Tr. J. of Mathematics
18 (1994) 23 - 41.

Graph Moduli Spaces and Cohomology Operations

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Abstract

In this paper we define a moduli space of "Graph flows" in a manifold, and use them to define analogues of Donaldson invariants. These take values in tensor products of the cohomology and homology of the manifold and can be interpreted as generalized cohomology operations. We show how to construct classical invariants such as the Stiefel - Whitney classes and the Steenrod operations in this way. We also give homotopy theoretic descriptions of these invariants which will allow the definition of higher order Donaldson type invariants.

The polynomial invariants defined by Donaldson [4] have had a dramatic impact on four dimensional differential topology in recent years. The simplest, zero degree invariant $q_0(X)$, of a closed, simply connected, smooth Riemannian four manifold X, is an integer which is given by counting (with sign) the components of $\mathcal{M}^0(X)$, the zero dimensional moduli space of anti- self dual connections on an $SU(2)$ bundle over X. (Recall that the dimension of $\mathcal{M}_k(X)$, the moduli space of ASD connections on the principal $SU(2)$ - bundle of Chern class $k \in H^4(X) \cong \mathbb{Z}$, is $8k-3(1+b^+(X))$ where $b^+(X)$ is the rank of the maximal positive definite subspace of the intersection form.)

For manifolds with boundary, the analogous invariants take values in the Floer (co)homology of the boundary. For example, if X is a four manifold with boundary a homology 3 - sphere, say $\partial X = Y$, then the zero degree invariant $q_0(X) \in HF_*(Y)$. Roughly, this invariant is defined as follows.

Given a connection $\gamma \in \mathcal{M}^0(X)$ with finite Yang - Mills energy, then it approaches a flat connection, say $\rho(\gamma)$, on the trivial SU(2) bundle over Y. After suitable perturbations of the metric this limiting connection ρ can be viewed as an element of the chain complex $CF_*(Y)$, of Floer's "instanton homology" of Y. (A sign is assigned to the flat connection coming from orientation considerations.) One can then define the resulting Donaldson invariant by

$$q_0(X) = \sum_{\gamma \in \mathcal{M}^0(X)} \rho(\gamma) \in CF_*(Y).$$

The second author was partially supported by a grant from the NSF

It is proved in [5] that this class is a cycle in the Floer chain complex and determines an element

$$q_0(X) \in HF_*(Y)$$

which is independent of the choices of perturbations and metrics, and is an invariant of the smooth structure on X. (Actually whether $q_0(X)$ lies in Floer homology or Floer *cohomology* depends on choice of normal vector field of Y.)

Now if X is an oriented four manifold with several boundary components, say

$$\partial X = Y_1 \sqcup \cdots \sqcup Y_k$$

where the Y_i's are homology spheres, then the Donaldson invariant takes values in the tensor product

$$q_0(X) \in HF^*(Y_1) \otimes \cdots \otimes HF^*(Y_j) \otimes HF_*(Y_{j+1}) \otimes \cdots \otimes HF_*(Y_k)$$

where $Y_1, \cdots Y_j$ are the boundary components whose normal bundles are oriented with an *inward* pointing vector field, and $Y_{j+1} \cdots Y_k$ are the boundary components oriented with *outward* pointing vector fields. When we take field coefficients and use the identity

$$HF^* \otimes HF_* \cong Hom(HF^*, HF^*)$$

we can think of these Donaldson invariants as operations in Floer cohomology.

Now similar invariants exist in the setting of pseudo-holomorphic curves in a symplectic manifold. These were first defined in [8] [7]. More specifically, let (M^{2n}, ω), be a closed, simply connected symplectic manifold. Let Σ be a Riemann surface with k - boundary components. Then let $\mathcal{M}(\Sigma, M)$ be the moduli space of pseudo - holomorphic maps

$$\gamma : \Sigma \longrightarrow M$$

which when restricted to a boundary component circle is a constant loop in M. Let $\mathcal{M}^0$ denote the zero dimensional component of this space. After suitable perturbations, an element $\gamma \in \mathcal{M}^0$ determines an element of the tensor product of Floer chain complexes,

$$\rho(\gamma) \in CF_*(M)^{\otimes k}$$

given by the restriction of γ to the boundary components. Here CF_* is the Floer chain complex of the symplectic action functional on the loop space $L(M)$. Again, the Donaldson invariant is defined by counting with sign

$$q_0(\Sigma, M) = \sum_{\gamma \in \mathcal{M}^0} \rho(\gamma) \in CF_*(M)^{\otimes k}.$$

which is shown to be a cycle in [1], and represents an element

$$q_0(\Sigma, M) \in HF_*(M)^{\otimes k}$$

which is an invariant of the symplectic structure on M. Again, this invariant may be interpreted as an operation on the Floer cohomology. This point of view was introduced by Witten in [8] in which he used this invariant to construct cup product structures on the Floer cohomology of the symplectic action on the loop space $L\mathbb{C}P^n$.

Our goal is to understand in more depth the relationship between Donaldson invariants, and more generally, the homotopy type of thes types of moduli spaces "with ends", and the Floer homology (and more generally the "Floer homotopy type" in the sense of [3]) of the ends.

In order to understand the basic structure of this relationship we will examine an example of a "moduli space with ends" where the resulting "Floer cohomology" of the ends is usual cohomology. This is a moduli space of graphs in a closed manifold, which was defined and studied in [1]. We will describe some of the results of [1] here and expand upon them. In particular we will show how the classical operations in the cohomology of a manifold, including characteristic classes and the Steenrod operations can be obtained as Donaldson invariants in this theory. We will also describe a homotopy theoretic approach to these Donaldson invariants which will allow us to view secondary (and higher order) Steenrod operations as invariants defined by these moduli spaces. The goal is to use this approach to define secondary and higher order Donaldson invariants for four manifolds and symplectic manifolds when certain primary Donaldson invariants vanish. This will be studied in a future paper.

This paper is organized as follows. In section one we will define the moduli space of graphs, the resulting Donaldson invariants, and prove a basic gluing theorem. Some examples will also be discussed. In section two we show how to produce equivariant analogues of these invariants using the symmetry groups of these graphs and show how the Steenrod operations are examples. These two sections summarize some of the results of the first author in his Ph.D thesis [1]. In section three we describe the beginning of a joint project in which we apply the categorical and homotopy theoretic viewpoint of Morse theory and Floer theory developed by the second author, J. Jones, and G. Segal [2], [3] to study these invariants and to define higher order invariants.

The second author would like to thank the organizers of the Gokova conference for the opportunity to participate in an exciting conference in a beautiful part of the world.

