied to any element z always give some element. It is only assumed to give a class when z is of the form (A,f) where $\forall x \in A.C\ell(fx)$. Similarly $\underline{i}_k z$ is always defined but it is only assumed we get a class when z is of the form $(\bar{y},\bar{A})$ where $\bar{y} = (y_1,\ldots,y_n)$, $\bar{A} = (A_1,\ldots,A_m)$ and $k = \ulcorner\phi(x,X^+; y_1,\ldots,y_n, A_1,\ldots,A_m)\urcorner$ with ϕ elementary.

2.6 Consequences of the applicative axioms (Refer to [F1] 3.3 for more details.)

(1) (Explicit definition). With each applicative term $\tau(x)$ is associated a term $\lambda x.\tau(x)$ such that

$$(\lambda x.\tau(x))\downarrow \wedge \forall y\,((\lambda x\,\tau(x))y \simeq \tau(y)).$$

Informally, $\lambda x.\tau(x)$ "exists as a rule" whether or not $\tau(x)\downarrow$ for any given x.

(2) (Zero, successor). Define $x' = (x,0)$. By I(v), (vi) we conclude $x' \neq 0$, $x' = y' \Rightarrow x = y$, $x = y' \Rightarrow y = p_1 x$.

(3) (Recursion theorem). By the usual diagonalization we can define r such that

$$\forall f[(r\,f\downarrow) \wedge \forall x[(r\,f)x \simeq f(r\,f)x]].$$

(4) (Non-extensionality) ([F1]3.4). We can disprove $\forall f,g[\forall x(fx \simeq gx) \Rightarrow f = g]$. The idea is to associate with each f an f^* with the same domain and which is identically 0 on that domain (use defn. by cases). Then f is total $\Leftrightarrow f^* = \lambda x.0$, if extensionality holds. Diagonalizing gives a contradiction.

2.7 Consequences of axiom III.

(1) (Elementary inductive definitions). Given elementary $\phi(x, X^+; \bar{y}; \bar{A})$, let $\cap X[\text{Clos}_\phi(X)] \simeq C$. Then

$$\forall x[x \in C \Leftrightarrow \forall X(\text{Clos}_\phi(X) \Rightarrow x \in X)].$$

For the proof of $\Rightarrow$, consider any X, apply $\text{Clos}_\phi(\hat{\psi}) \Rightarrow \psi(x)$ to $\psi(x) = (x \in X)$. For the proof of $\Leftarrow$, apply $\text{Clos}_\phi(C)$.

(2) (Elementary comprehension). Given elementary $\phi(x; \bar{y}; \bar{A})$ we have defined $\{x|\phi(x)\}$ as $\{x|\phi_c(x)\}$; call this C. Then $\text{Clos}_\phi(C)$ shows

$\forall x[\emptyset(x) \Rightarrow x \in C]$ and $\mathrm{Clos}_\emptyset(\hat{\emptyset})$ shows $\forall x \in C.\emptyset(x)$. Hence $\{x|\emptyset(x)\}$ is a class and

$$y \in \{x|\emptyset(x)\} \Leftrightarrow \emptyset(y).$$

(3) (<u>Class constructions</u>). The following are obtained directly as special cases of (2):

$$V = \{x|x=x\}\ , \quad \Lambda = \{x|x \neq x\}$$

$$\{y_1,\ldots,y_n\} = \{x|x=y_1 \vee \ldots \vee x=y_n\}$$

$$A \cap B = \{x|x \in A \vee x \in B\}\ , \quad A \cup B = \{x|x \in A \vee x \in B\}$$

$$-A = \{x|x \notin A\}$$

$$A \times B = \{x|\exists y \in A\, \exists z \in B\ \ x=(y,z)\}$$

$$(A \rightarrow B) = \{f|\ f : A \rightarrow B\}$$

$$f[A] = \{y|\exists x \in A\,(fx \simeq y)\}$$

$$\mathfrak{D} f = \{x|(fx\downarrow)\}.$$

(4) (<u>The natural numbers</u>). We introduce these by

$$N = \cap X[0 \in X \wedge \forall x(x \in X \Rightarrow x' \in X)]$$

i.e. as $\cap X[\mathrm{Clos}_\emptyset(X)]$ for $\emptyset(x,X) = [x=0 \vee \exists y \in X\,(x=y')]$. Then we have:

$$0 \in N, \quad x \in N \Rightarrow x' \in N, \ \text{ and}$$

$$\psi(0) \wedge \forall x(\psi(x) \Rightarrow \psi(x')) \Rightarrow \forall x \in N.\ \psi(x)\ , \quad \text{for any } \psi.$$

(5) (<u>Primitive recursion on</u> N). Using the recursion theorem and definition by cases we obtain existence of r_N satisfying

$$r_N(0,a,f) \simeq a\ , \quad r_N(x',a,f) \simeq f(x,r_N(x,a,f)).$$

Note that for any A, r_N: $N \times A \times A^{N \times A} \rightarrow A$. With explicit definition we can now generate all primitive recursive operators, in particular bounded minimum and bounded quantification.

(6) (<u>Partial recursion on</u> N). The unbounded minimum μf is defined as $g(f,0)$ where $g(f,x) \simeq (\mu y \leq x)(fy \simeq 0)$ if $\exists y \leq x(fy \simeq 0 \wedge \forall z < y(fz\downarrow))$ and $g(f,x) \simeq g(f,x')$ if $\forall y \leq x(fy\downarrow \wedge fy \neq 0)$ (g obtained by recursion theorem and def. by cases). Then we can get for each $k \in N$ existence of f_k with $f_k x \simeq \{k\}(x)$ for all $x \in N$. Also $\lambda(z,x).\{x\}(x)$ is obtained.

(7) (<u>Non-extensionality for classes</u>). Similarly to 2.6(4) we can disprove $\forall A, B(A \equiv B \Rightarrow A = B)$; cf. [F1]3.4.

2.8 <u>Consequences of the join axiom</u> IV. We write $\Sigma_{x \in A} fx$ for $j(A,f)$ so that

$$z \in \Sigma_{x \in A} fx \Leftrightarrow \exists x \in A\ \exists y(y \in fx \wedge z = (x,y))$$

whenever $\forall x \in A\, \exists X(fx \simeq X)$. Note that the defining property of $\Sigma_{x \in A} fx$ is not stratified.

(1) (<u>Product</u>). Suppose $\forall x \in A.C\ell(fx)$. The class

$$\Pi_{x \in A} fx =_{def} \{g \mid \forall x \in A((x,gx) \in C)\}$$
$$\text{where } C = \Sigma_{x \in A} fx$$

exists by join and elementary comprehension, and satisfies

$$g \in \Pi_{x \in A} fx \Leftrightarrow \forall x \in A\, (gx \in fx).$$

In other words, once we have Σ, the unstratified definition of Π can be reduced to the stratified (indeed elementary) definition given above. Note that if $fx = B$ for each $x \in A$ then $(\Sigma_{x \in A} fx) \equiv A \times B$ and $(\Pi_{x \in A} fx) \equiv (A \to B)$.

(2) (Union and intersection). Similarly we can infer existence of $\bigcup_{x \in A} fx$ and $\bigcap_{x \in A} fx$.

(3) (Membership on classes of classes). Suppose $\forall x \in A . C\ell(x)$, i.e. A is a class of classes. Then the class

$$E_A = \Sigma_{x \in A} x$$

represents the membership relation on A:

$$z \in E_A \Leftrightarrow \exists x \in A \, \exists y (z = (x,y) \wedge y \in x).$$

(4) (Non-existence of a class of all classes). Suppose there exists A such that $\forall x[x \in A \Leftrightarrow C\ell(x)]$. Using E_A we can form $C = \{x | x \in A \ \wedge x \notin x\}$, from which we get a contradiction. Thus we cannot in general introduce $\mathcal{P}(B)$, a class of all sub-classes of B.

(5) (Relative power class). Given any class A of classes, we can form

$$\mathcal{P}_A(B) = \{a | a \in A \wedge a \subseteq B\}, \text{ i.e. } \{a | a \in A \wedge \forall x((a,x) \in E_A \Rightarrow x \in B\}.$$

We shall make particular use of a generalization of $\mathcal{P}_S(B)$ later in the paper.

2.9 Classes with "equality" relations. A class A with equality I on it is a pair $a = (A,I)$ where $A^2 \subseteq I$ and $A^2 \cap I$ is an equivalence relation on A; if this holds we write $C\ell\text{-}Eq(a)$. Note that $A = p_1 a$ and $I = p_2 a$ in this case. Classes with equality relations arise naturally in the practice of explicit mathematics (cf. e.g. § 4 below) and are in any case essential for a non-extensional development. S will satisfy $\forall a \in S[C\ell\text{-}Eq(a)]$. Furthermore in the set-theoretic models of $T_0(S)$ and $T_1(S)$ (in §3) we shall show that $(A,I) \in S$ implies A/I is a set of the model.

(1) Notation. Given $C\ell\text{-Eq}(a)$, $a = (A, I)$, we write $\equiv_a$ or $\equiv_A$ for I and $x \in a$ for $x \in p_1 a$. By $(A, =)$ we mean A with the relation $I = \{(x,y) \mid x = y\}$; $a = (A, =)$ is called a discrete class in this case. The natural numbers will be dealt with as a discrete class, for example.

(2) (The subclass relation in $C\ell\text{-Eq}$). We write $a \subseteq_h b$ or $h : a \subseteq b$ for $h : A \to B \wedge \forall x, y \in A [x \equiv_A y \Leftrightarrow hx \equiv_B hy]$ when $a = (A, \equiv_A)$, $b = (B, \equiv_B)$. (Thus in the set-theoretic interpretation, h induces an injection of $A/\equiv_A$ into $B/\equiv_B$.) $a \subseteq b$ is written for $\exists h(a \subseteq_h b)$. a, b are isomorphic when there exist h_1, h_2 inverse to each other (on A, B resp.) such that $a \subseteq_{h_1} b \wedge b \subseteq_{h_1} a$.

(3) (Finitary operations on $C\ell\text{-Eq}$). Let $a = (A, \equiv_A)$, $b = (B, \equiv_B)$. We put

$$\text{(i)} \quad \begin{cases} a \times b = (A \times B, \equiv_{A \times B}), \text{ where} \\ (x_1, x_2) \equiv_{A \times B} (y_1, y_2) \Leftrightarrow x_1 \equiv_A y_1 \wedge x_2 \equiv_B y_2 . \end{cases}$$

We put

$$\text{(ii)} \quad \begin{cases} b^a = (B^A, \equiv_{B^A}), \text{ where} \\ f \in B^A \Leftrightarrow f : A \to B \wedge \forall x, y \in A [x \equiv_A y \Rightarrow fx \equiv_B fy], \text{ and} \\ f \equiv_{B^A} g \Leftrightarrow \forall x \in A [fx \equiv_B gx]. \end{cases}$$

The operation (i) is distinguished from Cartesian product on classes (2.7(3)) by the context.

