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HOLOMORPHIC SPHERES IN LOOP GROUPS AND BOTT PERIODICITY

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Abstract. In this paper we study the topology of spaces of holomorphic maps from the Riemann sphere $\mathbb{P}^1$ to infinite dimensional Grassmanian manifolds and to loop groups. Included in this study is a complete identification of the homotopy types of $Hol_k(\mathbb{P}^1, BU(n))$ and of $Hol_k(\mathbb{P}^1, \Omega U)$, where the subscript k denotes the degree of the map. These spaces are shown to be homotopy equivalent to the k^{th} Mitchell - Segal algebraic filtration of the loop group $\Omega U(n)$ [7], and to $BU(k)$, respectively.

One of the most important theorems in Topology and Geometry is the "Bott Periodicity Theorem". In its most basic form it states that there is a natural homotopy equivalence,

$$\beta : \mathbb{Z} \times BU \xrightarrow{\simeq} \Omega U.$$

Here U is the infinite unitary group, $U = \varinjlim_n U(n)$, $BU = \varinjlim_n BU(n)$ is the limit of the classifying spaces, and $\Omega U = C_0^\infty(S^1, U)$ is the space of smooth, basepoint preserving loops.

In a paper which first pointed to the deep relationship between the index theory of Fredholm operators and Algebraic Topology, Atiyah [1], defined a homotopy inverse to the Bott map β, which can be viewed as a map

$$\bar\partial : C_0^\infty(S^2, BU) \xrightarrow{\simeq} \mathbb{Z} \times BU.$$

This map was defined by studying the index of the family of operators obtained by coupling the $\bar\partial$ operator to a smooth map from S^2 to a Grassmannian.

Since both mapping spaces $C_0^\infty(S^2, BU)$ and ΩU have path components naturally identified with the integers, we denote $C_0^\infty(S^2, BU)_k$ and $\Omega_k U$ the path components consisting of degree k - maps. Thus Bott periodicity, together with Atiyah's results says that for each integer k, there are natural homotopy equivalences,

$$BU \xleftarrow[\bar\partial]{\simeq} C_0^\infty(S^2, BU)_k \xrightarrow[\beta]{\simeq} C_0^\infty(S^2, \Omega U)_k.$$

By using some of these classical constructions as well recent results on the topology of holomorphic maps of spheres to Grassmannians and loop groups, we will prove the following holomorphic version of this result in this paper.

Theorem 1. *For each positive integer k, there are natural homotopy equivalences*

$$BU(k) \xleftarrow[\bar\partial]{\simeq} Hol_k(\mathbb{P}^1, BU) \xrightarrow[\beta]{\simeq} Hol_k(\mathbb{P}^1, \Omega U).$$

Remark. It should be noted that this theorem states that the original Bott equivalence β and Atiyah's inverse map $\bar\partial$ restrict to give equivalences in the holomorphic category as well. What is perhaps surprising about this result is that in this setting one obtains

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homotopy equivalences with the classifying space of unitary groups whose rank corresponds to the degree of the holomorphic map. Finally, observe that this theorem states that the inclusion of holomorphic maps into all smooth maps (of degree k),

$$Hol_k(\mathbb{P}^1, BU) \hookrightarrow C_0^\infty(S^2, BU)_k$$

or,

$$Hol_k(\mathbb{P}^1, \Omega U) \hookrightarrow C_0^\infty(S^2, \Omega U)_k$$

is homotopy equivalent to the inclusion of classifying spaces,

$$BU(k) \hookrightarrow BU.$$

In future work we will study consequences of this theorem to the understanding of periodicity phenomena in holomorphic K - theory of a smooth, projective variety.

Our other main result in this paper has to do with an identification of the topology of subspaces $Hol_k(\mathbb{P}^1, BU(n))$ of $Hol_k(\mathbb{P}^1, BU)$. By theorem 1, this latter space is homotopy equivalent to $BU(k)$, and so the inclusion $BU(n) \subset BU$ induces a map

$$j : Hol_k(\mathbb{P}^1, BU(n)) \to BU(k).$$

On the other hand, the inclusion of holomorphic maps into smooth maps, together with the identification of $\Omega^2 BU(n)$ with $\Omega U(n)$ induces a map

$$\iota : Hol_k(\mathbb{P}^1, BU(n)) \to \Omega_k U(n).$$

Furthermore, it is not difficult to see that the maps j and ι are compatible when composed to $BU \simeq \Omega U$.

Now S. Mitchell in [7] described an algebraic filtration of the loop group $\Omega U(n)$ by compact, complex subvarieties:

$$F_{1,n} \subset F_{2,n} \subset \cdots \subset F_{k,n} \subset \cdots \subset F_{\infty,n} \simeq \Omega U(n).$$

The subspace $F_{k,n}$ have as their homology, precisely the intersection of $H_*(BU(k))$ and $H_*(\Omega U(n))$, viewed as subgroups of $H_*(BU) = H_*(\Omega U)$. In [8], Richter proved that this filtration stably splits, so that the loop group $\Omega U(n)$ is stably homotopy equivalent to a wedge of the subquotients, $F_{k+1,n}/F_{k,n}$. The following theorem was conjectured by Mann and Milgram in [6] after an analysis of the holomorphic mapping spaces $Hol_k(\mathbb{P}^1, Gr_m(\mathbb{C}^n))$:

Theorem 2. *There is a natural homotopy equivalence*

$$Hol_k(\mathbb{P}^1, BU(n)) \simeq F_{k,n}.$$

These theorems are proven by using the identification of the loop group $\Omega U(n)$ with a certain moduli space of holomorphic bundles over $\mathbb{P}^1$, together with holomorphic trivializations on a disk. This identification was established in [9]. We then identify the homotopy type of $Hol_k(\mathbb{P}^1, BU(n))$ with a subspace of this moduli space consisting of bundles that are "negative" in the sense that they are (holomorphically) isomorphic to a direct sum of line bundles, each of which has nonpositive first Chern class. The topology of these moduli spaces are then studied in two ways: homologically, using calcluations of [6], and Morse theoretically, by analyzing the gradient flow of the Dirichlet energy functional on $\Omega U(n)$.

This paper is organized as follows. In section 1 we review some results from the theory of loop groups. The main reference for this material is Pressley and Segal's book [9]. This theory will in particular allow us to define the terms and maps in the statements of the above theorems more carefully. In section 2 we give a proof of theorems 1 and 2.

The real Bott periodicity analogues of the above theorems (where the unitary groups are replaced by orthogonal and symplectic groups) will appear in the Stanford University Ph.D thesis of the second author written under the direction of the first author. The authors are grateful to Paulo Lima-Filho, Paul Norbury, and Giorgio Valli for helpful conversations regarding this work.

§1. Loop Groups.

In this section we recall some of the basic constructions from the theory of loop groups as developed in [9]. We will use these constructions to define the holomorphic structures necessary to define the spaces and maps in the theorems described in the introduction. In what follows we will work with the Lie groups $U(n)$, but everything we use has obvious analogues for arbitrary compact semi-simple Lie groups. Again, we refer the reader to [9] for details.

As defined in the introduction, let LG denote the space of smooth maps from the circle S^1 to a Lie group G. The loop group $LGL(n,\mathbb{C})$ has the following important subgroups.

(1) The group $L^+GL(n,\mathbb{C})$ of maps $\gamma : S^1 \to GL(n,\mathbb{C})$ that extend to holomorphic maps of the closed disk $D^2 \to GL(n,\mathbb{C})$.
(2) the group $L_{pol}GL(n,\mathbb{C})$ of loops whose matrix entries are finite Laurent polynomials in z. That is, loops γ of the form
$$\gamma(z) = \sum_{k=-N}^{N} A_k z^k$$
for some N, where the A_k's are $n \times n$ matrices.
(3) The based loop group $\Omega GL(n,\mathbb{C})$, and the corresponding subgroup of polynomial loops, $\Omega_{pol}GL(n,\mathbb{C})$.

In the theory of loop groups, there is an underlying Hilbert space H^n defined to be the space of square integrable functions from the circle to $\mathbb{C}^n$:
$$H^n = L^2(S^1,\mathbb{C}^n).$$

Notice that this Hilbert space has a natural polarization:
$$H^n \cong H^n_+ \oplus H^n_-$$
where H^n_+ consists of those functions whose negative Fourier coefficients are all zero; or equivalently those functions $f : S^1 \to \mathbb{C}^n$ that extend to a holomorphic map of the disk.

Observe that the loop group $LGL(n,\mathbb{C})$ has a natural representation on the Hilbert space $H^n = L^2(S^1,\mathbb{C}^n)$ given by matrix multiplication. In particular the Laurent polynomial ring $\mathbb{C}[z,z^{-1}]$ acts on H^n, via the action of the loops
$$z^k \to z^k \cdot I_{n\times n} \in LGL(n,\mathbb{C}).$$

Now recall the "restricted Grassmannian" of H^n as defined in [9].

Definition. *Define the space $Gr_\infty(H^n)$ to be the space of all closed subspaces $W \subset H^n$ such that*

(1) *the orthogonal projection $pr_+ : W \to H^n_+$ is a Fredholm operator and its image consists of smooth functions, and*
(2) *the orthogonal projection $pr_- : W \to H^n_-$ is a Hilbert-Schmidt operator and its image also consists of smooth functions.*

Consider also the following important submanifolds of $Gr = Gr_\infty(H^n)$.

(1) $Gr_0 = \{W \in Gr : \exists k \geq 0 \, \text{such that} \, z^k H_+ \subset W \subset z^{-k} H_+^n\}$
(2) $Gr^{(n)} = \{W \subset Gr : zW \subset W\}$
(3) $Gr_0^{(n)} = Gr^{(n)} \cap Gr_0$.

The action of the loop group $LGL(n, \mathbb{C})$ on H^n induces a transitive action on the restricted Grassmannian $Gr^{(n)}$. The subgroup $LU(n)$ also acts transitively. Consider the space $H_+^n \subset Gr^{(n)}$. Clearly the isotropy group of this space of the action of $LGL(n, \mathbb{C})$ consists of those loops whose Fourier expansions contain only nonnegative powers of z. These loops are exactly the boundary values of holomorphic maps of the disk. In the above notation we called this subgroup $L^+GL(n, \mathbb{C})$. We therefore have a diffeomorphism

$$LGL(n, \mathbb{C})/L^+GL(n, \mathbb{C}) \cong Gr^{(n)}.$$

Furthermore, if one restricts the action to $LU(n)$, one sees that the isotropy subgroup of H_+^n is given by the subgroup $LU(n) \cap L^+GL(n, \mathbb{C})$. This is the subgroup of loops of $U(n)$ that are boundary values of holomorphic maps of the disk. A generalization of the maximum modulus principle, as proved in [9] shows that this subgroup consists only of the constant loops, $U(n) \subset LU(n)$. This, together with the analogous argument using $Gr_0^{(n)}$ and polynomial loops proves the following.