1. The Moduli Space of Graph Flows

Let M be a closed, compact, smooth Riemannian manifold of dimension d, and let Let Γ be an oriented, finite, possibly non-compact, graph with m edges parametrized by $[0,1]$, $(-\infty, 0]$, and $[0, \infty)$. We call these edges "internal", incoming", and "outgoing" respectively. Let these edges be indexed $\{E_1, ..., E_m\}$ such that the first n are noncompact, and the rest are internal. Among the n noncompact edges the first n_1 are assumed to be incoming, the next $n_2 = n - n_1$ are are assumed to be outgoing. In this section we define the moduli space $\mathcal{M}(\Gamma, M)$ of "graph flows", and use these to define the analogues of Donaldson invariants mentioned in the introduction. We show that these invariants can be viewed as operations in $H^*(M)$, and compute some examples.

We begin by defining the notion of an *M-structure* for our graph Γ. The space of all such M-structures will play a significant roll in our constructions.

Definition 1. *Fix an oriented, parameterized graph Γ and a closed Riemannian manifold M as above. An M-structure σ on Γ consists of the following:*

1. A real number ℓ_i associated to each internal edge of E_i of Γ. We think of ℓ_i as the length of E_i, even though we allow $\ell_i \leq 0$.

2. A function $f_i \in C^\infty(M)$ associated to each edge E_i of Γ. We assume the f_i's are distinct.

The space of all M-structures will be denoted $\mathcal{S}(\Gamma, M)$. Notice that there is a homeomorphism

$$\mathcal{S}(\Gamma, M) \cong \mathbb{R}^{m-n} \times F_m(C^\infty(M))$$

where $F_m(X) \subset X^m$ is the configuration space of m distinct points in X.

For fixed choice of such a structure σ, we are now ready to define the moduli space $\mathcal{M}_\sigma(\Gamma, M)$ of "Γ - flows in M".

let $\gamma : \Gamma \to M$ be a continuous map, smooth on the edges. For each internal edge E_i let $\gamma_i : [0,1] \to M$ be the restriction of γ to E_i composed with the parameterization of E_i by $[0,1]$ given as part of the data of Γ. For the incoming and outgoing edges we define $\gamma_i : (-\infty, 0] \to M$ or $\gamma_i : [0, \infty) \to M$ similarly.

Definition 2. *γ lies in $\mathcal{M}_\sigma(\Gamma, M)$ if and only if for each edge E_i it satisfies the differential equation*

$$d\gamma_i/dt + \ \ell_i \ \nabla f_i = 0.$$

For the noncompact edges (i.e the incoming and outgoing edges) in this equation ℓ_i is assumed to be 1. Here ∇f_i is the gradient vector field. $\mathcal{M}_\sigma(\Gamma, M)$ is topologized as a subspace of $C^0(\Gamma, M)$.

We let $\mathcal{M}(\Gamma, M)$ be the union of the spaces $\mathcal{M}_\sigma(\Gamma, M)$ where the structures σ vary in $\mathcal{S}(\Gamma, M)$. $\mathcal{M}(\Gamma, M)$ is topologized so that natural the projection map

$$\pi : \mathcal{M}(\Gamma, M) \to \mathcal{S}(\Gamma, M)$$

is continuous.

Given $P \subset \mathcal{S}(\Gamma, M)$, let $\mathcal{M}_P(\Gamma, M) = \pi^{-1}(P)$. These spaces will be important in general, but in this chapter we restrict ourselves to studying $\mathcal{M}_\sigma(\Gamma, M)$, the moduli space associated to a single structure.

We now describe some basic properties of these moduli spaces. These properties are analogues of properties of moduli spaces of anti-self dual connections and of pseudo-holomorphic curves, and have similar (in fact easier) proofs. See [1] for details.

Again, fix a structure $\sigma \in \mathcal{S}(\Gamma, M)$. This defines a vector of labelling functions of the edges. Let $f = (f_1, ...f_n)$ be the n - tuple of functions labelling the noncompact edges. Observe that every $\gamma \in \mathcal{M}_\sigma(\Gamma, M)$ has the property that its restriction to each noncompact edge γ_i is a gradient flow line, so it therefore converges to a critical point, say a_i, of the function f_i. Thus γ can be associated to an n - tuple $\vec{a} = (a_1, \cdots, a_n)$ where a_i is a critical point of f_i. For a fixed n - tuple $\vec{a}$, let

$$\mathcal{M}_\sigma(\Gamma, M; \vec{a}) \subset \mathcal{M}_\sigma(\Gamma, M)$$

be the subspace of those $\gamma \in \mathcal{M}_\sigma(\Gamma, M)$ which converge on the ith edge to the critical point a_i.

Theorem 1. *For a generic choice of structure $\sigma \in \mathcal{S}(\Gamma, M)$, the moduli spaces $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$ are manifolds for every n- tuple of critical points $\vec{a}$. The dimension of $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$ is given by the formula*

$$dim(\mathcal{M}_\sigma(\Gamma, M; \vec{a})) = \textstyle\sum_{i=1}^{n_1}[index(a_i)] - \sum_{i=1}^{n_2}[index(a_{n_1+i})] - d(n_1 - 1) \\ - d \cdot dim(H_1(\Gamma, \mathbb{R}))$$

where, as above, n_1 and n_2 are the number of incoming and outgoing edges of Γ respectively. Furthermore an orientation on the manifold M induces orientations on the moduli spaces $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$.

The generic condition on the structure σ in this theorem is that the labelling functions f_i are Morse, and satisfy the "Morse - Smale transversality properties". That is, the stable and unstable manifolds of the critical points all intersect transversally.

Fix a structure σ satisfying this generic property. We will now construct a natural compactification of the space $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$. To do this we first recall the natural compactification of the space of gradient flow lines of a Morse function converging to two fixed critical points. We will refer to the space of flow-lines from critical point a_i to critical point b_i by $\mathcal{M}_\sigma(a_i, b_i)$. The following is a standard result in classical Morse theory. See [2] for example.

Lemma 2. *Let $\overline{\mathcal{M}}(a, b)$ denote the space of "piecewise flow lines" connecting connecting critical points a and b. That is*

$$\overline{\mathcal{M}}(a, b) = \bigcup_{a = a_0 > a_1 > \ldots > a_j = b} \mathcal{M}(a, a_1) \times \ldots \times \mathcal{M}(a_{j-1}, b),$$

where the union is taken over decreasing finite sequences of critical points. (The partial ordering is defined by $\alpha \geq \beta$ iff $\mathcal{M}(\alpha, \beta)$ is nonempty.) Then $\overline{\mathcal{M}}(a, b)$ is compact and contains $\mathcal{M}(a, b)$ as an open dense subspace.