(4) Infinitary operations on $C\ell\text{-Eq}$. Suppose A is a discrete class and that for each $x \in A$, fx is in $C\ell\text{-Eq}$, say $fx = b_x = (B_x, \equiv_x)$. Then we put

$$\text{(i)} \quad \begin{cases} \Sigma_{x \in A} fx = (\Sigma_{x \in A} B_x, \equiv_\Sigma) \text{ where} \\ (x_1, y_1) \equiv_\Sigma (x_2, y_2) \Leftrightarrow x_1 = x_2 \wedge y_1 \equiv_{x_1} y_2 . \end{cases}$$

Note that Bx is $p_1(fx)$ and $\equiv_x$ is $p_2(fx)$. Under the same conditions on A and f we define

$$\text{(ii)} \quad \begin{cases} \Pi_{x \in A} fx = (\Pi_{x \in A} Bx, \equiv_\Pi) \text{ where} \\ g \equiv_\Pi h \Leftrightarrow \forall x \in A [gx \equiv_x hx]. \end{cases}$$

We rely on the context to distinguish Σ and Π as operations on sequences of classes (2.8) or on sequences of classes with equality relations, as here.

<u>Remark</u>. The present operations can be generalized still further to define $\Sigma_{x \in a} fx$ and $\Pi_{x \in a} fx$ for $a = (A, \equiv_A)$ and $fx = (B_x, \equiv_x)$ under the following circumstances. Namely we must be provided with a system of maps $h_{x,y} : fx \subseteq fy$ for $x, y \in A$ with $x \equiv_A y$ such that $h_{x,y}$ and $h_{y,x}$ are inverses and $h_{x,z} = h_{y,x} \circ h_{x,y}$ when $x \equiv_A y \equiv_A z$. For full generality, closure under these extended operations could and should be included in the S-axioms; however, only the operations with discrete index classes will be used in the applications and, for simplicity, closure will be assumed only for these.

(5) (<u>Inductive separation</u>). Given $a_i = (A_i, \equiv_i)$ where $\equiv_i$ is I_i $(1 \leq i \leq m)$ we write $\bar{a}$ for $(a_1, \ldots, a_m)$ and $\phi(\ldots, \bar{a})$ for a formula which includes among its class parameters A_i and I_i $(1 \leq i \leq m)$. Given $b = (B, \equiv_B)$ and elementary $\phi(x, X^+ ; \bar{y}, \bar{a})$ we shall consider the process of separation applied to b, yielding

$$\text{(i)} \qquad (\{x \mid x \in B \wedge \phi_c(x, - ; \bar{y}, \bar{a})\}, \equiv_B)$$

when we make no change in the equivalence relation.

(6) (<u>Coarsening</u>). Suppose given a class with equality $a = (A, I)$. By a <u>coarsening of</u> a we mean a structure $a' = (A, I')$ where $A^2 \cap I \subseteq I'$. Only explicitly definable coarsenings will be considered below. (In the set-theoretic interpretation there is a natural map of A/I onto A/I'.)

We now formulate the remaining axioms of $T_o(S)$.

2.10 The S-axioms - (group V of $T_o(S)$).

(i) $a \in S \Rightarrow C\ell - Eq(a)$

(ii) $(N, =) \in S$

(iii) $a, b \in S \Rightarrow a \times b \in S \wedge b^a \in S$

(iv) $(A, =) \in S \wedge f : A \to S \Rightarrow \Sigma_{x \in A} fx \in S \wedge \Pi_{x \in A} fx \in S$.

(v) For each elementary $\phi(x, X^+ ; \bar{y}, \bar{a})$ with $\bar{a} = (a_1, \ldots, a_m)$ and $a_1, \ldots, a_m \in S$ and for each $b = (B, \equiv_B) \in S$ we have: $(\{x | x \in B \wedge \phi_c(x, - ; \bar{y}, \bar{a})\}, \equiv_B) \in S$.

(vi) Under the same hypothesis as (v) we have: if $I' = \{z | \phi_c(z, - ; \bar{y}, \bar{a})\}$ and $\forall x, y \in B [x \equiv_B y \Rightarrow (x,y) \in I]$ then $(B, I') \in S$.

Remark. These axioms are related to the ones for "bounded classes" given in [Fl] 7.3. An essential difference is that the predicate Bd is replaced here by the class constant S.

2.11 The system $T_1(S)$. This has only one additional axiom, which is really an expansion of the applicative axioms I. It involves a new constant $\underline{e}_N$ for the operation of existential quantification over N.

I(vii) ($\exists^N$ - axiom)

$$[\underline{e}_N f \simeq 0 \Leftrightarrow \exists x \in N (fx \simeq 0)] \wedge [\underline{e}_N f \simeq 1 \Leftrightarrow \forall x \in N (fx \simeq 1)].$$

2.12 Other axioms. It is natural to consider some further possible axioms. First of all, note that in the separation axiom V(v) for S, it was assumed the parameters of the definition are also in S. It is possible to strengthen this, at least up to $\equiv$, and most simply for discrete sets. We shall write $A \in S$ to mean $(A, =) \in S$.

V(vi) (Discrete separation). $\forall A, B [B \in S \Rightarrow \exists B_1 (B_1 \in S \wedge B_1 \equiv B \cap A)]$.

It will be shown in 3.6 how to get a model of $T_1(S)$ together with V(vi). The following will also be obtained in the same model:

V(vii) (Choice). $(B,\equiv_B)\in S \Rightarrow \exists C[C\subseteq B \wedge C\in S \wedge \forall x\in B\exists! y\in C\,(x\equiv_B y)]$.

By choice, discrete sets serve to represent all sets.

2.13 Stratified comprehension. Another strengthening to be considered is the principle, for any stratified $\phi(x,\bar{a},\bar{A})$:

$$\exists C[\{x|\phi(x)\}\simeq C \wedge \forall x(x\in C \Leftrightarrow \phi(x))].$$

Among other things this would allow us to introduce $\cap X[\mathrm{Clos}_\phi(X)]$ as an abstract, namely $\{x|\forall X(\mathrm{Clos}_\phi(X)\Rightarrow x\in X)\}$. (However this would not give the full strength of the elementary induction axiom, since it only yields proof by induction for stratified properties $\psi(x)$). It is also possible to model $T_1(S)$ with stratified comprehension.

Further special stronger axioms will be considered in connection with ordinals in § 5 below.

3. Models of the theories.

3.1 Outline and preliminaries. There are quite a variety of models to be considered. We describe here the general pattern of construction. By an applicative structure we mean any model $\mathfrak{A}=(A,\simeq,k,s,d,p,p_1,p_2,0)$ of the applicative axioms (I) of $T_0(S)$. 2/ Ordinary recursion theory and its generalizations provide a wealth of examples of such structures; some familiar ones are recalled in 3.2. For our purposes, a pairing structure is any structure $\mathfrak{A}_0=(A,P,P_1,P_2,0)$ where $P:A^2\xrightarrow{1-1}A$, $P_1(0)=0$, $P_2(0)=0$, $P_i(P(x_1,x_2))=x_i$, and $P(x_1,x_2)\neq 0$ all x_1,x_2. Any pairing structure generates an applicative model $\bar{\mathfrak{A}}_0$, as will be described in 3.3. More generally we can incorporate any pre-assigned collection of functions $\mathfrak{F}$; the result is denoted $\bar{\mathfrak{A}}_0(\mathfrak{F})$. In particular, given any model $\mathfrak{M}=(M,\epsilon_M)$ of set theory taken as a pairing structure in the standard way, we shall obtain applicative models $\bar{\mathfrak{M}}(\mathfrak{F})$ which range from ordinary recursion theory on $\mathfrak{M}$ to the incorporation of all set-functions (3.4).

2/ Following Friedman [Fr 1], I previously called these enumerative structures. Their source is in the Wagner-Strong axioms for abstract enumerative recursion theory.

Any pairing structure provides us with finite coding ability. First of all, the natural numbers are represented via the successor operation $x \mapsto x' = P(x,0)$. We may regard A as providing the alphabet for a symbolic system. Any word from A is represented by a code in A, then any finite sequence of such words is represented by another code, etc. We shall refer to coding procedures without giving specific details.

Given any applicative structure $\mathfrak{A}$ we shall show in 3.5 how to construct a model $\mathfrak{A}^*$ of $T_0(S)$, by interpreting the class variables to range over a certain collection of codes in A. Actually, interpretation of S and the membership relationship on S are explained first and these are then used to explain the interpretation of $C\ell$ and ε in general. The basic method of model building goes back to [F1]pp.104-107 for T_0.

Of special interest to us will be the case where we start with an applicative structure $\mathfrak{A}$ over a model $\mathfrak{M} = (M, \varepsilon_M)$ of set theory. By feeding in a code for each set of M in S, we can arrange that $\mathfrak{A}^*$ is a model of $T_0(S)$ in which S is a system of representatives of all ordinary sets(3.6). Taking $\mathfrak{A} = \overline{\mathfrak{M}}(\mathfrak{F})$ for various $\mathfrak{F}$ from 3.4, we can thus compare ordinary recursion theory, hyperarithmetic theory and full set-function theory as operative in a full set-theoretical situation.

There is only one additional point to be made for the theory $T_1(S)$. We shall call $\mathfrak{A}$ plus e_N (in A) an $\mathbb{E}^N$-applicative model if it satisfies as well the $\mathbb{E}^N$-axiom I(vii). Here the relation $x \varepsilon N$ is to be given its standard interpretation i.e.: x belongs to the smallest subset of A which contains 0 and is closed under the "successor" operation $u \mapsto (u,0)$. (Note that N itself will appear as a code for this set in A). Then $\mathfrak{A}^*$ is automatically a model for $T_1(S)$ if it is a model of $T_0(S)$ and $\mathfrak{A}$ is $\mathbb{E}^N$-applicative.

3.2 Familiar applicative models.

(1) (Ordinary recursion theory). We write $\mathrm{Rec}(\omega)$ for the applicative model $\mathfrak{A} = (\omega, \simeq, k, s, d, p, p_1, p_2, 0)$ where $xy \simeq z \Leftrightarrow \{x\}(y) \simeq z$ and the constants are suitably chosen. It is convenient for the following to assume that all of ω is generated from 0 by the operation $x, y \mapsto pxy$ and that $p_1 0 = p_2 0 = 0$. The subset N is in effective 1-1 correspondence with ω and the functions $\lambda y.fy$ for which $f : N \to N$ are the images of the recursive functions under this correspondence.