Theorem 1.1. *There are diffeomorphisms*

$$\begin{aligned} \Omega U(n) \cong LU(n)/U(n) &\cong LGL(n, \mathbb{C})/L^+GL(n, \mathbb{C}) \cong Gr^{(n)} \\ \Omega_{pol} U(n) \cong L_{pol} U(n)/U(n) &\cong L_{pol} GL(n, \mathbb{C})/L^+_{pol} GL(n, \mathbb{C}) \cong Gr_0^{(n)} \\ &\textit{given by} \quad \gamma \to \gamma H_+^n \end{aligned}$$

These diffeomorphisms determine the complex structure on the infinite dimensional manifolds $\Omega U(n)$ and $\Omega_{pol} U(n)$ that are used in the statement of theorem 1 in the introduction.

Given an element $W \in Gr^n$ define the **virtual dimension** of W, $vd(W)$, to be the Fredholm index of the projection of the projection map

$$pr_+ : W \to H_+^n.$$

So in particular we have

$$vd(W) = dim\left(W \cap H_-^n\right) - dim\left(W^\perp \cap H_+^n\right).$$

As we will see below, much of the analysis necessary to prove theorem 1 comes from studying the situation when the projection operator $pr_+ : W \to H_+^n$ has trivial kernel, and hence $vd(W) = -dim(coker(pr_+))$.

Now notice that since every element $W \in Gr_0$ has the property that $z^k H_+^n \subset W \subset z^{-k} H_+^n$ and hence by considering the projection $\bar{W} \subset z^{-k}H_+^n / z^k H_+^n$, one has that Gr_0 is the union of finite dimensional Grassmannians,

$$Gr_0 = \bigcup_p Gr(z^p H_+^n / z^{-p} H_+^n).$$

and is therefore topologically equivalent to $\mathbb{Z} \times BU$. The integer denotes the virtual dimension of W.

Via the identification of $Gr^{(n)}$ with the loop group, the virtual dimension determines the path component of the loop. More specifically, the path component of particular loop $\gamma \in \Omega U(n)$ is determined by the winding number of the composite loop

$$S^1 \xrightarrow{\gamma} U(n) \xrightarrow{det} U(1) = S^1.$$

By abuse of terminology we refer to the winding number of this composite loop as the winding number of γ, which we denote by $wdg(\gamma)$.

The following is a straightforward exercise that we leave to the reader.

Proposition 1.2. *Let $\gamma \in \Omega U(n)$ correspond to $W \in Gr^{(n)}$. That is, $W = \gamma H_+^n$. Then the virtual dimension of W and the winding number of γ are related by*

$$v.d(W) = -wdg(\gamma).$$

Notice furthermore that if $\gamma \in \Omega_{pol} U(n)$ is a polynomial loop, then this composite $det \circ \gamma$ is also a polynomial loop in $S^1 \in \mathbb{C}$. The only such polynomial maps (i.e polynomials in one variable of constant unit length) are $z \to z^k$ for some $k \in \mathbb{Z}$. Therefore we may conclude that the polynomial loop group in $SU(n)$ consists precisely of those polynomial loops in $\Omega_{pol} U(n)$ with winding number zero. We again refer the reader to [9] for details.

Recall that there was another description of the loop group $\Omega U(n)$ in [9]that is given in terms of holomorphic bundles over the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \infty = D_0 \cup D_\infty$, where $D_0 = \{z : |z| \leq 1\}$ and $D_\infty = \{z : |z| \geq 1\}$.

Proposition 1.3. *Let $\mathcal{C}_{k,n}$ denote the space*

$$\begin{aligned} \mathcal{C}_{k,n} = \{ & \textit{isomorphism classes of pairs } (E,\phi), \textit{ where } E \to \mathbb{P}^1 \textit{ is an} \\ & n \textit{ - dimensional holomorphic bundle of Chern class } c_1(E) = -k, \\ & \textit{and } \phi \textit{ is a holomorphic framing of } E \textit{ restricted to } D_\infty \} \end{aligned}$$

Then there is a natural diffeomorphism

$$\mathcal{C}_{k,n} \xrightarrow{\cong} \Omega_k U(n).$$

The diffeomorphism in this proposition can be described as follows. Let $(E, \phi) \in \mathcal{C}_{k,n}$. So $\phi : E|_{D_\infty} \to \mathbb{C}^n$ is a fixed trivialization. Let $\sigma : E|_{D_0} \to \mathbb{C}^n$ be any holomorphic trivialization of the restriction of E to the other disk, D_0. On the intersection $S^1 = D_0 \cap D_\infty$, the trivializations ϕ and σ differ by a loop $\gamma \in LGL(n, \mathbb{C})$. Of course the loop γ depends on the choice of trivialization σ, but if a different trivialization, say σ' were used to construct a different loop, say γ^1, then $\gamma \cdot (\gamma')^{-1}$ would be a loop that extends to a holomorphic map $\sigma \cdot (\sigma')^{-1} : D_0 \to GL(n, \mathbb{C})$. Thus $\gamma \cdot (\gamma')^{-1} \in L^+GL(n, \mathbb{C})$. This procedure therefore gives a well defined map

$$\mathcal{C}_{k,n} \to LGL(n, \mathbb{C})/L^+GL(n, \mathbb{C}) \cong \Omega U(n).$$

The fact that this map yields the diffeomorphism claimed in the proposition follows from the Bruhat factorization theorem for loops. We refer the reader to [9] for details of this argument.