There is a similar compactification for the moduli spaces of Γ - flows. Namely, let

$$\overline{\mathcal{M}}_\sigma(\Gamma, M; \vec{a}) = \bigcup_{\vec{b}} \mathcal{M}_\sigma(\Gamma, M; \vec{b}) \times \overline{\mathcal{M}}_\sigma(b_1, a_1) \times \cdots \times \overline{\mathcal{M}}_\sigma(a_n, b_n).$$

Whether we use $\overline{\mathcal{M}}_s(b_i, a_i)$ or $\overline{\mathcal{M}}_s(a_i, b_i)$ in the above union depends on whether the ith edge is incoming or outgoing. $\overline{\mathcal{M}}_\sigma(\Gamma, M; \vec{a})$ consists of Γ - flows that are allowed to be piecewise flows on the noncompact edges. We refer to these as "piecewise Γ - flows". There is an obvious way to topologize $\overline{\mathcal{M}}_\sigma(\Gamma, M; \vec{a})$. The proof of the following is simply an adaptation of the proof of the above lemma, and is carried out in [1].

Theorem 3. *The space $\overline{\mathcal{M}}_\sigma(\Gamma, M; \vec{a})$ is compact and contains $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$ as an open dense subspace.*

This result can be improved in such a way so as to identify the ends of the moduli space $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$. To do this we set up the following notation. For n - tuples of critical points $\vec{a}$ and $\vec{b}$ associated to the structure σ, consider the oriented spaces of flow lines

$$\begin{aligned} \mathcal{M}_i &= \mathcal{M}_{f_i}(b_i, a_i) \quad \text{for incoming} \quad E_i \\ \mathcal{M}_i &= \mathcal{M}_{f_i}(a_i, b_i) \quad \text{for outgoing} \quad E_i. \end{aligned}$$

Theorem 4. *There exist "gluing" maps*

$$\Phi_{\vec{a}, \vec{b}} : \overline{\mathcal{M}}_\sigma(\Gamma, M; \vec{a}) \times \prod_{a_i \neq b_i} \overline{\mathcal{M}}_i \times [0, 1) \to \overline{\mathcal{M}}_\sigma(\Gamma, M; \vec{b}),$$

that are orientation preserving homeomorphisms onto disjoint images. Moreover the complement of the images,

$$\overline{\mathcal{M}}_\sigma(\Gamma, M; \vec{b}) - \bigcup_{\vec{a}} \Phi_{\vec{a}, \vec{b}}$$

is compact.

In this paper we will be primarily concerned with the moduli spaces of dimension zero and one, $\mathcal{M}^0_\sigma(\Gamma, M; \vec{a})$ and $\mathcal{M}^1_\sigma(\Gamma, M; \vec{a})$. These theorems tell us that $\mathcal{M}^0_\sigma(\Gamma, M; \vec{a}) = \overline{\mathcal{M}}^0_\sigma(\Gamma, M; \vec{a})$ is a finite set of points with signs (orientation). Moreover if an end of one of these isolated Γ - flows glues to an isolated flow line, then the pair forms one end of a compact interval of Γ - flows. The other end of this interval is modelled by another such pair.

This information will allow us to define a Donaldson - type invariant for these moduli spaces, which we now proceed to do.

Fix a generic structure $\sigma \in \mathcal{S}(\Gamma, M)$ as above. Given our Morse-Smale functions f_i, let $C_*(M, f_i)$ be the associated Morse-Smale chain complex generated by the critical points, and let $C^*(M, f_i)$ be the dual cochain complex.

We define a class $q(\Gamma, M)$ to be an element of the complex

$$\bigotimes_{1\leq i\leq n_1} C^*(M, f_i) \bigotimes_{n_1+1\leq i\leq n} C_*(M, f_i)$$

in the following manner. Consider those n - tuples of critical points $\vec{a}$ such that $dim(\mathcal{M}_\sigma(\Gamma, M; \vec{a})) = 0$. These spaces contain a finite number of oriented points which can be counted with sign (if M is oriented - otherwise this is well defined mod 2, and we take coefficients to be $\mathbb{Z}_2$).

Definition 3.

$$q(\Gamma, M) = \sum \#\mathcal{M}_\sigma(\Gamma, M; \vec{a})[\vec{a}] \in \bigotimes_{1\leq i\leq n_1} C^*(M, f_i) \bigotimes_{n_1+1\leq i\leq n} C_*(M, f_i).$$

Using the gluing theorem above and the definition of the boundary and coboundary operators in the Morse-Smale complex, one can show the following (see [1]):

Lemma 5.

$$dq = 0.$$

We shall therefore view $q(\Gamma, M)$ as an element of the associated homology,

$$q(\Gamma, M) \in H^*(M)^{\otimes n_1} \otimes H_*(M)^{\otimes n_2}.$$

The following says that $q(\Gamma, M)$ is indeed an invariant of M.

Theorem 6. *The homology class $q(\Gamma, M)$ does not depend on the choice of structure $\sigma \in \mathcal{S}(\Gamma, M)$.*

Sketch of Proof: Since the space of generic structures inside $\mathcal{S}(\Gamma, M)$ is connected we can find curves connecting any two generic structures. The induced paths give chain homotopy equivalences that preserve the $q(\Gamma, M)$'s $\square$

We now describe four basic examples of these invariants.

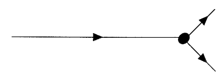
Example 1. $\Gamma =$

In this case $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$ has dimension zero if and only if $\vec{a} = (a)$ is a maximum. Thus $q(\Gamma, M) \in H_d(M)$, and it can easily be seen to be the fundamental class. (Coefficients should be taken in $\mathbb{Z}_2$ if M is not orientable).

Example 2. $\Gamma =$

In this case $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$ has dimension $ind(a_1) + ind(a_2) - d$, where $\vec{a} = (a_1, a_2)$. Thus $q(\Gamma, M) \in \oplus_q H^q(M) \otimes H^{d-q}(M)$, which defines an element in $\oplus_q Hom(H^q(M), H_{d-q}(M))$. This is the Poincare duality isomorphism, given by taking the cap product with the fundamental class.

Example 3. $\Gamma =$

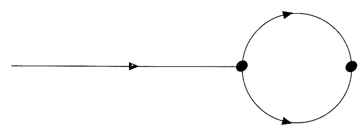


In this case $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$ has dimension $ind(a_1) - ind(a_2) - ind(a_3)$, where $\vec{a} = (a_1, a_2, a_3)$. Thus

$$q(\Gamma, M) \in \oplus_{r \leq k} H^k(M) \otimes H_r(M) \otimes H_{k-r}(M)$$

and defines an element in $\oplus_{r \leq k} Hom(H^r(M) \otimes H^{k-r}(M), H^k(M))$. This is the cup product operation.

Example 4. $\Gamma =$



In this case $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$ has dimension $ind(a) - d$, where $\vec{a} = (a)$. Thus $q(\Gamma, M) \in H^d(M)$. It is easily seen to be the Euler class (or Stiefel - Whitney class w_d if M is not orientable).

We end this section by discussing some basic structure properties of the invariants $q(\Gamma, M)$. (See [1] for details.) In particular the following three results say that the four examples above can be used to compute the invariant for any graph.