(2) ($\exists^N$-recursion theory). Here again $\mathfrak{A}$ has domain ω. Take a Π^1_1 predicate $\emptyset(x,y,z)$ which for $x = 0, 1, 2, \ldots$ enumerates all Π^1_1 partial functions (z as a function of y), and put $xy \simeq z \Leftrightarrow \emptyset(x,y,z)$. ($\emptyset$ may be obtained by uniformization; cf. [Ro]§ 16.5). We may choose a number e_N to satisfy

$$e_N f \simeq u \Leftrightarrow u = 0 \wedge \exists y \in N\,(fy \simeq 0) \vee u = 1 \wedge \forall y \in N\, \exists z (z \neq 0 \wedge fy \simeq z),$$

since the defining condition is arithmetical. (We assume the same effective pairing and projection functions as in (1).) Thus $\mathfrak{A}$ with e_N is an $\exists^N$-applicative structure, denoted $\exists^N\text{-}\mathrm{Rec}(\omega)$. The total functions here are exactly the hyperarithmetic functions, as are the total functions on N. (The relation $\simeq$ is not quite the same as obtained from Kleene recursion in ${}^2E$ [K] with $xy \simeq z \Leftrightarrow \{x\}({}^2E, y) \simeq z$, since ${}^2E(\lambda y.\{f\}({}^2E,y))$ is defined only when $\lambda y\{f\}({}^2E,y)$ is total. The total functions generated are the same.)

(3) (Admissible recursion theory) [Ba]. Let A be an admissible set in which we have Σ_1 uniformization, e.g. when $A = L_\alpha$ with α admissible, or more generally when A has a Σ_1 global well-ordering. Using a Σ_1 enumeration of the Σ_1 partial functions we then obtain an applicative model $\mathfrak{A} = (A, \simeq, \ldots)$. We write $\Sigma_1\text{-}\mathrm{Rec}(A)$ for $\mathfrak{A}$, and $\Sigma_1\text{-}\mathrm{Rec}(\alpha)$ when $A = L_\alpha$. When $\alpha > \omega$, $\Sigma_1\text{-}\mathrm{Rec}(\alpha)$ is $\exists^N$-applicative. For $\alpha = \omega_1^c$ (the least "non-constructive" ordinal) the Σ_1 partial functions from N to N in $\Sigma_1\text{-}\mathrm{Rec}(\omega_1^c)$ coincide with the $\exists^N$-partial recursive functions of (2).

3.3 Generating applicative models. Let $\mathcal{A}_0 = (A, P, P_1, P_2, 0)$ be a pairing structure, i.e. $P: A^2 \xrightarrow{1-1} A$, $P_i(P(x_1,x_2)) = x_i$, $P(x_1,x_2) \neq 0$ and $P_1(0) = P_2(0) = 0$. Let $\mathcal{F}$ be any family of unary partial functions on A with $\text{card}(\mathcal{F}) \leq \text{card}(A)$. Choose codes $k, s, d, p, p_1, p_2, f_F (F \in \mathcal{F})$, $k_x, s_x, s_{xy}, d_a, d_{ab}, d_{abx}, p_x$ which are distinct from 0 and from each other for all a, b, x, y in A. Then we take $\simeq$ to be the least relation satisfying: $kx \simeq k_x$, $k_x y \simeq y$, $sx \simeq s_x$, $s_x y \simeq s_{xy}$, $s_{xy} z \simeq u$ whenever $xz \simeq w$, $yz \simeq v$ and $wv \simeq u$, $da \simeq d_a$, $d_a b \simeq d_{ab}$, $d_{ab} x \simeq d_{abx}$, $d_{abx} y \simeq a$ if $x = y$, $d_{abx} y \simeq b$ if $x \neq y$, $px \simeq p_x$, $p_x y \simeq P(x,y)$, $p_1 x \simeq P_1(x)$, $p_2 x \simeq P_2(x)$, and $f_F x \simeq F(x)$ for each F in $\mathcal{F}$. The resulting structure is an applicative model, denoted $\overline{\mathcal{A}_0}(\mathcal{F})$, such that each F in $\mathcal{F}$ is represented by an element f_F. Similarly we can define $\overline{\mathcal{A}_0}(\mathsf{E}^N, \mathcal{F})$ which is the E^N - applicative structure generated from $\mathcal{F}$. When $\mathcal{F}$ is empty we obtain applicative models $\overline{\mathcal{A}_0}$ and $\overline{\mathcal{A}_0}(\mathsf{E}^N)$, respectively.

Given $\mathcal{A}_0$, let $A_0 \subseteq A$ and $\text{Gen}_P(A_0)$ = the closure of A_0 under pairing. A_0 is said to be an atomic base for $\mathcal{A}_0$ if $P: A^2 \to (A - A_0)$ and $A = \text{Gen}_P(A_0)$ and $P_1(x) = x$ for $x \in A_0$. We get a nice mapping from $\overline{\mathcal{A}_0}$ to $\overline{\mathcal{B}_0}$ when both $\mathcal{A}_0, \mathcal{B}_0$ have atomic bases A_0, B_0, resp. and we have $H: A_0 \to B_0$ with $H(0) = 0$. H extends canonically to $H: A \to B$ with $H(P(x_1,x_2)) = P(H(x_1), H(x_2))$; it is seen that $H(P_i(x)) = P_i(H(x))$ for $i = 1,2$. For simplicity fix $k, s, d, p, p_1, p_2 \in \text{Gen}_P(\{0\})$ ($\subseteq \text{Gen}_P(A_0)$ since $0 \in A_0$), and fix $kx \simeq (k,x)$, $(k,x)y \simeq x$, $sx \simeq (s,x)$, $(s,x)y \simeq ((s,x),y)$, $((s,x),y)z \simeq xz(yz)$, etc. in the same way both in $\overline{\mathcal{A}_0}$ and $\overline{\mathcal{B}_0}$. Thus $H(k) = k$, $H(s) = s$, etc.; it is then proved by induction that

$$(*) \qquad xy \simeq z \Rightarrow (Hx)(Hy) \simeq Hz.$$

Hence if f determines a total function $F: N \to N$ in $\overline{\mathcal{A}_0}$ then Hf determines the same function in $\overline{\mathcal{B}_0}$. We apply this in particular to $\overline{\mathcal{B}_0} = (\text{Gen}_P(\{0\}), \simeq, \ldots)$, which is effectively isomorphic to $\text{Rec}(\omega)$. It follows that the total functions from N to N in $\overline{\mathcal{A}_0}$ represent just the ordinary recursive functions. In this sense $\overline{\mathcal{A}_0}$ is a conservative lifting of ordinary recursion theory to $A = \text{Gen}_P(A_0)$.

(It is really a form of Moschovakis' prime computability theory [M] on the pure domain A_o since $\mathrm{Gen}_P(A_o) \cong A_o^*$.)

All of the preceding is directly extended to $\overline{\mathfrak{A}_o}(\mathbb{E}^N)$ and $\overline{\mathfrak{B}_o}(\mathbb{E}^N)$ when $\mathfrak{A}_o$, $\mathfrak{B}_o$ have atomic bases A_o, B_o, resp. It is proved that (*) continues to hold by showing that e_N behaves in $\overline{\mathfrak{B}_o}(\mathbb{E}^N)$ on Hf just as e_N behaves on f in $\overline{\mathfrak{A}_o}(\mathbb{E}^N)$. In particular, by taking $B_o = \{0,1\}$ we obtain that $\overline{\mathfrak{A}_o}(\mathbb{E}^N)$ <u>is a conservative lifting of $\mathbb{E}^N$-recursion theory to</u> $A = \mathrm{Gen}_P(A_o)$. Hence the total functions $\lambda f.fy$ from N to N in $\mathbb{E}^N\text{-Rec}(\mathfrak{A}_o)$ are just the hyperarithmetic functions.

3.4 <u>Applicative models on set-theoretical structures</u>. We now simply specialize the preceding to the pairing structure $\mathfrak{A}_o = (M, P, P_1, P_2, 0)$ obtained from a model $\mathfrak{M} = (M, \in_M)$ of set theory by taking the standard set-theoretical pairing and projection functions. Using well-foundedness it follows that $\mathfrak{A}$ has atomic base consisting of the elements of M which are not pairs. We write $\mathrm{Rec}(\mathfrak{M})$ and $\mathbb{E}^N\text{-Rec}(\mathfrak{M})$ for $\mathrm{Rec}(\mathfrak{A}_o)$, $\mathbb{E}^N\text{-Rec}(\mathfrak{A}_o)$, resp. These structures thus constitute conservative extensions of ordinary, resp. $\mathbb{E}^N$-recursion theory to M.

In addition to the preceding we wish also to consider the applicative model $\overline{\mathfrak{A}}_o(\mathfrak{F})$ where $\mathfrak{F}$ is the collection of all partial functions from M to M whose graph is a set in M. We denote this by $\text{Set-Fun}(\mathfrak{M})$. When $\mathfrak{M} = (M, \in_M)$ is a standard model, say $M = V_\alpha$ for limit α, and $F = \lambda x.fx$ is a partial function in $\overline{\mathfrak{A}}_o(\mathfrak{F})$, i.e. $f \in M$ and $fx \simeq F(x)$ for $x \in \mathrm{dom}(F)$, then the restriction of F to any set in M is in $\mathfrak{F}$. Note for $\alpha > \omega$ that $\text{Set-Fun}(\mathfrak{M})$ is also an $\mathbb{E}^N$-applicative model.

For illustrative purposes in the following we shall concentrate on the applicative models $\mathrm{Rec}(\omega)$, $\mathbb{E}^N\text{-Rec}(\omega)$ and (for standard $\mathfrak{M} = (M, \in_M)$) $\mathrm{Rec}(\mathfrak{M})$, $\mathbb{E}^N\text{-Rec}(\mathfrak{M})$ and $\text{Set-Fun}(\mathfrak{M})$.

3.5 <u>Generating models of $T_o(S)$ and $T_1(S)$</u>.