We note that with respect to the diffeomorphisms

$$\mathcal{C}_{k,n} \cong \Omega_k U(n) \cong (Gr^{(n)})_k$$

the Chern class of the bundle in $\mathcal{C}_{k,n}$ corresponds to the virtual dimension of the subspace $W \in Gr^{(n)}$, which, as observed above, corresponds to the opposite of the winding number of the loop in $\Omega U(n)$.

These models of the loop groups will prove very important in our study of holomorphic mapping spaces. One aspect of these models that is quite useful is that they come equipped with filtrations by holomorphic subvarieties. That is, there is a sequence of compact, complex subvarieties:

$$F_{1,n} \hookrightarrow \cdots F_{k,n} \hookrightarrow F_{k+1,n} \hookrightarrow \cdots F_{\infty,n} = \Omega_{pol} SU(n) \simeq \Omega SU(n).$$

The homotopy theory of these filtrations has been studied in great detail by Mitchell [7], and by Richter in [8]. In particular one has the following properties (see [7])

Proposition 1.4. *If one takes the limit over the rank n, then there is a natural homotopy equivalence*

$$\lim_{n\to\infty} F_{k,n} \simeq BU(k).$$

Moreover the following diagram homotopy commutes:

$$\begin{array}{ccc} F_{k,n} & \xrightarrow{n\to\infty} & BU(k) \\ \cap\big\downarrow & & \big\downarrow\cap \\ \Omega U(n) & \xrightarrow[\hookrightarrow]{} & \Omega U \simeq BU. \end{array}$$

Furthermore the homology of this filtration is given by

$$H_*(F_{k,n}) \cong H_*(\Omega U(n)) \cap H_*(BU(k)) \subset H_*(BU).$$

In [8] Richter also showed that this filtration stably splits. That is, there is a stable homotopy equivalence

$$\Omega SU(n) \simeq_s \bigvee_k F_{k,n}/F_{k-1,n}.$$

As was stated in the introduction, (theorem 2) one of our main goals in this paper is to prove that $F_{k,n}$ is homotopy equivalent to the holomorphic mapping spaces $Hol_k(\mathbb{P}^1, BU(n))$.

The filtration $F_{k,n}$ can be defined as follows. As in [11], define the sub - semigroup $\tilde{\Omega}$ of the space of polynomial loops $\Omega_{pol}U(n)$ to consist of loops that only involve non-negative powers of z in their Fourier expansions. Furthermore, we write

$$\tilde{\Omega} = \coprod_{k \geq 0} \tilde{\Omega}_k$$

where $\tilde{\Omega}_k$ consists of loops of winding number k.

The space $F_{k,n}$ is homeomorphic to $\tilde{\Omega}_k$, but in order to see how these spaces give a filtration of $\Omega SU(n)$, we consider the loop

$$\begin{aligned} \lambda : S^1 &\to U(n) \\ z &\to \begin{pmatrix} z^{-1} & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ & & \cdots & \\ & & & 1 \end{pmatrix} \end{aligned}$$

Clearly λ is a polynomial loop of winding number -1. Therefore if we define

$$F_{k,n} = \lambda^k \tilde{\Omega}_k$$

then $F_{k,n}$ consists of polynomial loops of winding number zero. That is, $F_{k,n} \subset \Omega_{pol}SU(n)$. Moreover $F_{k,n} \subset F_{k+1,n}$ and

$$\Omega_{pol}SU(n) = \bigcup_{k \geq 1} F_{k,n}.$$

(see [7].) Now the spaces of loops $\tilde{\Omega}_k$ can also be interpreted in the above Grassmannian models in the following way. Since $\gamma \in \tilde{\Omega}_k$ has only positive terms in its Fourier expansion, the space $W = \gamma H^n_+$ is a subspace of H^n_+. Moreover since γ has winding number k, the subspace W has virtual dimension $-k$, (proposition 1.2). This means that the dimension of H^n_+/W is equal to k. Thus we have the following.

Proposition 1.5. *Under the diffeomorphism described in theorem 1.1, we have that*

$$\begin{aligned}\tilde{\Omega}_k &= \{W \in Gr_0^{(n)} : W \subset H_+^n \text{ and } dim\, H_+^n/W = k\} \\ &= \{W \in Gr_0^{(n)} : vd(W) = -k, \text{ and } ker(pr_+ : W \to H_+^n) = 0\}.\end{aligned}$$

It will be helpful to have a description of $\tilde{\Omega}_k$ (or equivalently $F_{k,n}$) in terms of holomorphic bundles as well. To do this we need to introduce the notion of a **negative** holomorphic bundle.