Proposition 7. *If Γ_1 and Γ_2 are homotopy equivalent via a homotopy that preserves orientations on their end, then $q(\Gamma_1, M) = q(\Gamma_2, M)$.*

Now let Γ_1 and Γ_2 be oriented graphs. Let $\Gamma_{1,2}^{i\#j}$ be the oriented graph obtained by gluing incoming edge i of Γ_1 to outgoing edge j of Γ_2.

Proposition 8.

$$q(\Gamma_{1,2}^{i\#j}, M) = q(\Gamma_1, M) \Diamond^{i,j} q(\Gamma_2, M),$$

where $\Diamond^{i,j}$ denotes tensorial contraction of cohomology in the ith coordinate with homology in the jth coordinate.

Corollary 9. *Changing the orientation of a non-compact edge induces the Poincare duality isomorphism on the relevent tensor coordinate of the invariant* $q(\Gamma, M)$.

2. The Equivariant Invariants

In this section we will generalize the construction of section one to moduli spaces that are associated to families of M-structures. The invariants that we define will be elements of an equivariant (co)homology of the n-fold product of M. In basic cases we extract operations from $H^*(M)$ to $H^*(M)$ which are associated to elements of the homology of the orientation preserving symmetry group of Γ, Σ_Γ. In particular we will show how the classical Steenrod operations arise this way. The details of the results in this section can be found in [1].

Let M be a closed Riemannian manifold of dimension d and Γ an oriented graph as in the last secton. Recall the projection onto the space of structures

$$\pi : \mathcal{M}(\Gamma, M) \to \mathcal{S}(\Gamma, M).$$

The group of automorphisms of the oriented graph Σ_Γ acts naturally on both $\mathcal{M}(\Gamma, M)$ and $\mathcal{S}(\Gamma, M)$ and the map π is equivariant. Notice that these actions are free because a structure associates distinct functions to different edges. Furthermore, since $\mathcal{S}(\Gamma, M)$ is contractible, $\mathcal{S}(\Gamma, M)/\Sigma_\Gamma$ is a classifying space $B\Sigma_\Gamma$.

We will be considering the induced map on orbit spaces

$$\pi : \mathcal{M}(\Gamma, M)/\Sigma_\Gamma \to \mathcal{S}(\Gamma, M)/\Sigma_\Gamma \simeq B\Sigma_\Gamma.$$

We will be interested in the pull back under π of certain families of structures. We choose to find these families in smaller structure spaces which are easier to deal with.

Let $f : M \to \mathbb{R}$ be a Morse-Smale function and let $U_f \subset C^\infty(M)$ be a contractible neighborhood of f that contains only Morse-Smale functions. Consider the structure space $\mathcal{S}_f(\Gamma, M)$ defined by only allowing functions from this smaller set, U_f. The Σ_Γ action restricts to $\mathcal{S}_f(\Gamma, M) \subset \mathcal{S}(\Gamma, M)$, so the quotient $\mathcal{S}_f(\Gamma, M)/\Sigma_\Gamma$ is well defined. We note that the inclusion of $\mathcal{S}_f(\Gamma, M)/\Sigma_\Gamma$ in $\mathcal{S}(\Gamma, M)/\Sigma_\Gamma$ is a homotopy equivalence and both spaces are homotopy equivalent to $B\Sigma_\Gamma$.

Each Morse-Smale function $g \in U_f$ has a set of isolated critical points which we call $Crit(g)$. Using a contraction of the neighborhood U_f one can define a fixed bijective correspondence $\phi_g : Crit(g) \to Crit(f)$.

As in section one we distinguish elements of a moduli space, $\mathcal{M}_f(\Gamma, M) = \pi^{-1}(\mathcal{S}_f(\Gamma, M)/\Sigma_\Gamma)$, by the asymptotic behavior of each Γ-flow along its ends. To do this notice that the sets of critical points to which a Γ - flow converge are permuted by the Σ_Γ action. Given $\gamma \in \mathcal{M}_f(\Gamma, M)$, let $\vec{a}$ denote the orbit of the set of critical points. We call this the critical orbit of γ. Let

$$\mathcal{M}_f(\Gamma, M; \vec{a}) \subset \mathcal{M}_f(\Gamma, M)/\Sigma_\Gamma$$

be the subspace of those $\gamma \in \mathcal{M}_f(\Gamma, M)$ whose critical orbits correspond to $\vec{a}$ under the correspondence ϕ above.

Now given a singular i-simplex, $\delta : \Delta^i \to \mathcal{S}_f(\Gamma, M)/\Sigma_\Gamma$, define the pullback moduli space

$$\mathcal{M}_\delta(\Gamma, M; \vec{a}) = \delta^*(\mathcal{M}_f(\Gamma, M; \vec{a}))$$

which projects onto the standard i-simplex, Δ^i.

The following are basic properties of these moduli spaces, and are analogous to ones proved in section one.

Theorem 10. *For a generic singular i-simplex $\delta : \Delta^i \to \mathcal{S}_f(\Gamma, M)/\Sigma_\Gamma$, the moduli space $\mathcal{M}_\delta(\Gamma, M; \vec{a})$ is a manifold. The dimension of $\mathcal{M}_\delta(\Gamma, M; \vec{a})$ is given by the formula*

$$dim(\mathcal{M}_\delta(\Gamma, M; \vec{a})) = dim(\mathcal{M}_\sigma(\Gamma, M; \vec{a})) + i.$$

Furthermore, orientations of the manifold M and of the simplex Δ^i induce orientations on the moduli spaces $\mathcal{M}_\delta(\Gamma, M; \vec{a})$.

Lemma 11. *The zero dimensional moduli spaces $\mathcal{M}^0_\delta(\Gamma, M; \vec{a})$ are compact, and thus finite.*

Let $\#\mathcal{M}^0_\delta(\Gamma, M; \vec{a})$ be the number of points in the moduli space counted with $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients. We must count with $\mathbb{Z}_2$ coefficients when the moduli space is not oriented. We will use these moduli spaces to construct $Q(\Gamma, M)$, an element of

$$H^*(E\Sigma_\Gamma \times_{\Sigma_\Gamma} (M)^{n_1} \times (M)^{n_2}),$$

where Σ_Γ acts on $(M)^{n_1} \times (M)^{n_2}$ via the map

$$\Sigma_\Gamma \longrightarrow \Sigma_{n_1} \times \Sigma_{n_2}$$

which assigns to a symmetry the associated permutation of the ends.

First note that the inclusion of any finite subcomplex P of $\mathcal{S}_f(\Gamma, M)/\Sigma_\Gamma$ can be perturbed so that each simplex is generic in the sense of Theorem 1. This complex is covered by another complex $\tilde{P} \subset \mathcal{S}_f(\Gamma, M)$. Over $\mathcal{S}_f(\Gamma, M)$, each Γ - flow γ as associated to it (via the correspondence ϕ) a well defined n-tuple of critical points of f.