Let $\mathfrak{A}$ be any applicative model and let binary E_o be given. This determines sets $\{x : xE_o a\}$ for each $a \in A$. We shall build a model of $T_o(S)$

in which each such set is represented by some member of S. First we attend to codes. Let $i_n z = (1,n,z)$, $jz = (2,z)$, and $c_a = (3, a)$ for each a; these are thus total operations. For any elementary $\phi(x,X^+; y_1,\ldots,y_n; Y_1,\ldots,Y_m)$ and $\bar{y} = (y_1,\ldots,y_n)$, $\bar{a} = (a_1,\ldots,a_m)$ the object $\{x|\phi_c(x, -; \bar{y}; \bar{a})\}$ at $Y_j \to a_j$ (in other words, $\cap X[\forall x(\phi(x,X;\bar{y},\bar{a}) \Rightarrow x \in X)]$) is taken by definition to be the code $i_{\ulcorner\phi\urcorner}(\bar{y},\bar{a})$. In particular this is the code for $\{x|\phi(x;\bar{y};\bar{a})\}$ when X does not occur in ϕ. We shall define S inductively and along with this inductive definition the membership relation $x \in a$ for each a in S. When $\phi(x;\bar{y};\bar{Y})$ is elementary the variables Y_i only occur to the right of $\in$ in ϕ and no class quantifiers are used (nor does 'S' appear in ϕ). Thus if membership in a_j has been determined for each $j = 1,\ldots,m$, it is automatically determined for $\{x|\phi(x;\bar{y};\bar{a})\}$ by $z \in \{x|\phi(x;\bar{y};\bar{a})\} \Leftrightarrow \phi(z;\bar{y},\bar{a})$. More generally, membership $z \in \cap X[\forall x(\phi(x,X;\bar{y},\bar{a})) \Rightarrow x \in X]$ is determined to hold just in case z belongs to every subset X of A (in the real world) which satisfies the closure condition shown. In particular membership in N is determined in the standard way. Next, suppose membership in a is determined as well as membership in fx for each $x \in a$; then membership in $j(a,f)$ (or $\Sigma_{x \in a} fx$) is determined so as to satisfy the join axiom IV. From this we define $\Pi_{x \in a} fx$ and membership in it as in 2.8(1). Let $(a,=)$ be $(a,\{x|x = (p_1x, p_2x) \wedge p_1x = p_2x\})$. Define St as the smallest subset of A which (i) contains $(c_a, =)$ for each $a \in S$, (ii) contains $(N, =)$, (iii) contains $(a \times b, \equiv_{a \times b})$ and $(b^a, \equiv_b a)$ whenever it contains $(a, \equiv_a)$, $(b, \equiv_b)$ (iv) contains $\Sigma_{x \in a} fx$ and $\Pi_{x \in a} fx$ whenever it contains $(a, =)$ and fx for each $x \in a$, (v) contains $(\{x|x \in b \wedge \phi_c(x,-;\bar{y}; (a_i, I_i)_{i \leq m}\}, \equiv_b)$ whenever it contains $(a_1,I_1),\ldots,(a_m,I_m)$ and $(b, \equiv_b)$, and (vi) contains (b,I') whenever it contains $(b, \equiv_b)$ and $I' = \{x|\phi_c(x, -; \bar{y}, (a_i,I_i)_{i \leq m}\}$ and $b^2 \cap I'$ is an equivalence relation on b. Simultaneously with this inductive generation we determine $x \in a$ and $x \equiv_a y$ for each $(a, \equiv_a)$ in St by the procedure described above combined with the explanations in 2.9(3) - (6); to begin with put $x \in c_a \Leftrightarrow (x,a) \in E_o$.

Next we give St a code S in A, and put $a \in S$ if it belongs to St. To complete the construction of the model, we simply take $C\ell$ to be the smallest subset of A such that (i) $St \subseteq C\ell$, (ii) $\{x|\phi_c(x, -; \bar{y}; \bar{a})\}$ is in $C\ell$

whenever $\phi(x, X^+; \bar{y}; \bar{Y})$ is elementary and each a_j is in $C\ell$, and (iii) $\Sigma_{x\in a} fx$ is in $C\ell$ whenever a and fx are in $C\ell$, for each $x \in a$. Again this is accompanied by the definition of the membership relation $x \in a$ for each a in $C\ell$, by the determination procedure described above.

It is now readily checked that $(\mathfrak{A}, C\ell, \epsilon, S)$ is a model of $T_0(S)$. The only point to be observed in the inductive generation axiom III is that for $C = \{x | \phi_c(x, -)\}$ and any $\psi(x)$ we have $Clos_\phi(\hat{\psi}) \Rightarrow \forall x \in C.\psi(x)$. This is because $z \in C$ iff z belongs to every subset of A in the real world which is closed under ϕ, and in particular to $\{x : \psi(x)\}$ when $Clos_\phi(\hat{\psi})$, even if there is no member of $C\ell$ which represents that set. (To distinguish real set formation from the code $\{x|\phi(x)\}$ we write $\{x : \psi(x)\}$ in the first case.)

We write $\mathfrak{A}^*/E_0$ for the structure $(\mathfrak{A}, C\ell, \epsilon, S)$ just constructed (or simply $\mathfrak{A}^*$ if E_0 is empty). When $\mathfrak{A}$ is an $\mathbb{E}^N$-applicative structure then $\mathfrak{A}^*/E_0$ is a model of $T_1(S)$. In particular

(1) $$Rec(\omega)^* \models T_0(S)$$

and

(2) $$\mathbb{E}^N\text{-}Rec(\omega)^* \models T_1(S).$$

Remark. By a modification of this construction using the technique given in [F1] p.134, we can obtain a model $\mathfrak{A}^+/E_0$ of $T_0(S)$ + (Stratified comprehension). The idea is to start with the full 2nd order structure over $\mathfrak{A}$, introduce Skolem functions for the formulas in this structure and then close under codes for these functions when generating $C\ell$.

3.6 Models of $T_0(S)$ and $T_1(S)$ over set-theoretical structures.

For simplicity, take $\mathfrak{M} = (M, \epsilon_M)$ with $M = V_\alpha$ where α is inaccessible. Thus full (2nd order) replacement holds in $\mathfrak{M}$, i.e. if a is a set in M and $F : a \to M$ is a subfunction of M then $F[a] = \{F(x) : x \in_M a\}$ belongs to M. Let

$\mathfrak{a} = (M, \simeq, k, s, d, p, p_1, p_2, 0)$ be an applicative structure over M using the standard set-theoretical pairing and projective functions, and let E_0 be the membership relation ϵ_M on M. Thus $\mathfrak{a}^*/\epsilon_M$ is a model of $T_0(S)$ such that:

(1) <u>for each</u> a <u>in</u> M, <u>we have</u> $(c_a, =) \in S$ <u>and</u> $x \in c_a \Leftrightarrow x \in_M a$.

We shall now associate with each $a = (A, \equiv_A) \in S$ a function H_a and a set $\hat{a}$ such that

(2)
(i) $\hat{a} \in M$ and
(ii) $H_a : (A/\equiv_A) \to \hat{a}$ is one-one and onto.

The definition of $\hat{a}$ and H_a is by induction on the generation of S in 3.5. We shall only follow the former, the latter accompanying it in a natural manner. For convenience we also write $a^\wedge$ for $\hat{a}$. When $(A, =) \in S$ we write $\hat{A}$ for $\hat{a}$. The definition is:

(3)
(i) $(c_a, =)^\wedge = a$

(ii) $(N, =)^\wedge = N$ (the smallest set in M containing 0 and closed under $x \to (x, 0)$).

(iii) $(a \times b)^\wedge = \hat{a} \times \hat{b}$ and
$(b^a)^\wedge = \{F : F$ is a function in M from $\hat{a}$ to $\hat{b}$ and for some f, $\forall x \in A\,(F(H([x])) = H([fx]))\}$.

(iv) $(\Sigma_{x \in A} fx)^\wedge = \Sigma_{x \in \hat{A}} (fx)^\wedge$ and
$(\Pi_{x \in A} fx)^\wedge = \{G \in \Pi_{x \in A} (fx)^\wedge :$ for some g, $\forall x \in A(G(H_A(x)) = H_{fx}([gx]))\}$

(v) $(\{x \mid x \in B \wedge \phi_c(x, -; \bar{y}, \bar{a}\}, \equiv_B)^\wedge$ is the smallest subset X of $\hat{b}$ such that $\forall x[x \in_M \hat{b} \wedge \phi(x, X; \bar{y}, \bar{a}^\wedge) \Rightarrow x \in_M X]$.

(vi) $(B, I')^\wedge$, for $I' = \{x \mid \phi_c(x, -; \bar{y}, \bar{a})\}$ with $b = (B, I)$, $I \subseteq I'$ and $B^2 \cap I'$ an equivalence relation, is the image in M of $b^\wedge$ under the equivalence relation $x_1 \equiv x_2 \Leftrightarrow \forall x\,[\phi((x_1, x_2), X; \bar{y}, \bar{a}^\wedge) \Rightarrow (x_1, x_2) \in_M X]$.

Full replacement for M is used in (iv) and full separation is used in (v) and (vi) to show that the resulting $\hat{a}$ belongs to M.

(1) and (2) may be summarized by saying that <u>the sets of M are exactly the subsets of M represented in $\mathfrak{A}^*/\epsilon_M$ by the members of S.</u> For illustrative purposes, we shall concentrate in the following on the three structures $\mathfrak{A}^*/\epsilon_M$ obtained by starting with the three applicative models $\mathfrak{A} = \mathrm{Rec}(\mathfrak{M})$, $\mathrm{E}^N\text{-Rec}(\mathfrak{M})$ and $\text{Set-Fun}(\mathfrak{M})$ of 3.4, which will be more simply designated as follows:

(4) (i) $\mathfrak{M}^*_{\mathrm{Rec}} = (\mathrm{Rec}(M))^*/\epsilon_M$ (which satisfies $T_0(S)$)

(ii) $\mathfrak{M}^*_{\mathrm{E}^N\text{-Rec}} = (\mathrm{E}^N\text{-Rec}(M))^*/\epsilon_M$ (which satisfies $T_1(S)$)

(iii) $\mathfrak{M}^*_{\text{Set-Fun}} = (\text{Set-Fun}(M))^*/\epsilon_M$ (which satisfies $T_1(S)$).

In the first of these, $(N \to N)$ consists of codes of the recursive functions, in the second it consists of codes of the hyperarithmetic functions and in the third of codes of all set-theoretical functions from N to N. In all of these, S consists of codes of all the sets of M.

We now show that $\mathfrak{A}^*/\epsilon_M$ is also a model of the further axioms of Discrete Separation and of Choice formulated in 2.12. For the first of these V(vi), note that discrete sets can be generated only from the c_a's and N by $\times, \Sigma$ and inductive separation, since $\to$, Π and (proper) coarsening never lead to discrete sets. We can then prove by induction that

(5) if $a = (A, \equiv) \in S$ then $x \in_M \hat{A} \Leftrightarrow x \in A$.

To establish V(vi) from this, given discrete B and any class A, form the subset $a = \{x : x \in B \cap A\}$ of M. Then $x \in_M a \Leftrightarrow x \in_M \hat{B} \wedge x \in A$, so $a \in M$ by full separation. Then $c_a \in S$ and $c_a \equiv B \cap A$ as required. To prove the choice axiom V(vii), consider any $(B, \equiv_B) \in S$. Since $B \subseteq M$ and $B/\equiv_B$ is equivalent to a set $\hat{b}$ in M, there is a choice set a in M for B, i.e. $\forall x[x \in_M a \Rightarrow x \in B]$

and $\forall x \in B \exists ! y \in_M a(x \equiv_B y)$. Then c_a is a discrete choice set for B in the model $\mathcal{C}^* / \in_M$.

This completes our model-theoretic work. We now turn to an outline of several recursion-theoretic applications of the theories $T_0(S)$, $T_1(S)$ via the models of 3.5 and 3.6.