By a well known theorem of Grothendieck [4], every holomorphic bundle over $\mathbb{P}^1$ is isomorphic to a direct sum of holomorphic line bundles of the form $O(j)$, where $c_1(O(j)) = j \in \mathbb{Z}$. Furthermore this direct sum decomposition is unique (up to order). A holomorphic bundle $E \to \mathbb{P}^1$ is **negative** if

$$E \cong O(j_i) \oplus \cdots \oplus O(j_m)$$

with each $j_i \leq 0$. Notice that the dimension of E is m, and the first Chern class is given by

$$c_1(E) = \sum_{i=1}^{m} j_i.$$

A **positive** holomorphic bundle has the analogous definition.

Define the sub - moduli space

$$\mathcal{C}_{k,n}^- \subset \mathcal{C}_{k,n}$$

to consist of those $(E, \phi) \in \mathcal{C}_{k,n}$ such that E is negative.

Now an easy exercise verifies that with respect to the diffeomorphism $\mathcal{C}_{k,n} \cong Gr^{(n)}$ described above, a holomorphic bundle E being negative corresponds to a subspace W of H^n lying entirely in H_+^n. Equivalently this means that the projection operator $pr_+ : W \to H_+^n$ has zero kernel, and so $vd(W) = -dim\, coker(pr_+)$. By proposition 1.4 we therefore have the following.

Proposition 1.6. *With respect to the diffeomorphisms $\mathcal{C}_{k,n} \cong \Omega_k U(n) \cong (Gr^{(n)})_k$, we have that*

$$\begin{aligned}\tilde{\Omega}_k &\cong \mathcal{C}_{k,n}^- \cap \Omega_{pol} U(n) \\ &\cong \mathcal{C}_{k,n}^- \cap Gr_0^{(n)}.\end{aligned}$$

We refer to this space as $\mathcal{C}_{k,n}^-(pol)$.

§2. Proofs of Theorems 1 and 2.

In this section we give proofs of theorems 1 and 2, as stated in the introduction. We begin with theorem 2, which asserts that $F_{k,n}$ is homotopy equivalent to the holomorphic mapping space $Hol_k(\mathbb{P}^1, BU(n))$.

In view of proposition 1.6, it is sufficient to prove the following two theorems:

Theorem 2.1. *There is a natural homotopy equivalence*

$$Hol_k(\mathbb{P}^1, BU(n)) \simeq \mathcal{C}^-_{k,n}.$$

Theorem 2.2. *The inclusion*

$$\mathcal{C}^-_{k,n}(pol) \subset \mathcal{C}^-_{k,n}$$

is a homotopy equivalence.

We begin with a proof of theorem 2.1. We start by describing a model (up to diffeomorphism) of $Hol_k(\mathbb{P}^1, BU(n))$.

Proposition 2.3. *Let V be an infinite dimensional complex vector space and $V_\infty \subset V$ a fixed n - dimensional subspace. Define $\mathcal{M}_{k,n}$ to be the following moduli space:*

$$\begin{aligned}\mathcal{M}_{k,n} = \{&\text{isomorphism classes of pairs } (E, j), \text{ where } E \to \mathbb{P}^1 \text{ is a negative} \\ &n \text{ - dimensional holomorphic bundle of Chern class } c_1(E) = -k, \\ &\text{and } j : \Gamma(E^*)^* \to V \text{ is a linear embedding taking the fiber at } \infty,\ E_\infty \text{ to } V_\infty\}\end{aligned}$$

Then there is a natural homeomorphism

$$h : Hol_k(\mathbb{P}^1, Gr_n(V)) \xrightarrow{\cong} \mathcal{M}_{k,n}.$$

Proof. Let $E \to \mathbb{P}^1$ be an n - dimensional negative holomorphic bundle of Chern class $c_1(E) = -k$. Let $E^* \to \mathbb{P}^1$be its dual bundle. Then clearly E^* is a positive holomorphic bundle. By the Riemann - Roch theorem, the dimension of the space of holomorphic sections $\Gamma(E^*)$ is $n + k$. Given a point $x \in \mathbb{P}^1$, consider the adjoint mapping of vector spaces

$$\iota_x : E_x \to \Gamma(E^*)^*$$

defined by

$$\iota_x(v)(\sigma) = \sigma(x)(v)$$

where $v \in E_x$ and $\sigma \in \Gamma(E^*)$. The fact that E is negative assures that ι_x will be a linear embedding for all $x \in \mathbb{P}^1$. This process defines a holomorphic map

$$\iota : \mathbb{P}^1 \to Gr_n(\Gamma(E^*)^*).$$

Given a linear embedding $j : \Gamma(E^*)^* \hookrightarrow V$ taking E_∞ to V_∞, then ι defines a holomorphic map $\mathbb{P}^1 \to Gr_n(V)$ which is based in that it sends ∞ to V_∞.

This then defines the map

$$\mathcal{M}_{k,n} \to Hol_k(\mathbb{P}^1, Gr_n(V)).$$

The fact that it is a diffeomorphism follows from the fact that the pull back of the universal bundle over $Gr_n(V)$ via a holomorphic map is clearly negative.

Theorem 2.1 will now follow from the following.