We can then define the class

$$Q_P(\Gamma, M) \in Hom_{\Sigma_\Gamma}(C_*(\tilde{P}); C^*(M, f)^{\otimes n_1} \otimes C_*(M, f)^{\otimes n_2})$$

in the following manner. For each simplex δ in $\tilde{P}$ consider those n - tuples of critical points $\vec{a}$ such that $dim\, \mathcal{M}_\delta(\Gamma, M; \vec{a}) = 0$. These spaces contain a finite number of points which can be counted with sign (if M is not we count mod 2). Taking all these simplices together, we construct the desired element $Q_P(\Gamma, M)$.

Definition 4.

$$Q_P(\Gamma, M)(\delta) = \sum \#\mathcal{M}^0_\delta(\Gamma, M; \vec{a})[\vec{a}] \in C^*(M, f)^{\otimes n_1} \otimes C_*(M, f)^{\otimes n_2}.$$

$Q_P(\Gamma, M)$ is clearly an equivariant homomorphism and so lies in

$$Hom_{\Sigma_\Gamma}(C_*(\tilde{P}); C^*(M, f)^{\otimes n_1} \otimes C_*(M, f)^{\otimes n_2}).$$

This (tri) graded group is a cochain complex (i.e has a natural coboundary operator). The analogue of Lemma 2 in section one is the following.

Lemma 12. *$Q_P(\Gamma, M)$ is a Σ_Γ invariant cocycle. Thus it represents a cohomology class*

$$Q_P(\Gamma, M) \in H^*(\tilde{P} \times_{\Sigma_\Gamma} (M)^{n_1} \times (M)^{n_2}).$$

Since this construction is valid for every finite subcomplex of $\mathcal{S}_f(\Gamma, M)/\Sigma_\Gamma$ and is easily seen not to depend on the choices of perturbations, this process actually defines a cohomology class

$$\begin{aligned} Q_P(\Gamma, M) &\in H^*(\mathcal{S}_f(\Gamma, M) \times_{\Sigma_\Gamma} (M)^{n_1} \times (M)^{n_2}) \\ &= H^*(E\Sigma_\Gamma \times_{\Sigma_\Gamma} (M)^{n_1} \times (M)^{n_2}). \end{aligned}$$

Notice that this class does not depend on the original function f. In order to produce cohomology operations from the class $Q(\Gamma, M)$ we apply it to elements in $H_*(B\Sigma_\Gamma)$.

Definition 5. *For a class $\alpha \in H_*(B\Sigma_\Gamma)$ define the invariant $q_\alpha(\Gamma, M)$ to be the equivariant homology class,*

$$q_\alpha(\Gamma, M) \in H^*_{\Sigma_{n_1}}((M)^{n_1}) \otimes H^{\Sigma_{n_2}}_*((M)^{n_2})$$

given by evaluating $Q(\Gamma, M)$ on α and using the homomorphism $\Sigma_\Gamma \to \Sigma_{n_1} \times \Sigma_{n_2}$ described above.

Note that the same construction works for any subgroup $G \subset \Sigma_\Gamma$. The invariants of section one are those that arise when G is trivial.

We now describe two examples of these equivariant invariants.

Example 1. Let Γ be the graph used in section one to obtain the Euler class. The symmetry group $\Sigma_\Gamma = \mathbb{Z}_2$ which acts by exchanging the two internal edges. Since the group acts trivially on the noncompact end, the class $Q(\Gamma, M)$ takes its value in the $\mathbb{Z}_2$ cohomology of the product, $B\mathbb{Z}_2 \times M$. Thus for each generator

$$\alpha_i \in H_i(B\mathbb{Z}_2; \mathbb{Z}_2) \cong \mathbb{Z}_2$$

we have $q_{\alpha_i}(\Gamma, M) \in H^{d-i}(M;\mathbb{Z}_2)$. It is not difficult to see that this is the Stiefel-Whitney class $\omega_{d-i}(M)$.

Example 2. Let Γ be the 3-ended graph used to obtain the cup product in section 1. Again the symmetry group $\Sigma_\Gamma = \mathbb{Z}_2$ which acts by exchanging the two outgoing ends. In this case for the class $\alpha_i \in H_i(B\mathbb{Z}_2;\mathbb{Z}_2)$ the invariant

$$q_{\alpha_i}(\Gamma, M) \in \oplus_j \left(H_j^{\mathbb{Z}_2}(M \times M;\mathbb{Z}_2) \otimes H^{j-i}(M;\mathbb{Z}_2) \right)$$

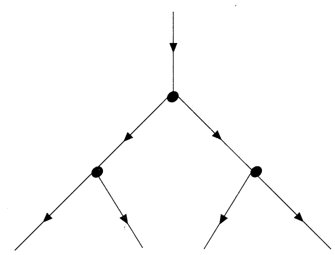
and so may be viewed as a homomorphism

$$q_{\alpha_i}(\Gamma, M) : H^j_{\mathbb{Z}_2}(M \times M;\mathbb{Z}_2) \longrightarrow H^{j-i}(M;\mathbb{Z}_2).$$

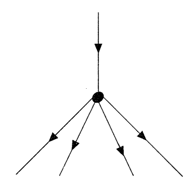
This is the *cup* $- i$ product. The Steenrod squaring operation Sq^{k-i} occurs when precomposing this operation with the natural inclusion

$$\begin{array}{lcl} H^k(M;\mathbb{Z}_2) & \longrightarrow & H^{2k}_{\mathbb{Z}_2}(M \times M;\mathbb{Z}_2) \\ \beta & \longrightarrow & \beta \otimes \beta. \end{array}$$

The composition of two Steenrod squares can be obtained here by considering a graph of the following type,



Here the symmetry group is the dihedral group D_4. It is easy to see that the moduli space of this graph is homotopy equivalent to the moduli space of the following graph



This graph has larger symmetry group, Σ_4. Thus the above dihedral group invariant factors through the symmetric group invariant. This produces the Adem relations among the Steenrod squares.

A similar situation occurs when we compose the Steenrod squares with the cup product. In this case we need only consider the $\mathbb{Z}_2$ action on the first vertex of the big graph. This gives the operations $Sq^i(\alpha \cup \beta)$. Shrink the two internal edges and consider the larger symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$. The original symmetry maps into this one by the diagonal map. The kernel of the associated homology map is exactly the Cartan relations.