4. Bishop's constructive measure theory in $T_0(S)$.

4.1 Introduction. It was claimed in [F1] §5.1 that all of Bishop's constructive analysis [Bl] could be formalized in T_0, where Bishop's basic notion of operation f applied to an element x is read fx and where one takes for the notion of set $(A, =_A)$ ($=_A$ being an equality relation on A) pairs (A,E) with A,E classes for which $E \subseteq A^2$ is an equivalence relation on A. In other words, in the terminology of 2.9 above we are dealing with members of $C\ell$-Eq. Bishop's notion of function

$$f: (A, =_A) \rightarrow (B, =_B)$$

is formally expressed by

$$(f : A \rightarrow B) \wedge \forall x \in A \; \forall y \in A[x =_A y \Rightarrow fx =_B fy] ,$$

i.e. by $f \in B^A$ in the sense of 2.9(3) above. It is a direct matter to proceed from this basis to transcribe the work of [Bi 1] into T_0. This will be modified to an interpretation of [Bi 1] into $T_0(S)$ in 4.2; the reason for passing to $T_0(S)$ will be given in a moment. Some elaboration of general approach and points involved has been given in [F3]. [3/] We wish here to concentrate on aspects of the constructive theory of measure and so only relevant preliminary notions will be mentioned in 4.2. The treatment in [Bi1] was superseded by that in Bishop and Cheng [Bi,C], which is both more natural and more powerful. It was also claimed in [F1] that the latter could be formalized in T_0. Literally speaking this is not correct, since as will be seen below the abstract notion of integration space

[3/] Unpublished notes, a published version of which is eventually planned.

in [Bi,C] requires, prima facie, a power set operation $X \mapsto \mathcal{P}(X)$. That has also been an obstacle to other formal representations of Bishop's work such as given by Myhill [My] and Friedman [Fr 2] in extensional systems, in consequence of which they argued for modifying the mathematics to fit the systems. (In any case, there is no problem for T_0 or these other systems to deal with concrete constructive measure and integration theories such as Lebesgue measure on Euclidean spaces $\mathbb{R}^n$, because only the notions of being measurable and integrable are then needed.) It will be shown here how to formalize the abstract theory of [Bi,C] in $T_0(S)$, using a form of the operation $X \mapsto \mathcal{P}_S(X)$. The possible significance of this for constructive and recursive mathematics will be discussed in 4.5 - 4.6.

4.2 Basic concepts. We shall work informally within $T_0(S)$, calling members of $C\ell$ the classes and members of S the sets. Following [Bi 1] we shall write $=_A$ instead of $\equiv_A$ and we shall talk about sets A rather than $(A, =_A)$ (as is frequent in mathematics, one designates a structure by its domain). Thus, instead of using lower case letters for sets as in 2.9, we here use capital letters and write $A \times B$, B^A for the operations defined in 2.9(3) and $\Sigma_{x \in A} B_x$ $\Pi_{x \in A} B_x$ for the operations defined in 2.9(4) when A is discrete. Also $A \subseteq B$ is, as defined in 2.9(1), given by a function $i \in B^A$ such that $ia_1 =_B ia_2 \Rightarrow a_1 =_A a_2$; i is called an inclusion map in this case and A a subset of B. Our classes do not need to have equality relations attached to them, though every class A of mathematical interest does in fact have an $=_A$ prescribed for it. We now describe how various further notions from [Bi 1] are to be treated in $T_0(S)$.

First, from [Bi 1] Ch.2, the integers Z are defined by separation from $N \times N$ with $=_Z$ a coarsening of $=_{N \times N}$, and then the rationals Q are defined by separation from $Z \times Z$ with $=_Q$ the usual coarsening of $=_{Z \times Z}$. The arithmetical operations are extended to Z and Q. Z^+ can be identified with the discrete set $\{x \mid x \in N \wedge x \neq 0\}$. Given any set X, the sequences $x = \{x_n\}_{n \in Z^+}$ or $x = \{x_n\}$ from X are simply the members of X^{Z^+} (writing

x_n for xn). The class of all these sequences thus forms a set. The set $\mathbb{R}$ of <u>real numbers</u> is defined to be the set of <u>regular sequences</u> of rationals $x = (x_n)$, i.e. for which $\forall m, n \in Z^+(|x_m - x_n| \leq \frac{1}{m} + \frac{1}{n})$. $\mathbb{R}$ is thus defined by separation from Q^{Z^+}. Equality of reals $(x_n) =_{\mathbb{R}} (y_n)$ is defined by $\forall n \in Z^+(|x_n - y_n| < \frac{2}{n})$ (which is a coarsening of equality of sequences of rationals). $\mathbb{R}^+$ is the subset of $\mathbb{R}$ which consists of the pairs (x,n) where $x \in \mathbb{R}$, $n \in Z^+$ and $x_n > \frac{1}{n}$; the inclusion map is $i(x,n) = x$. In other words, these are reals with an explicit positive lower bound $\frac{1}{n}$. Bishop continually stresses the requirement of such explicit <u>witnessing or side information</u>, but for notational simplicity mostly does not show it in practice. This is potentially ambiguous, e.g. when we speak about a real number x being in $\mathbb{R}^+$ without specifying n for the lower bound. However, the context determines what additional information is to be understood as supplied - e.g. when talking about reals in $\mathbb{R}^+$. We shall follow [Bi 1] in this practice of casual designation.

The relation $y < x$ (or $x > y$) is defined to hold if $(x-y)$ is in $\mathbb{R}^+$, and $x \neq y$ if $y < x$ or $x < y$; clearly, both of these relations require witnessing information, e.g. $y < x$ (by n) if $(x-y, n) \in \mathbb{R}^+$. For each pair of real numbers a,b with $a < b$, the open interval (a,b) is defined as a subset of $\mathbb{R}$; its members are those x with $a < x$ and $x < b$ (together with the appropriate witnessing information). Closed or partially closed intervals are treated similarly. The set of all sequences of real numbers $x = (x_n)$ is $\mathbb{R}^{Z^+}$. The relation $\lim_{n \to \infty} x_n = y$ is defined to hold with the side-information (n_k) when $\forall n \geq n_k(|x_n - y| \leq \frac{1}{k})$; then $\sum_{n=1}^{\infty} x_n = y$ is defined as usual. The remainder of [Bi 1] Ch.2 is devoted to a constructive development of the calculus, which we do not need to follow.

We next look at some set-theoretical notions from Ch.3 of [Bi 1]. Some of these have already been dealt with in 2.9 and at the beginning of this section. A <u>family of subsets</u> of a set B with <u>index set</u> A is given by an operation f which associates with each $x \in A$ a subset B_x of B, in such a way that equal subsets are associated with equal elements of A. To be more precise we are to be supplied with an i which gives for each x an inclusion map i_x (or ix) of B_x in B and a j which gives for each $x,y \in A$ with $x =_A y$ a map $j_{x,y}$ (or $j(x,y)$) which is an inclusion map of Bx in By, and where all these maps commute appropriately. When A is discrete only the maps i_x are needed; we shall only have to deal with families over discrete sets. What Bishop denotes by $\bigcup_{x \in A} B_x$ is here written $\Sigma_{x \in A} B_x$, which is a subset of B by the inclusion map $h(x,y) = i_x y$. On the other hand, $\bigcap_{x \in A} B_x$ is defined to consist of the members g of $\Pi_{x \in A} B_x$ such that $\forall x,y \in A\,(i_x(gx) = i_y(gy))$.

Given a set X and an apartness relation $\neq$ on X(i.e. one which satisfies conditions like those of $\neq$ on $\mathbb{R}$) we call a pair of subsets (A^1, A^2) of X complemented if $(x \in A^1 \wedge y \in A^2 \Rightarrow x \neq y)$. (For example if $a < b < c$ then the pair consisting of the open intervals (a,b), (b,c) is complemented.) For each complemented set (A^1, A^2) we can associate a characteristic function χ_A to $A = A^1 \cup A^2$, i.e. to $A = \Sigma_{n \in \{1,2\}} A^n$. This is simply given by $\chi_A(x,1) = 1$ and $\chi_A(x,2) = 0$. The complement of a complemented set (A^1, A^2) is (A^2, A^1). The union of a countable family $A_n = (A^1_n, A^2_n)$ of complemented sets is defined to be $\vee_n A_n = (\cup_n A^1_n, \cap_n A^2_n)$ where $\cup$ and $\cap$ are as defined in the preceding paragraph. Similarly the intersection is taken to be $\wedge_n A_n = (\cap_n A^2_n, \cup_n A^2_n)$. The class of Borel sets generated from a given class B_0 of complemented sets is the smallest class B which includes B_0 and which is closed under countable unions and intersections $\vee_n$ and $\wedge_n$.

4.3 Abstract constructive integration theory. We now turn to [Bi,C] p.1. It is assumed that X is a set with an apartness relation $\neq$. The following definition is given loc. cit. : " let $F(X)$ be the set of all ordered pairs $(f,D(f))$ such that $D(f)$ is a subset of X and f is a function from $D(f)$ to the set of real numbers $\mathbb{R}$, with the property that $x \neq y$ whenever $f(x) \neq f(y)$." Here is where the operation $X \mapsto \mathcal{P}(X)$ makes a prime-facie appearance. In place of it we shall use in $T_o(S)$ what will be defined as the <u>class of all subsets of</u> X, where we are taking the notion of subset in its wider mapping sense, i.e. as a pair (a,i) with $a=(A,=_A)\in S$ and $a\subseteq_i X$. To define this class, we make use of the membership relation on S given by

(1) $$E_S^1 = \sum_{a\in S} p_1 a .$$

Thus for $a=(A,=_a)$ or $(A,=_A)$ we have $(a,x)\in E_S^1 \Leftrightarrow a\in S \wedge x\in A$. We shall also write $x\in a$ for $x\in p_1 a$ in this case. The corresponding sum of equality relations gives us

(2) $$E_S^2 = \sum_{a\in S} p_2 a ,$$

so that for $a=(A,=_a)$ we have $(a,(x,y))\in E_S^2 \Leftrightarrow a\in S \wedge x =_a y$. It follows that the property (of a and i) $a\subseteq_i X$ is expressible by the elementary formula

(3) $$\forall x[(a,x)\in E_S^1 \Rightarrow ix\in X] \wedge \forall x,y[(a,x)\in E_S^1 \wedge (a,y)\in E_S^1 \Rightarrow \{(a,(x,y))\in E_S^2 \Leftrightarrow ix =_X iy\}].$$

Hence the class

(4) $$\rho_S(X) = \{(a,i)\,|\,a\in S \wedge a\subseteq_i X\}$$

exists by the Elementary Comprehension axiom schema III. However, it need not exist as a set since it is not obtained by separation and since it involves the parameter S.

The definition of $F(X)$ in $T_o(S)$ is given similarly:

(5) $$F(X) = \{(f,(a,i))\,|\,a\in S \wedge a\subseteq_i X \wedge \forall x[(a,x)\in E_S^1 \Rightarrow fx\in \mathbb{R}]$$
$$\wedge \forall x,y((a,x)\in E_S^1 \wedge (a,y)\in E_S^1 \Rightarrow [(a,(x,y))\in E_S^2 \Rightarrow fx =_{\mathbb{R}} fy]$$
$$\wedge [fx \neq_{\mathbb{R}} fy \Rightarrow ix \neq iy])\} .$$

Again $F(X)$ exists as a class in $T_o(S)$, which we designate more simply by

(5)' $F(X) = \{(f,(a,i)) | a \in S \wedge a \subseteq_i X \wedge f: a \to \mathbb{R} \wedge$

$$\forall x,y[x \in a \wedge y \in a \wedge fx \neq_{\mathbb{R}} fy \Rightarrow ix \neq iy]\}.$$

As in [Bi,C], we write $D(f)$ for (a,i) when $(f,(a,i)) \in F(X)$ and we write $x \in D(f)$ for $\exists y(y \in a \wedge iy = x)$. We also write f alone for $(f, D(f))$.