Lemma 2.4. *The moduli spaces $\mathcal{C}^-_{k,n}$ and $\mathcal{M}_{k,n}$ are homotopy equivalent.*

Proof. We actually prove that the moduli spaces $\mathcal{M}_{k,n}$ and $\mathcal{C}^-_{k,n}$ are both homotopy equivalent to an intermediate space $\mathcal{X}_{k,n}$ defined by

$$\begin{aligned}\mathcal{X}_{k,n} = \{&\text{isomorphism classes of pairs } (E, j, \theta), \text{ where } (E, j) \in \mathcal{M}_{k,n}, \\ &\text{and } \theta : E_{|D_\infty} \to D_\infty \times V_\infty \text{ is a holomorphic trivialization}\}.\end{aligned}$$

Forgetting the trivialization θ defines a map from $\mathcal{X}_{k,n}$ to $\mathcal{M}^-_{k,n}$. This is a homotopy equivalence because it is clearly a fibration, whose fiber at a given $(E, j) \in \mathcal{M}_{k,n}$ is the space of choices of trivialization θ, which is a contractible space, $Hol(D_\infty, GL(n, \mathbb{C}))$.

Now the projection map $(E, j, \theta) \to (E, \theta)$ defines a map

$$\mathcal{X}_{k,n} \to \mathcal{C}^-_{k,n}.$$

This map is also a homotopy equivalence because it is a fibration whose fiber at a given $(E, \theta) \in \mathcal{C}^-_{k,n}$ is the contractible Stiefel variety of linear embeddings of a finite dimensional vector space, $\Gamma(E^*)^*$, into the infinite dimensional space V. This proves the lemma, and therefore completes the proof of theorem 2.1.

We now proceed to prove theorem 2.2. We will give two proofs of this theorem. The first is homological in nature and will rely on the calculations of the homology of the spaces $F_{k,n}$ by Mitchell [7] and of spaces of holomorphic maps from $\mathbb{P}^1$ to Grassmannians by Mann and Milgram [6]. The second proof will be Morse theoretic in nature, and will rely on the dynamics of a flow of a natural C^*- action on loop groups studied in [9]. This proof is more geometric in nature, and has the feature that it will give an alternative proof of Mann and Milgram's calculation of $H_*(Hol_k(\mathbb{P}^1, Gr_n(V)))$.

Homological Proof. Let $\beta : F_{k,n} \to Hol_k(\mathbb{P}^1, Gr_n(V))$ be the composition

$$\beta : F_{k,n} \cong \tilde{\Omega}_k \cong \mathcal{C}^-_{k,n}(pol) \subset \mathcal{C}^-_{k,n} \simeq Hol_k(\mathbb{P}^1, Gr_n(V)).$$

Of course it suffices to prove that β induces an isomorphism in homology.

Notice from the construction of the map β that if one composes with the inclusion of holomorphic maps into smooth maps,

$$F_{k,n} \xrightarrow{\beta} Hol_k(\mathbb{P}^1, BU(n)) \hookrightarrow \Omega^2_k BU(n) \simeq \Omega_k U(n) \simeq \Omega SU(n)$$

then this map is homotopic to the inclusion of the Mitchell filtration $F_{k,n} \subset \Omega SU(n)$.

Also, consider the composition of β with the map $Hol_k(\mathbb{P}^1, Gr_n(V)) \to BU(k)$ given by

$$\delta : Hol_k(\mathbb{P}^1, Gr_n(V)) \simeq \mathcal{C}^-_{k,n} \to Gr_k(H^n_+)$$

where the last map assigns to a loop $\gamma \in \Omega U(n)_k$ whose associated holomorphic bundle E_γ is negative (and therefore lies in $\mathcal{C}^-_{k,n}$) the cokernel of the projection operator pr_γ viewed as a subspace $coker(pr_\gamma) \subset H^n_+$. The composition

$$F_{k,n} \xrightarrow{\beta} Hol_k(\mathbb{P}^1, Gr_n(V)) \xrightarrow{\delta} Gr_k(H^n_+) \simeq BU(k)$$

is homotopic to the map described in proposition 1.4. We therefore have the following homotopy commutative diagram:

$$\begin{tikzcd}
F_{k,n} \arrow[r, "\beta"] & Hol_k(\mathbb{P}^1, Gr_n(V)) \arrow[r, "\delta"] \arrow[d] & BU(k) \arrow[d, "\cap"] \\
 & \Omega_k U(n) \arrow[r, hook] & \Omega U \simeq BU
\end{tikzcd}$$

where the outer maps $F_{k,n} \xrightarrow{\delta\circ\beta} BU(k)$ and $F_{k,n} \to \Omega_k U(n)$ are the maps described in proposition 1.4. Now by this proposition, the composite map $F_{k,n} \to BU$ maps injectively in homology, with image equal to the intersection $H_*(\Omega U(n)) \cap H_*(BU(k)) \subset H_*(BU)$. Thus by the commutativity of this diagram, the homomorphism

$$\beta_* : H_*(F_{n,k}) \to H_*(Hol_k(\mathbb{P}^1, Gr_n(V)))$$

is injective. To prove surjectivity, we use a result of Mann and Milgram [7] that says the inclusion of the holomorphic maps into all smooth maps