3. A Categorical Approach

In this section we describe a more homotopy theoretic way of viewing the invariants. We describe categories $\mathcal{C}_{\mathcal{M}(\Gamma,M)}$ and $\mathcal{C}_{\mathcal{E}}$ that have as their classifying spaces the moduli space $\mathcal{M}(\Gamma, M)$ and the space of ends of $\mathcal{M}(\Gamma, M)$ respectively. We then define an equivariant "ends" functor

$$\mathcal{E} : \mathcal{C}_{\mathcal{M}(\Gamma,M)} \longrightarrow \mathcal{C}_{\mathcal{E}}$$

whose equivariant homological properties determine the invariants $q(\Gamma, M)$. The feature of studying the invariants this way is that it gives a clear method of calculation, and the homotopy nature of this definition will allow for the definition of higher order invariants defined when the primary invariants $q(\Gamma, M)$ vanish. The details of the constructions in this section and the proofs of their properties will appear elsewhere in due course.

We begin by recalling from [2] a categorical approach to Morse theory. The constructions in this section may be viewed as generalizations (or applications) of the results in [2].

Let $f : M \to \mathbb{R}$ be a smooth map on a closed, finite dimensional Riemannian manifold. Let $\mathcal{C}_f$ be the topological category whose objects are the critical points of f, and the morphisms between two critical points $Mor(a, b)$ is given by

$$Mor(a, b) = \overline{\mathcal{M}}(a, b),$$

the compactification of the space of gradient flow lines given by "piecewise flow lines" as discussed in section one. Let $B\mathcal{C}_f$ be the classifying space of this category. This is a standard construction which defines a simplicial space which one k - simplex for every k -tuple of composable morphisms. The following is the main result of [2].

Theorem 13. *There is a natural homotopy equivalence*

$$\phi : B\mathcal{C}_f \simeq M.$$

For a generic f, (i.e a Morse function satisfying the Morse - Smale transversality conditions) ϕ is a homeomorphism.

This theorem will be useful when we are dealing with the moduli spaces with a fixed structure, $\mathcal{M}_\sigma(\Gamma, M)$. However, as seen in the last section we often need to allow the structures to vary. For this we will need the following corollary of this theorem.

Given a closed manifold M, let $\mathcal{C}_M$ denote the topological category whose objects are pairs (f, a), where $f : M \to \mathbb{R}$ is a smooth function and $a \in M$ is a critical point of f. For morphisms we set

$$Mor((f_1, a_1), (f_2, a_2)) = \begin{cases} \emptyset & \text{if} \quad f_1 \neq f_2, \\ \overline{\mathcal{M}}_f(a, b) & \text{if} \quad f_1 = f_2 \end{cases}$$

where $\overline{\mathcal{M}}_f(a,b)$ is the space of piecewise gradient flow lines of the function $f = f_1 = f_2$. We then have the following result.

Corollary 14. *There is a natural homotopy equivalence*

$$B\mathcal{C}_M \simeq M.$$

Sketch of Proof. Let $C^\infty(M)$ denote the constant category whose objects are smooth functions on M. (By constant we mean a category whose only morphisms are the identity morphisms.) There is a clear functor

$$\psi : \mathcal{C}_M \to C^\infty(M)$$

defined on objects by sending (f,a) to f. The fiber category at any object f is $\mathcal{C}_f$. By the theorem, on the classifying space level all these fibers are homotopy equivalent to M. This will be the key fact in proving that ψ is a quasi- fibration of categories. That is, on the classifying space level $\psi : B\mathcal{C}_M \to C^\infty(M)$ is a quasi-fibration. Since $C^\infty(M)$ is contractible, the result will follow $\square$

We will now use these results to study the moduli space $\mathcal{M}(\Gamma, M)$ and the invariants $q(\Gamma, M)$ categorically.

First, consider the compactified moduli space $\overline{\mathcal{M}}(\Gamma, M)$ described in section one. Recall that an element $\gamma \in \overline{\mathcal{M}}(\Gamma, M)$ is a continuous map $\gamma : \Gamma \to M$ and a M - structure σ, so that when restricted to any internal edge γ is a gradient flow line, and restricted to any noncompact edge γ is a *piecewise* flow with respect to the structure σ. Recall that we called such elements "piecewise Γ - flows".

Notice that when viewed simply as a set, there is a natural partial ordering on the elements of $\overline{\mathcal{M}}(\Gamma, M)$. Namely $\gamma_1 < \gamma_2$ if γ_2 is obtained from γ_1 by gluing on piecewise flows on the noncompact (incoming or outgoing) edges. Notice therefore that a minimal element in this partial ordering is any actual Γ - flow

$$\gamma \in \mathcal{M}(\Gamma, M) \subset \overline{\mathcal{M}}(\Gamma, M).$$

Now any partially order set X can be viewed as a category, where the objects are the elements in X. There exists a unique morphism between points x and y if and only if $x \leq y$. If X is a topological space the resulting category may be viewed as a topological category.

Let $\mathcal{C}_{\mathcal{M}(\Gamma,M)}$ be the topological category associated to this partial ordering on the space $\overline{\mathcal{M}}(\Gamma, M)$. We then have the following.

Theorem 15. *There is a natural homotopy equivalence between the classifying space of the category $\mathcal{C}_{\mathcal{M}(\Gamma,M)}$ and the moduli space $\mathcal{M}(\Gamma, M)$,*

$$B\mathcal{C}_{\mathcal{M}(\Gamma,M)} \simeq \mathcal{M}(\Gamma, M).$$

Sketch of Proof. Given any partially ordered set X viewed as a category, we can consider the quotient category $X/\equiv$ whose space of objects is X modulo the equivalence relation generated by the partial ordering. There are no morphisms other than the identity morphisms. Notice that there is a natural projection functor

$$\pi : X \longrightarrow X/\equiv .$$

In the case of the category $\mathcal{C}_{\mathcal{M}(\Gamma,M)}$ defined from the partial ordering on $\overline{\mathcal{M}}(\Gamma, M)$, the quotient category $\overline{\mathcal{M}}(\Gamma, M)/\equiv$ is homeomorphic to $\mathcal{M}(\Gamma, M)$ (that is the space of objects is homeomorphic to $\mathcal{M}(\Gamma, M)$ which can be viewed as a constant category). This is because every element in $\overline{\mathcal{M}}(\Gamma, M)/\equiv$ is uniquely represented by an element $\gamma \in \mathcal{M}(\Gamma, M)$. This in turn is because every piecewise Γ - flow $\tilde{\gamma}$ is $\geq$ to a unique Γ - flow $\gamma \in \mathcal{M}(\Gamma, M)$. This fact also implies that if one considers the fiber category $\mathcal{C}_{\mathcal{M}(\Gamma,M)}(\gamma)$ of the projection functor

$$\pi : \mathcal{C}_{\mathcal{M}(\Gamma,M)} \longrightarrow \overline{\mathcal{M}}(\Gamma, M)/\equiv \quad \cong \quad \mathcal{M}(\Gamma, M)$$

over any Γ - flow γ, then $\mathcal{C}_{\mathcal{M}(\Gamma,M)}(\gamma)$ is a partially ordered set with a unique minimal element. Therefore its classifying space is contractible. Then by applying Quillen's theory [6] one can conclude that the projection map $\pi : \mathcal{C}_{\mathcal{M}(\Gamma,M)} \rightarrow \mathcal{M}(\Gamma, M)$ induces a homotopy equivalence on classifying spaces $\Box$

We are now ready to define our "ends functor" with which we will define the invariants $q(\Gamma, M)$. First we define the "end category" $\mathcal{C}_{\mathcal{E}}(\Gamma)$ as follows. Suppose the oriented graph Γ has n_1 incoming (noncompact) edges and n_2 outgoing edges, where $n_1 + n_2 = n$. Define the category $\mathcal{C}_{\mathcal{E}}$ as follows.