Following this on p.2 in [Bi,C] we meet the basic definition (1.1) of <u>integration space</u>. This is read in $T_0(S)$ as follows: a triple (X,L,I) is an integration space if X is a non-empty set with an apartness relation and L is a subclass of $F(X)$ and $I: L \to \mathbb{R}$ has the properties (1)-(4) of [Bi,C]1.1. Here I is a partial function in the sense of $T_0(S)$ (we are using a capital letter so as to follow the notation of [Bi,C]). The idea of 1.1 is that L is an initial stock of integrable functions each defined "almost everywhere" and that $I(f)$ is the integral of f. (More precisely this is $I(f,(a,i))$ where $(f,(a,i)) \in L$.) An <u>integrable function</u> is then defined (1.6) to be a pair $(f,\{f_n\}_{n=1}^{\infty})$ for which $f \in F(X)$, $\{f_n\}$ is a sequence in L, $\Sigma_n I(|f_n|)$ exists and $fx = \Sigma_n f_n x$ holds whenever $\Sigma_n |f_n x|$ converges. Let L_1 be the class of all such pairs; $I_1(f,\{f_n\})$ is defined on L_1 by $I_1 f = \Sigma_n I_1(f_n)$. The first main result is that (X, L_1, I_1) is again an integration space; furthermore it is shown to have good completeness properties. Constructive <u>Lebesgue integration theory</u> falls out as the special case $(\mathbb{R}, L_1, I_1)$ for suitable initial $(\mathbb{R}, L, I)$.

4.4 <u>Measure theory in arbitrary integration spaces</u> (X,L,I) ([Bi,C] § 2). A complemented set $A = (A^1, A^2)$ is said to be <u>integrable</u> (or <u>measurable</u>) if its characteristic function χ_A is integrable; in this case the <u>measure</u> of A is defined to be $\mu(A) = I(\chi_A)$. Calculations with μ may be carried out as usual. Two of the results of [Bi,C] worth noting are 2.6 and 2.10:

(1) <u>if</u> $A = (A^1, A^2)$ <u>is an integrable set with</u> $\mu(A) > 0$ <u>then</u> A^1 <u>contains at least one element, and</u>

(2) *if* $(A_k)_{k=1}^{\infty}$ *is a sequence of integrable sets and* $\alpha = \lim_{n \to \infty} \mu(\vee_{k=1}^{n} A_k)$ *exists then* $\vee_k A_k$ *is integrable with measure* α.

4.5 $T_o(S)$ *and constructive mathematics*. Following these lines, all of [Bi,C] can be formalized in $T_o(S)$. Indeed, only intuitionistic logic need be used in the process. Thus if $T_o(S)^{(i)}$, i.e. intuitionistic $T_o(S)$ is regarded as constructively justified, it provides us with a constructive formalization of the whole of [Bi 1] and [Bi,C]. While I argued in [F1] that $T_o^{(i)}$ is constructively justified, I am a little hesitant about extending this claim to $T_o(S)^{(i)}$, though I think a case can also be made for that.5/ However, I do believe it has the character of a theory which would both be recognized on direct grounds as constructively valid and be adequate to the body of mathematics in [Bi 1], [Bi,C] and the further publications continuing Bishop's mathematical program.

4.6 $T_o(S)$ *and recursive measure theory*. Using models of $T_o(S)$ such as $(Rec(\omega))^*$ (3.5) and $\mathfrak{M}^*_{Rec}$ (3.6) in which the functions in $(N \to N)$ are just the recursive functions, every notion and result of $T_o(S)$ has a recursion-theoretic interpretation. In particular, the members of $\mathbb{R}$ are just the recursively regular sequences of rational numbers, which are one form of the *recursive real numbers*. Borel sets over $\mathbb{R}$ are what would otherwise be called *recursively coded Borel sets*. Every such set (regarded as a complemented set) is definable in $T_o(S)$. Hence we can apply the conclusion of [Bi,C] 2.6 noted in 4.4 above to obtain:

(1) *if a recursive Borel set* $A = (A^1, A^2)$ *is integrable with* $\mu(A) > 0$ *then* A^1 *contains some recursive real number*.

The potential utility of (1) is limited by the hypothesis of integrability, which is strong. It is not true (as might first be expected) that every recursive Borel set is integrable. Indeed, let x_n be a monotone increasing recursive

5/ It is only weak evidence that $T_o(S)$ is consistent with Church's thesis $\forall f \in N \to N\, \exists e \in N \forall x \in N (fx \simeq \{e\}x)$, by the model $(Rec(\omega))^*$.

sequence of recursive reals $0 \leq x_n \leq 1$ such that $\lim_{n \to \infty} x_n$ is not recursive. Take disjoint open intervals A_n with $\mu(A_n) = x_{n+1} - x_n$. Then $\vee_n A_n$ is recursively open but not measurable. To verify integrability of a recursive Borel set A in general, one must verify recursiveness of μ applied at each step of the build-up of A. (No doubt, with this understanding, one could give a relatively simple direct statement and proof of (1) which does not need to pass through the formalization in $T_0(S)$ outlined above.) It is of interest to compare (1) with recursion-theoretic basis results, e.g. [S], where one gets information about existence for definable A of definable members when $\mu(A) > 0$. In these cases the conclusions are much weaker than (1) (we cannot get recursive members) but so also are the hypotheses, since μ is read there in terms of standard measure theory.

4.7 Remarks on $T_0(S)$ and recursive mathematics. Some expectations about the relations of T_0 to recursive mathematics were formulated in [F1] 5.2; these are continued and in certain respects better borne out by $T_0(S)$. As explained in the introduction, the idea is simply that a theorem $\phi^{(set)}$ of classical set-theoretic mathematics which has a positive recursive analogue $\phi^{(rec)}$ can be assimilated to a common general form ϕ which is provable in $T_0(S)$. The classical result can be then read off by the interpretation of $T_0(S)$ in $\mathfrak{m}^*_{Set\text{-}Fun}$ while the recursive one is given by the specialization to $(Rec(\omega))^*$ or $\mathfrak{m}^*_{Rec}$. On the other hand, if $\phi^{(rec)}$ turns out negatively, use of the latter models shows the independence of ϕ from $T_0(S)$. Note that there is no reason to expect intuitionistic logic to play any special role here, and none to suppose - contrary to [F1] - that in the positive case ϕ is already provable in $T_0(S)^{(i)}$. This point is relevant to attempts to give a constructive redevelopment of classical algebra, which has turned out to be surprisingly difficult. To begin with, as mentioned in [Bi 2], it is not constructively true that the ring Z is Noetherian. Given an ideal A in Z we cannot in general decide whether $A = \{0\}$ or not, and even if $A \neq \{0\}$ is known, we cannot find a finite basis for A. On the other hand, if we allow use of classical logic in $T_0(S)$, then we can prove that Z is a principal ideal domain. This follows from the statement in $T_0(S)$ that every non-empty subset A of N contains a least element $\min(A)$. Of course this is weaker than the would-be constructive statement, since it is not asserted that the map $A \mapsto \min(A)$ is provided by a function of the system. While these distinctions are clear in principle, it is a matter of detailed study to see how much mathematics of recursion-theoretic interest can be formulated in $T_0(S)$ which is not already derivable in $T_0(S)^{(i)}$.

5. Accessible ordinals and number classes in $T_0(S)$ and $T_1(S)$.

5.1 Introduction. Various recursion-theoretic analogues of the set theoretical notion of accessible ordinal have been developed, most extensively by Richter [R]. This involves defining classes $\mathcal{O}_n$ of numbers by a complicated inductive definition which regulates the choice of n from "previous classes" as well as the generation of each class separately. We provide here an abstract development in $T_0(S)$ which is much like that in set-theory: we first define the class $\mathcal{O}_S$ of accessible ordinals and then a map $x \mapsto \Omega_x$ for $x \in \mathcal{O}_S$ where Ω_x is the ordinal number class associated with x (5.2). Classical and recursive ordinal number theories come out as special cases in the models of $T_0(S)$ (5.3). For further applications of ordinals we show that a selection operator for Ω_1 can be found in certain of these models which satisfy $T_1(S)$. This and other principles and applications are discussed at the conclusion in 5.4.

5.2 Definitions of the concepts. Throughout this section we work informally in $T_0(S)$. We shall first set up the general definition of $\mathcal{O}_A$, where A is any class of classes with equality. $\mathcal{O}_A$ is supposed to be the closure under 0, $'$ and $\sup_{x \in a} fx$ for a in A and unbounded $f: a \to \mathcal{O}_A$. This will then be specialized to $A = S$.

(1) Suprema. These are given by codes using the definition

$$\sup_a f = (f, a, 1),$$

for any f, a. The '1' distinguishes the result from that of successor $x' = (x, 0)$. By projections we can uniquely recover both f, a from $\sup_a f$. We also write

$$\sup_{x \in a} fx = \sup_a f .$$

(2) <u>The class</u> Θ_A <u>of ordinals with suprema from</u> A. Suppose A is a class of classes with equality $a = (p_1 a, \equiv_a)$. We continue to write $x \in a$ for $x \in p_1 a$. The classes Θ_A and L_A will be defined by a simultaneous inductive definition. We write $x \leq_{\Theta_A} y$ or simply $x \leq y$ for $(x,y) \in L_A$. Then we put $x < y \Leftrightarrow x \leq y \wedge y \leq x$ and $x < y \Leftrightarrow x' \leq y$. The clauses of the inductive definition are as follows:

(i) $0 \in \Theta_A$

(ii) $x \in \Theta_A \Rightarrow x' \in \Theta_A$

(iii) $a \in A \wedge \forall x \in a (fx \in \Theta_A) \wedge \forall x, y \in a [x \equiv_a y \Rightarrow fx \equiv fy]$
$\wedge \forall x \in a \exists y \in a (fx < fy) \Rightarrow \sup_a f \in \Theta_A$.

(iv) $x \in \Theta_A \Rightarrow 0 \leq x$

(v) $x, y \in \Theta_A \wedge x \leq y \Rightarrow x' \leq y'$

(vi) $y \in \Theta_A \wedge \sup_a f \in \Theta_A \wedge \forall x \in a (fx \leq y) \Rightarrow \sup_a f \leq y'$

(vii) $y \in \Theta_A \wedge \sup_a f \in \Theta_A \wedge \exists x \in a (y \leq fx) \Rightarrow y' \leq \sup_a f$

(viii) $\sup_a f \in \Theta_A \wedge \sup_b g \in \Theta_A \wedge \forall x \in a \exists y \in b (fx \leq gy) \Rightarrow \sup_a f \leq \sup_b g$.