$$Hol_k(\mathbb{P}^1, Gr_n(\mathbb{C}^m)) \hookrightarrow \Omega^2_k Gr_n(\mathbb{C}^m)$$

induces an injection in homology for every Grassmannian $Gr_n(\mathbb{C}^m)$. This implies that the map

$$H_*(Hol_k(\mathbb{P}^1, Gr_n(V))) = H_*(Hol_k(\mathbb{P}^1, BU(n))) \to H_*(\Omega^2 BU(n)) = H_*(\Omega U(n))$$

is injective. But since the inclusion $H_*(\Omega U(n)) \to H_*(\Omega U) \cong H_*(BU)$ is injective, that means that the composition

$$H_*(Hol_k(\mathbb{P}^1, Gr_n(V))) \to H_*(\Omega U(n)) \to H_*(BU)$$

is injective. Also, by the commutativity of the diagram, we know that the image lies in the intersection $H_*(\Omega U(n)) \cap H_*(BU(k)) \subset H_*(BU)$. But as remarked above this intersection is the image of (and isomorphic to) $H_*(F_{k,n})$. Hence both maps

$$H_*(Hol_k(\mathbb{P}^1, Gr_n(V))) \to H_*(\Omega U(n)) \cap H_*(BU(k))$$

and

$$H_*(F_{k,n}) \to H_*(\Omega U(n)) \cap H_*(BU(k))$$

are isomorphsms. Thus by the commutativity of the above diagram

$$\beta_* : H_*(F_{k,n}) \to H_*(Hol_k(\mathbb{P}^1, Gr_n(V)))$$

is an isomorphism. Now since both these spaces are known to be simply connected ([7], [6]), this implies β is a homotopy equivalence.

Morse theoretic Proof. The second proof that the inclusion $\mathcal{C}^-_{k,n}(pol) \hookrightarrow \mathcal{C}^-_{k,n}$ is a homotopy equivalence is by studying the dynamics of a $\mathbb{C}^*$ - action on $\Omega U(n)$ described in [9].

To be more specific, consider the natural circle action on $H^n = L^2(S^1, \mathbb{C}^n)$. This action preserves the polarization and therefore induces an action on the Grassmannian $Gr(H^n)$. It was shown in [9] that this action extends to a smooth action of the units in $\mathbb{C}$ lying in the unit disk

$$\mathbb{C}^*_{\leq 1} \times Gr^{(n)} \to Gr^{(n)}$$

and to all of $\mathbb{C}^*$ on the polynomial Grassmannian,

$$\mathbb{C}^* \times Gr^{(n)}_0 \to Gr^{(n)}_0.$$

The induced flow given by the action of the reals was shown to be the gradient flow of the energy function on $\Omega U(n)$:

$$\begin{aligned} E : \Omega U(n) &\to \mathbb{R} \\ \gamma &\to \frac{1}{4\pi} \int_0^{2\pi} |\gamma(t)^{-1}\gamma'(t)|^2 dt. \end{aligned}$$

The critical points of this action are homomorphisms $\lambda : S^1 \to U(n)$, and hence the critical levels are indexed by the conjugacy class of the homomorphism; namely a partition $\bar{a} = (a_1, \cdots, a_n)$. We write $\lambda_{\bar{a}}$ to denote the critical level given by conjugates of the homomorphism

$$z \to (z^{a_1}, \cdots, z^{a_n}) \in U(1) \times \cdots \times U(1) \subset U(n).$$

The resulting Morse decomposition of $\Omega U(n) = LGL(n, \mathbb{C})/L^+$ was studied in great detail in [9]. In particular it was shown that the stable and unstable manifolds of the critical levels $\lambda_{\bar{a}}$ are given by the orbits under the (left) action of L^- and L^+ respectively. (Here, as above, L^+ denotes those loops that are boundary values of holomorphic maps of the disk D_0, and L^- dentoes those loops that are boundary values of holomorphic maps of the disk around infinity, D_∞.)

Recall that with respect to a generic metric, a Morse function (or a Morse - Bott function, whose critical points form nondegenerate critical submanifolds) satisfies the Morse - Smale transversality conditions. That is, the unstable and stable manifolds intersect each other transversally. In this case the critical points have a partial ordering, where one says that $a \geq b$ if there is a flow line emanating from a and converging to b. The resulting ordering of critical points of this flow on $\Omega U(n)$ was described in [9] in the following way.

Let $S \subset \mathbb{Z}$ be a set of integers whose symmetric difference with the nonnegative integers, $\mathbb{Z}^+$ is finite. That is,

$$S \Delta \mathbb{Z}^+ = \{S - \mathbb{Z}^+\} \cup \{\mathbb{Z}^+ - S\}$$

is a finite set. The *virtual cardinal* of such a set S, $v.c(S)$ is defined to be

$$v.c(S) = \#\{S - \mathbb{Z}^+\} - \#\{\mathbb{Z}^+ - S\}.$$

A set S with virtual cardinal $v.c(S) = k$ will be of **type** n**,** if for every $s \in S$,

$$s + n \subset S.$$

It was shown in [9] that the critical manifolds of the above flow when restricted to the component $\Omega_k U(n)$, are indexed by sets S of type n having virtual cardinal $= -k$. The relationship with the above indexing of critical points via conjugacy classes of homomorphisms $\lambda : S^1 \to U(n)$, of winding number k, and thus partitions $\bar{a} = (a_1, \cdots, a_n)$, with $\sum_{i=1}^n a_i = k$, is given as follows.