Definition 6. *Define $\mathcal{C}_{\mathcal{E}}(\Gamma)$ to be the product category*

$$\mathcal{C}_{\mathcal{E}}(\Gamma) = \prod_{n_1} \mathcal{C}_M^{op} \times \prod_{n_2} \mathcal{C}_M$$

where $\mathcal{C}_M^{op}$ denotes the "opposite category", that is the category whose objects are the same as those of $\mathcal{C}_M$, but $Mor_{\mathcal{C}^{op}}(a, b) = Mor_{\mathcal{C}}(b, a)$.

We now define a functor

$$\mathcal{E} : \mathcal{C}_{\mathcal{M}(\Gamma,M)} \longrightarrow \mathcal{C}_{\mathcal{E}}(\Gamma)$$

as follows. An object $\gamma \in \mathcal{C}_{\mathcal{M}(\Gamma,M)}$ is a piecewise Γ -flow with respect to some M - structure $\sigma \in \mathcal{S}(\Gamma, M)$. As described in section one γ converges to critical points of the functions labelling the incoming and outgoing edges that are given as part of the data of the structure σ. Thus γ determines an n tuple $(f_1, a_1), \cdots, (f_{n_1}, a_{n_1}), (f_{n_1+1}, a_{n_1+1}), \cdots, (f_n, a_n)$ of labelling functions and critical points, where the first n_1 f_i's label the incoming edges and the next $n_2 = n - n_1$ label the outgoing edges. Notice that this n - tuple is an object in the category $\mathcal{C}_{\mathcal{E}}(\Gamma)$. Therefore on objects we define

$$\mathcal{E}(\gamma) = (f_1, a_1), \cdots, (f_{n_1}, a_{n_1}), (f_{n_1+1}, a_{n_1+1}), \cdots, (f_n, a_n).$$

To define $\mathcal{E}$ on morphisms recall that in $\mathcal{C}_{\mathcal{M}(\Gamma,M)}$ there is a unique morphism from γ_1 to γ_2 if and only if $\gamma_2 \geq \gamma_1$. That means there exists an n - tuple of piecewise flow lines $(\alpha_1, \cdots, \alpha_n)$ of the functions $f_1, \cdots, f_n$ respectively, so that γ_2 is obtained by gluing the α_i's onto the ends of γ_1. The first n_1 of the α_i's are glued with an incoming orientation and the next n_2 are glued with an outgoing orientation. Notice that the n - tuple $(\alpha_1, \cdots \alpha_n)$ is a morphism in $\mathcal{C}_{\mathcal{E}}(\Gamma) = \prod_{n_1} \mathcal{C}_M^{op} \times \prod_{n_2} \mathcal{C}_M$. Therefore if $\Theta(\gamma_1, \gamma_2)$ denotes the unique morphism from γ_1 to γ_2 in $\mathcal{C}_{\mathcal{M}(\Gamma,M)}$, we define

$$\mathcal{E}(\Theta(\gamma_1, \gamma_2)) = (\alpha_1, \cdots, \alpha_n).$$

As in section two let Σ_Γ be the symmetry group of the oriented graph Γ. Notice that Γ acts freely on the category $\mathcal{C}_{\mathcal{M}(\Gamma,M)}$. Furthermore the product of the symmetric groups $\Sigma_{n_1} \times \Sigma_{n_2}$ acts freely on the end category $\mathcal{C}_{\mathcal{E}}(\Gamma)$. The following is a straightforward check of definitions.

Proposition 16. *The map*

$$\mathcal{E} : \mathcal{C}_{\mathcal{M}(\Gamma,M)} \longrightarrow \mathcal{C}_{\mathcal{E}}(\Gamma)$$

is a well defined equivariant functor with respect to the homomorphism $\Sigma_\Gamma \to \Sigma_{n_1} \times \Sigma_{n_2}$ given by sending a symmetry to the associated permutation of the ends.

To define the invariants $q(\Gamma, M)$ in this context we will be studying the map induced by $\mathcal{E}$ in equivariant homology. Specifically, consider the composition

$$\begin{aligned}
Q : H_k^{\Sigma_\Gamma}(B\mathcal{C}_{\mathcal{M}(\Gamma,M)}) = \quad & H_k^{\Sigma_\Gamma}(\mathcal{M}(\Gamma, M)) \\
\to H_k^{\Sigma_{n_1} \times \Sigma_{n_2}}(B\mathcal{C}_{\mathcal{E}}(\Gamma)) \quad & = \oplus_{k_1+k_2=k} H_{k_1}^{\Sigma_{n_1}}(\textstyle\prod_{n_1} B\mathcal{C}_M^{op}) \otimes H_{k_2}^{\Sigma_{n_2}}(\prod_{n_2} B\mathcal{C}_M) \\
& \cong \oplus_{k_1+k_2=k} H^{dn_1-k_1}_{\Sigma_{n_1}}((M)^{n_1}) \otimes H_{k_2}^{\Sigma_{n_2}}((M)^{n_2}) \\
& \to \oplus_{k_1+k_2=k} Hom\left(H^{k_2}_{\Sigma_{n_2}}((M)^{n_2}); H^{dn_1-k_1}_{\Sigma_{n_1}}((M)^{n_1})\right).
\end{aligned}$$

In the previous section we discussed how to define an invariant $q_\alpha(\Gamma, M)$ given any element $\alpha \in H_*(B\Sigma_\Gamma)$. Here will describe how $q_\alpha(\Gamma, M)$ is simply the image under the above map Q of an equivariant homology class $\theta(\alpha) \in H_*^{\Sigma_\Gamma}(\mathcal{M}(\Gamma, M))$. The first step is to study the relation of the equivariant moduli space to the space of ends equivariantly. For this consider the map to the structure space

$$\pi : \mathcal{M}(\Gamma, M) \longrightarrow \mathcal{S}(\Gamma, M)$$

studied previously. By abuse of notation we will let $\mathcal{S}(\Gamma, M)$ also denote the constant topological category whose objects are the space of structures. Then the map π can be realized on the categorical level

$$\pi : \mathcal{C}_{\mathcal{M}(\Gamma,M)} \longrightarrow \mathcal{S}(\Gamma, M)$$

which is Σ_Γ - equivariant. Consider the induced functor on orbit categories,

$$\pi : \mathcal{C}_{\mathcal{M}(\Gamma,M)}/\Sigma_\Gamma \longrightarrow \mathcal{S}(\Gamma, M)/\Sigma_\Gamma \simeq B\Sigma_\Gamma.$$

The following result is the main technical result, but is proved by relatively straightforward transversality arguments.