We may write $\Theta_A = \{x \mid (x, 0) \in W_A\}$ and $L_A = \{z \mid (z, 1) \in W_A\}$ where $W_A = \{x \mid \phi_c(x, - ; A, E_A^1, E_A^2)\}$ for suitable elementary $\phi_c(x, X^+ ; A, B, C)$. Here $E_A^1 = \sum_{a \in A} p_1 a$ and $E_A^2 = \sum_{a \in A} p_2 a$ so for $a \in A$ we can replace the unstratified conditions $(x \in a)$ and $(x \equiv_a y)$ by the stratified conditions $(a, x) \in E_A^1$ and $(a, (x, y)) \in E_A^2$, resp.

(3) Induction on Θ_A. If we fix $\leq$ as $\leq_{\Theta_A}$ in (i) - (iii), then Θ_A is equivalent to the least class which satisfies the closure conditions (i)-(iii). Thus we can carry out proof by induction to show $\forall x \in \Theta_A . \psi(x)$ for any property ψ

which satisfies the closure conditions (i) - (iii) in place of $\mathcal{O}_A$. In particular we can prove the converses of (i)-(iii), i.e.

(i) $z \in \mathcal{O}_A \Rightarrow z = 0$ or $z = x'$ for $x \in \mathcal{O}_A$ or $z = \sup_a f$ where $a \in A$, $f \in (\mathcal{O}_A)^a$ and $\forall x \in a \exists y \in a(fx < fy)$.

Of course we can also carry out proof by induction on $\leq_{\mathcal{O}_A}$. Using (i) just established, the conclusions of (2) (iv)-(viii) give all possible comparisons of elements of the form $0, x', \sup_a f$; in particular, we obtain $x \leq 0$ only if $x = 0$. Some properties that can be proved of $\leq$, $<$ and $\equiv$ are:

(ii) $x \in \mathcal{O}_A \Rightarrow x \leq x$

(iii) $x,y,z \in \mathcal{O}_A \wedge x \leq y \wedge y \leq z \Rightarrow x \leq z$.

(iv) $\equiv$ is an equivalence relation on $\mathcal{O}_A$

(v) $x,y \in \mathcal{O}_A \Rightarrow x < x' \wedge (y < x' \Rightarrow y \leq x)$

(vi) $y \in \mathcal{O}_A \wedge \sup_a f \in \mathcal{O}_A \Rightarrow [\sup_a f \leq y \Leftrightarrow \forall x \in a(fx \leq y)]$
$\wedge\ [y < \sup_a f \Leftrightarrow \exists x \in a(y \leq fx)]$.

(4) <u>Recursion on</u> $\mathcal{O}_A$. We can carry out definition by transfinite recursion on $\mathcal{O}_A$ by applying the recursion theorem 2.6(3). Given any g_o, g_1, g_2 we find h such that

(i) $ho = g_o$

(ii) $hx' \simeq g_1(x,h)$

(iii) $h(\sup_a f) \simeq g_2(a,f,h)$

for any x,a,f. (If g_1, g_2 are total then h can be chosen total.) This can also be done uniformly in a parameter:

(i)' $h(z,0) \simeq g_o z$

(ii)' $h(z,x') \simeq g_1(z,x,h)$

(iii)' $h(z, \sup_a f) \simeq g_2(z,a,f,h)$.

Usually for applications the functions g_1, g_2 take the form $g_1(x,h) = g_1^*(x,hx)$ and $g_2(a,f,h) = g_2^*(a,f,\lambda x.h(fx))$, and similarly in (ii)', (iii)'.

For example we can define ordinal addition by these means, giving:

(iv) $z + 0 = z$

$z + x' \simeq (z + x)'$

$z + \sup_a f \simeq \sup_{x \in a} (z + fx) = \sup_a \lambda x(z + fx)$.

Then for any (class of classes) A we can prove by induction on $\mathcal{O}_A$:

(v) $$z, x \in \mathcal{O}_A \Rightarrow (z + x) \in \mathcal{O}_A \; .$$

Similarly further familiar ordinal functions may be introduced and treated as usual.

Remark. The process of recursion is independent of A in (i)-(iii) or (i)'-(iii)'. Of course it is only for suitable g_1, g_2 that we will be sure that $\mathcal{O}_A$ is closed under h , as in (v).

(5) The accessible ordinals $\mathcal{O}_S$ and the regular number classes $\Omega_x^{(r)}$.
$\mathcal{O}_S$ is simply the special case of $\mathcal{O}_A$ for $A = S$.

The regular number classes $\Omega_x^{(r)}$ with equality relations $\equiv_x^{(r)}$ are defined by recursion as follows:

(i) $$\begin{cases} \Omega_0^{(r)} = N \text{ and } \equiv_0^{(r)} \text{ is the relation } = ; \\ \text{for } x \in \mathcal{O}_S \text{ with } x \neq 0,\ \Omega_x^{(r)} = \mathcal{O}_{B_x} \text{ and } (\equiv_x^{(r)}) \text{ is } (\equiv_{\mathcal{O}_{B_x}}) \\ \text{where } B_x = \{(\Omega_z^{(r)}, \equiv_z^{(r)}) \mid z < x\}. \end{cases}$$

Here $<$ is the relation $<_{\mathcal{O}_S}$. This recursion is justified by the principles in (4). We are to obtain a function h such that for each $x \in \mathcal{O}_S$,

(ii) $$hx \simeq (\Omega_x^{(r)} , \equiv_x^{(r)}) \; .$$

Recall from (2) that for any class B of classes with equality $a = (p_1 a, \equiv_a)$, $\mathcal{O}_B = \{x \mid (x,0) \in W_B\}$ and $L_B = \{z \mid (z,1) \in W_B\}$ where $W_B = \{x \mid \phi_c(x, -; B, E_B^1, E_B^2)\}$ for suitable elementary $\emptyset$, and where $E^i_B = \Sigma_{a\epsilon B} p_1 a$ for $i = 1,2$. Further, $x \leq_{\Theta_B} y \Leftrightarrow (x,y) \epsilon L_A$ and $x \equiv_{\Theta_B} y \Leftrightarrow (x,y) \epsilon L_B \wedge (y,x) \epsilon L_B$. Thus we want

$$\text{(iii)} \quad \begin{cases} ho \simeq (N, =) \text{ and, for } x \neq 0, \\ hx \simeq (\{x \mid (x,0) \epsilon W_{Ax}\}, \{(x,y) \mid ((x,y,1) \epsilon W_{Ax} \wedge ((y,x),1) \epsilon W_{Ax}\}) \\ \text{where } Ax \simeq \{u \mid \exists z (z < x \wedge u \simeq hz)\}. \end{cases}$$

Now for any b, whether or not it is a class of classes with equality, both

$$\text{(iv)} \qquad e^1 b = j(b,p_1) = \Sigma_{x\epsilon b} p_1 a \quad \text{and} \quad e^2 b = j(b,p_2) = \Sigma_{a\epsilon b} p_2 a$$

are always defined by axiom II(ii). Further

$$\text{(v)} \qquad wb = i_{\ulcorner\emptyset\urcorner}(b, e^1 b, e^2 b)$$

is defined by the same axiom. Thus for any class B of classes with equality, certainly

$$\text{(vi)} \qquad wB \simeq W_B .$$

Further, for suitable elementary θ_1 and θ_2 we have

$$\text{(vii)} \qquad i_{\ulcorner\theta_1\urcorner}(wb) = \{x \mid (x,0) \epsilon wb\} \quad \text{and}$$

$$i_{\ulcorner\theta_2\urcorner}(wb) = \{(x,y) \mid ((x,y),1) \epsilon wb \wedge ((y,x),1) \epsilon wb\}.$$

Finally, for suitable elementary ψ we have

$$\text{(viii)} \qquad i_{\ulcorner\psi\urcorner}(x,h) = \{u \mid \exists z (z <_{\Theta_S} x \wedge u \simeq hz)\}.$$

Combining these we put

$$\text{(ix)} \qquad g(x,h) = (i_{\ulcorner\theta_1\urcorner}(w\, i_{\ulcorner\psi\urcorner}(x,h)), i_{\ulcorner\theta_2\urcorner}(w\, i_{\ulcorner\psi\urcorner}(x,h))),$$

so that (iii) can be rewritten as

$$\text{(iii)}' \quad \begin{cases} ho \simeq (N, =) & \text{and} \\ hx \simeq g(x,h) & \text{for } x \neq 0. \end{cases}$$

Such h is found directly by the recursion theorem. Then it is proved by induction on $\mathfrak{O}_S$ that for each $x \in \mathfrak{O}_S$, hx is a class with equality and $B_x = \{hz \mid z < z\}$ is a class of such classes, interrelated as required by (i).

A case from the definition (i) of special interest below is $\Omega_1^{(r)}$ which is $\mathfrak{O}_{\{(N,=)\}}$, i.e. is the closure under successor and countable suprema, starting with 0.

<u>Remark</u>. It is not claimed as might be expected that the $(\Omega_x^{(r)}, \equiv_x^{(r)}) \in S$ for $x \in \mathfrak{O}_S$. The reasons are twofold. First the parameter S is used in the definition (i) with the $<$ relation. Secondly, the classes $\mathfrak{O}_{B_x}$ are not obtained from given sets by inductive separation, but rather by full inductive comprehension. This leads us to the following:

<u>Question</u>. Is there a reasonable extension of $T_0(S)$ (or of $T_1(S)$) which has the same kinds of models as that theory and in which we have $(\Omega_x^{(r)}, \equiv_x^{(r)}) \in S$ for each $x \in \mathfrak{O}_S$?

(6) <u>The accessible number classes</u> Ω_x. Obviously if $y \leq x$ then $B_y \subseteq B_x$ in (5)(i) (here, inclusion under the identity injection). We thus have $\Omega_y^{(r)} \subseteq \Omega_x^{(r)}$ and the relation $(\equiv_y^{(r)})$ is contained in $(\equiv_x^{(r)})$. This permits us to define the number classes Ω_x as follows.

$$\text{(i)} \quad \begin{cases} \Omega_x = \Omega_x^{(r)} & \text{for } x \in N \\ \Omega_{x'} = \Omega_x^{(r)} & \text{for } x \notin N \text{ and } x \in \mathfrak{O}_S \\ \Omega_{\sup_a f} = \bigcup_{x \in a} \Omega_{fx}^{(r)} & \text{for } \sup_a f \in \mathfrak{O}_S . \end{cases}$$

Thus

$$\text{(ii)} \qquad \Omega_{\sup_a f} = \bigcup_{y < \sup_a f} \Omega_y^{(r)} = \bigcup_{y < \sup_a f} \Omega_y .$$

The relations $\equiv_x$ on Ω_x are defined correspondingly.