Theorem 17. *Let $N \subset \mathcal{S}(\Gamma, M)/\Sigma_\Gamma \simeq B\Sigma_\Gamma$ be a closed submanifold of dimension p. Let $\mathcal{C}_{\mathcal{M}(\Gamma,M)}(N) = \pi^{-1}(N) \subset \mathcal{C}_{\mathcal{M}(\Gamma,M)}/\Sigma_\Gamma$ be the subcategory of $\mathcal{C}_{\mathcal{M}(\Gamma,M)}/\Sigma_\Gamma$ whose objects have structures lying in N. Then for a generic embedding of N in $\mathcal{S}(\Gamma, M)/\Sigma_\Gamma$ the classifying space $B\mathcal{C}_{\mathcal{M}(\Gamma,M)}(N)$ has the homotopy type of a closed manifold of dimension*

$$dim\, B\mathcal{C}_{\mathcal{M}(\Gamma,M)}(N) = p - (r-1)d$$

where, as before, $r = rank H_1(\Gamma)$ and $d = dim\, M$.

We use this result as follows. Let $N_\alpha \hookrightarrow \mathcal{S}(\Gamma, M) \simeq B\Sigma_\Gamma$ be a closed manifold of dimension p representing $\alpha \in H_p(B\Sigma_\Gamma)$. Let

$$\theta(\alpha) \in H^{\Sigma_\Gamma}_{p-(r-1)d}(\mathcal{M}(\Gamma, M))$$

be the homology represented by the inclusion $B\mathcal{C}_{\mathcal{M}(\Gamma,M)}(N_\alpha) \subset B\mathcal{C}_{\mathcal{M}(\Gamma,M)} \simeq \mathcal{M}(\Gamma, M)$ given by the above theorem. Consider the class

$$Q(\theta(\alpha)) \in \oplus_k Hom(H^k_{\Sigma_{n_2}}((M)^{n_2}); \quad H^{k+d(n_1+r-1)-p}_{\Sigma_{n_1}}((M)^{n_1}))$$

defined as the image of $\theta(\alpha)$ under the composite map Q defined above. The following is the main result of this section, and is proved using standard techniques that describe intersections in terms of homology.

Theorem 18. *Given any $\alpha \in H_*(B\Sigma_\Gamma)$,*

$$Q(\theta(\alpha)) = q_\alpha(\Gamma, M).$$

The main effect of this result is that the invariants $q_\alpha(\Gamma, M)$ can be defined globally in terms of equivariant homology. In particular one does not need to perturb metrics or functions to define these invariants. This may have calculational advantages. We end this paper by describing how, in practice, these invariants may be computed.

Any oriented graph Γ is homotopy equivalent relative to its (oriented) ends to a graph Γ' with a single vertex v_0, with n_1 incoming noncompact edges, n_2 outgoing noncompact edges, and r loops (internal edges which both start and end at v_o) for some n_1, n_2, $r = rank\, H_1(\Gamma)$. The symmetry goup of Γ' is given by

$$\Sigma_{\Gamma'} = \Sigma_{n_1} \times \Sigma_{n_2} \times \Sigma_r.$$

Via this homotopy equivalence an element $\alpha \in H_*(B\Sigma_\Gamma)$ determines an element

$$\alpha' \in H_*(B(\Sigma_{n_1} \times \Sigma_{n_2} \times \Sigma_r)).$$

Now by pinching the r looped internal edges in the graph Γ' to a point, one gets a projection map to the simply simply connected graph Γ^0 that has a single vertex v_0, n_1 incoming noncompact edges, n_2 outgoing noncompact edges, and no internal edges. The map of graphs $\Gamma \simeq \Gamma' \to \Gamma^0$ induces a homomorphism of their symmetry groups

$$\Sigma_\Gamma \to \Sigma_{\Gamma^0} = \Sigma_{n_1} \times \Sigma_{n_2}$$

which is the homomorphism described above given by sending a symmetry of Γ to the induced permutation of the ends. Thus $\alpha \in H_*(B\Sigma_\Gamma)$ is mapped to a class $\alpha^0 \in H_*(B(\Sigma_{n_1} \times \Sigma_{n_2}))$. Also the class $\theta(\alpha) \in H_*^{\Sigma_\Gamma}(\mathcal{M}(\Gamma, M))$ is mapped to

$$\theta^0 \in H_*^{\Sigma_{n_1} \times \Sigma_{n_2}}(\mathcal{M}(\Gamma^0, M))$$

. By the naturality of the construction we see that $q_\alpha(\Gamma, M)$ is given by

$$Q(\theta^0) \in Hom(H^*_{\Sigma_{n_2}}((M)^{n_2}); \quad H^*_{\Sigma_{n_1}}((M)^{n_1}))$$

where Q here is the homomorphism in equivariant homology induced by the end functor

$$\mathcal{E} : B\mathcal{C}_{\mathcal{M}(\Gamma^0;M)} \to B\mathcal{C}_{\mathcal{E}}(\Gamma) = (B\mathcal{C}_M^{op})^{n_1} \times (B\mathcal{C}_M)^{n_2}$$

as above. Thus what remains is to study the equivariant homotopy type of this map. This is done as follows.

Theorem 19. *There is a natural equivariant homeomorphism*

$$h : \mathcal{M}(\Gamma^0, M) \cong \mathcal{S}(\Gamma^0, m) \times M$$

defined by sending γ to $(\sigma, \gamma(v_0))$ where σ is the structure associated to γ. Furthermore, after passing to orbit spaces the realization of the end functor is homotopy equivalent to the equivariant diagonal map:

$$\Delta : B(\Sigma_{n_1} \times \Sigma_{n_2}) \times M \longrightarrow \left(E\Sigma_{n_1} \times_{\Sigma_{n_1}} (M)^{n_1}\right) \times \left(E\Sigma_{n_2} \times_{\Sigma_{n_2}} (M)^{n_2}\right).$$

Notice that this theorem and the above observation says that calculation of an invariant $q_\alpha(\Gamma, M)$ reduces to computing the above equivariant diagonal maps Δ in homology. These can be viewed as generalized cup_i products and their calculations are equivalent to the calculation in homology of groups of the inclusion of products of symmetric groups into larger symmetric groups. These calculations are well known.

Notice also that since the invariants $q_\alpha(\Gamma, M)$ can be defined entirely in terms of the equivariant properties of the induced map on classifying spaces of the end functor $B\mathcal{E}$, then by taking the homotopy fiber of this map we can define functional and higher order operations. We will return to this topic in another paper.

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