5.3 Interpretation in the models of $T_o(S)$, $T_1(S)$.

(1) Assignment of set-theoretic ordinals. Given any model $\mathcal{K}$ of $T_o(S)$ and any A in $\mathcal{K}$ which is a class of classes with equality relations we can inductively assign to each $x \in \mathfrak{O}_A$ a set-theoretic ordinal $|x|_{\mathfrak{O}_A}$, or simply $|x|$, by the expected conditions:

$$\text{(i)} \qquad |0| = 0, \quad |x'| = |x| + 1, \quad |\sup_a f| = \sup_{x \in a} |fx| .$$

Then it is proved by induction on W_A that

$$\text{(ii)} \quad x \leq_{\mathfrak{O}_A} y \Leftrightarrow |x| \leq |y|, \text{ hence } x <_{\mathfrak{O}_A} y \Leftrightarrow |x| < |y| \text{ and } x \equiv_{\mathfrak{O}_A} y \Leftrightarrow |x| = |y|,$$

for $x, y \in \mathfrak{O}_A$. It follows that

$$\text{(iii)} \quad a = (p_1 a, \equiv_a) \in A \wedge x, y \in a \wedge x \equiv_a y \wedge \sup f \in \mathfrak{O}_A \Rightarrow |fx| = |fy| .$$

Next, given any $B \subseteq \mathfrak{O}_A$ closed under successor, define

$$\text{(iv)} \qquad |B| = \sup_{x \in B} |x| .$$

We then obtain from 5.2 (5), (6)

$$\text{(v)} \quad \begin{cases} |\Omega_o^{(r)}| = \omega \\ |\Omega_x^{(r)}| = |\mathfrak{O}_{\{(\Omega_z^{(r)}, \equiv_z^{(r)}) | z < x\}}| \quad \text{for } x \in \mathfrak{O}_S, \ x \neq 0. \end{cases}$$

$$\text{(vi)} \quad \begin{cases} |\Omega_o| = \omega \\ |\Omega_{x'}| = |\mathfrak{O}_{\{(\Omega_z, \equiv_z) | z \leq x\}}|, \quad \text{for } x \in \mathfrak{O}_S \\ |\Omega_{\sup_a f}| = \sup_{x \in a} |\Omega_{fx}|, \quad \text{for } \sup_a f \in \mathfrak{O}_S . \end{cases}$$

(2) Interpretation in the full set-theoretical models. Let $\mathcal{K} = \mathfrak{m}^*_{\text{Set-Fun}}$, where $\mathfrak{m} = (V_{\alpha_o}, \in)$ for some inaccessible α_o. It may be seen in this case that

(i) $|\Theta_S|$ = the least inaccessible ordinal

and

(ii) $|\Omega_x| = \omega_{|x|}$ for each $x \in \Theta_S$,

while

(iii) $|\Omega_x^{(r)}|$ enumerates the accessible regular ordinals.

In the proof of $|\Omega_x| \leq \omega_{|x|}$ we use that every $a \in S$ represents a set $a/\equiv_a$ in M (3.6) and in the proof that $\omega_{|x|} \leq |\Omega_x|$ we use that every set-function in $\mathfrak{M}$ is represented by a function in $\mathcal{K}$ (3.4).

(3) Interpretation in the models. $Rec^*(\omega)$, $\mathfrak{M}^*_{Rec}$. In $Rec^*(\omega)$ all the functions met are partial recursive, so Ω_1 is just another form of the first Church-Kleene number class O_1 , and so

$$(i) \quad |\Omega_1| = \omega_1^c = \tau_1 ,$$

where τ_α lists the admissible (or recursively regular) ordinals [Ba]. Further it may be seen, e.g. by Richter [R] that

$$(ii) \quad |\Omega_n| = |\Omega_n^{(r)}| = \tau_n \text{ for } n \in N .$$

This suggests our first conjecture here:

$$(C_1) \qquad |\Omega_x^{(r)}| = \tau_{|x|} \quad \text{for each } x \in \Theta_S .$$

The second conjecture is that

$$(C_2) \qquad |\Theta_S| = \text{the least recursively inaccessible ordinal.}$$

The situation in $\mathfrak{M}^*_{Rec}$ should come out the same, using the homomorphic mapping of the applicative structure $Rec(\mathfrak{M})$ onto $Rec(\omega)$ from 3.3 and 3.4. This is clear for Ω_1 , but the details remain to be worked out for the higher number classes. We may read $(C_1),(C_2)$ equally well as conjectures applying to $\mathfrak{M}^*_{Rec}$.

(4) Interpretation in the models $\Xi^N\text{-Rec}^*(\omega)$, $\mathfrak{m}^*_{\Xi^N\text{-Rec}}$. In these cases all the functions are partial Π^1_1 and the total ones on N are hyperarithmetic. The version of O_1 obtained using hyperarithmetic functions gives no new ordinals so again we have

(i) $$|\Omega_1| = \omega_1^c = \tau_1 \quad \text{and} \quad |\Omega_n| = |\Omega_n^{(r)}| = \tau_n \quad \text{for} \quad n \in N .$$

One would expect the conjectures C_1 and C_2 to hold for the models $\Xi^N\text{-Rec}^*(\mathfrak{m})$, $\mathfrak{m}^*_{\Xi^N\text{-Rec}}$, if they hold in (3).

Remark. If the conjectures of (3) (or (4)) are correct, the development of 5.2 provides a conceptually superior way of introducing the recursively accessible ordinals and initial numbers: first, because it follows the set-theoretic pattern and second, because it provides a simultaneous generalization of both the classical and recursion-theoretic cases.

5.4 Further principles and applications.

(1) The ancestor relation in Ω_1. Define $\prec$ to be the least relation such that

(i) $x \prec x'$

(ii) $x \in N \wedge y \prec fx \Rightarrow y \prec \sup_N f$

(iii) $x \prec y \wedge y \prec z \Rightarrow x \prec z$.

Write $y \preccurlyeq x$ for $y \prec x \vee y = x$. It may be seen that we have a function e ("enumeration") so that

(iv) $$x \in \Omega_1 \Rightarrow ex : N \underset{\text{onto}}{\rightarrow} \{y \mid y \preccurlyeq x\}.$$

Further for $x \in \Omega_1$, $\{y \mid y \preccurlyeq x\}$ is linearly ordered by the $\prec$ relation, and finally

(v) $$x, y \in \Omega_1 \wedge y \leq x \Rightarrow \exists y_1 (y_1 \preccurlyeq x \wedge y \equiv y_1).$$

The relation $\prec$ corresponds to the r.e. relation fequently used on O_1.

(2) Consequences in $T_1(S)$. In this theory we have $\underline{e}_N$ which provides for quantification over N as an operation; using (1)(iv), this transfers uniformly to quantification over any set $\{y \mid y \preccurlyeq x\}$. Then further if $f: \Omega_1 \to V$ and $x_1 \equiv x_2 \Rightarrow fx_1 = fx_2$ we can decide (by an operation) whether $\exists y \leq x(fy = 0)$ since this is equivalent to $\exists y \preccurlyeq x\ (fy = 0)$. Hence bounded quantification on Ω_1 is decidable in this sense in $T_1(S)$.

(3) The selection axiom for Ω_1 is the following statement, which makes use of a new constant $\underline{c}$ ("choice"):

$$(\mathrm{Sel}_{\Omega_1}) \qquad x \in \Omega_1 \wedge fx \simeq 0 \Rightarrow (\underline{c}f) \in \Omega_1 \wedge f(\underline{c}f) \simeq 0.$$

We claim that for suitable choice of $\underline{c}$,

(i) Sel_{Ω_1} is true in $\exists^N\text{-Rec}^*(\omega)$ and in $\mathfrak{M}^*_{\exists^N\text{-Rec}}$

The reason is that in these models Ω_1 is a Π^1_1 set; then Sel_{Ω_1} is a consequence of Π^1_1-uniformization.

(4) Relations with the theory $T_1^{(\Omega)}$. The theory $T_1^{(\Omega)}$ set up in [F2] is an extension of T_1 in which the class Ω of ordinals is linearly ordered by $\prec$. It has a model in $\exists^N\text{-Rec}^*(\omega)$ or $\mathfrak{M}^*_{\exists^N\text{-Rec}}$ obtained by taking a Π^1_1 path thru O as a system of unique representatives. The theory $T_1(S) + (\mathrm{Sel}_{\Omega_1})$ can be used for all the same purposes as $T_1^{(\Omega)}$ (cf. (5) next). (We do not have an evident translation of the latter theory into the former since there are no obvious abstract means of defining a path through Ω_1 ordered by $\prec$.) As with the theory $T_1^{(\Omega)}$, a form of the continuum hypothesis CH is satisfied in the $\exists^N$-models, i.e. there is a map $g: \Omega_1 \xrightarrow[\text{onto}]{} N^N$ which takes equivalent ordinals onto equivalent functions. This is by a Π^1_1 enumeration of the hyperarithmetic functions.

(5) Actual and possible applications to model theory. It was shown in [F2] how certain portions of set-theoretical model theory dealing with models which are

countable or Ω_1 - enumerated can be carried out in $T_1^{(\Omega)} \pm (CH)$, thereby generalizing both the classical case and certain hyperarithmetic versions due to Cutland. The same can of course be done here using the theory $T_1(S) + (Sel_{\Omega_1}) \pm (CH)$. The advantage offered by the latter theory now is a systematic way of dealing with the higher number classes. Thus one would hope that this could be used as a means to draw further hyperarithmetic consequences. Promising areas of investigation would be Morley-Shelah theory, the logics $L(Q_\alpha)$, and the stationary logic of [B,K,M]. (I have learned from Barwise that his student E. Wimmers has recently given an abstract treatment of the logic $L(Q_1)$ in $T_1^{(\Omega)}$ which generalizes both the classical case and admissible versions found by Bruce and Keisler [Br, K].)

(6) <u>Possible applications to long hierarchies of ordinal functions.</u>
Bachmann had introduced hierarchies of normal "critical" functions on higher number classes which were used eventually to define "large" countable ordinals. In unpublished work by myself and Aczel, new and somewhat simplified hierarchies were proposed as substitutes. Match-up with the Bachmann ordinals in various cases was accomplished by Bridge [Br] and Buchholz [Bu]. Further, it was shown in these special cases that the countable ordinals generated are recursive, by detailed work with explicit order relations on the terms. One would like to obtain a theoretical reason for this outcome. It is possible that the development of the number classes Ω_x in $T_1(S)$ initiated in 5.2 provides a means to do this. The idea would be to show that the hierarchies in question can be established abstractly on the basis of these principles as a continuation of 5.2(5). Then under the interpretation of $T_1(S)$ in $\mathfrak{H}^N\text{-Rec}^*(\omega)$ or $\mathfrak{m}^*_{\mathfrak{H}^N\text{-Rec}}$, every specific countable ordinal generated is $< \omega_1^c$, hence recursive. Schütte [Sch] has isolated the set-theoretic properties of ordinals and functions which are sufficient for a development of the "long" hierarchies. This work may be a useful starting point for the proposal just made.

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