Given such a partition $\bar{a} = (a_1, \cdots, a_n)$, then let $b_i = na_i + i$, and let $S_{\bar{a}}$ be the type n set genererated by $\{b_1, \cdots, b_n\}$. That is,

$$S_{\bar{a}} = \bigcup_{q \in \mathbb{Z}^+} \{b_1 + qn, b_2 + qn, \cdots, b_n + qn\}.$$

One reason that this indexing of the critical levels is useful, is that, as was observed in [9], every set S with virtual cardinal $v.c(S) = m$, can be written as an ordered sequence

$$S = \{s_{-m}, s_{-m+1}, \cdots, s_q, \cdots\}$$

with $s_{-m} < s_{-m+1} < \cdots, s_q, \cdots$, and $s_m = m$ for all m sufficiently large. When written this way, the sets of virtual cardinal k have a natural partial ordering given by

$$S \geq S' \quad \text{if} \quad s_j \leq s'_j \quad \text{for all } j.$$

The following was proved in [9].

Lemma 2.5. *The above partial ordering on the type n sets S of virtual cardinal $-k$ corresponds to the Morse - Smale partial ordering of the critical levels of the energy functional resticted to the component $\Omega_k U(n)$.*

As we did with holomorphic bundles, we refer to a type n set S of virtual cardinal $-k$ as *negative*, if $S \subset \mathbb{Z}^+$. This terminology might be somewhat confusing , but we use it because in the definition $v.c(S) = \#\{S - \mathbb{Z}^+\} - \#\{\mathbb{Z}^+ - S\}$, a *negative* set S is one for which $\#\{S - \mathbb{Z}^+\} = 0$ and hence $v.c(S) = -\#\{\mathbb{Z}^+ - S\}$.

For a given k and n, define the type n set of virtual cardinal $-k$, S_k, to be

$$S_k = \{k, k+1, k+2, \cdots\}.$$

Write the integer k in the form $k = mn + i$, where $0 \leq i \leq n-1$. Then it is seen from the definition that partition that S_k corresponds to is $(m, m, \cdots, m, m+1, \cdots, m+1)$, where there are $n - i$ copies of m and i copies of $m+1$ in this partition of k.

S_k therefore corresponds to the homomorphism

$$\begin{aligned} \lambda_k : S^1 &\to U(1) \times \cdots \times U(1) \subset U(n) \\ z &\to (z^m, \cdots, z^m, z^{m+1}, \cdots, z^{m+1}) \end{aligned}$$

or, equivalently to the holomorphic bundle given by the direct sum

$$\lambda_i = \oplus_{n-i} O(-m) \quad \oplus \quad \oplus_i O(-(m+1)).$$

The following is an exercise with the above definitions.

Lemma 2.6. *The sets S_k are minimal in the sense that there if S is a type n set of virtual cardinal $-k$ with*

$$S_k \geq S,$$

then $S = S_k$. Furthermore, a type n set S of virtual cardinal $-k$ satisfies

$$S \geq S_k$$

if and only if S is negative.

This leads quite quickly to the following result.

Proposition 2.7. *The moduli space $\mathcal{C}^-_{k,n}$, when viewed as a subspace of $\Omega U(n)_k$, is equal to the union of the stable manifolds of the critical levels indexed by type n sets S of virtual cardinal $-k$, satisfying $S \geq S_k$. That is, $\mathcal{C}^-_{k,n}$ is the closure of the stable manifold of the critical level consisting of homomorphisms conjugate to λ_k, as defined above.*

Proof. As described above, the stable manifold of the critical level of the space of homomorphisms conjugate to $\lambda_{\bar{a}}$ is given by the orbit under the left action of $L^-(GL(n, \mathbb{C}))$.

Now the Bruhat factorization (see [9]) of a loop says that every loop $\gamma \in LGL(n, \mathbb{C})$ has a unique decomposition as

$$\gamma = \gamma_- \cdot \lambda \cdot \gamma_+$$

where $\gamma_- \in L^-$ and $\gamma_+ \in L^+$, and $\lambda : S^1 \to U(n)$ is a homomorphism. The corresponding holomorphic bundle E_γ is isomorphic to E_λ which is a direct sum of line bundles, and hence gives the Grothendieck decomposition. This then says that under the left action of L^-, the Grothendieck type of the corresponding holomorphic bundle is preserved. Thus the stable manifold of a critical set $\lambda_{\bar{a}}$ is exactly the space of loops γ whose corresponding holomorphic bundles E_γ are isomorphic to the direct sum of line bundles given by the partition $\bar{a}$. The proposition then follows from lemma 2.9.

The action of $L^+GL(n, \mathbb{C})$ on the polynomial loop group $\Omega_{pol}U(n)_k$ was studied in detail in [9] as well. As mentioned above, the orbits of this action give the unstable manifolds of the flow of energy functional, when restricted to the polynomial loop group. The following is result can be viewed as the dual of 2.7.

Proposition 2.8. *The space $\mathcal{C}^-_{k,n}(pol)$, when viewed as a subspace of $\Omega_{pol}U(n)_k$, is equal to the union of the unstable manifolds of the critical levels indexed by type n sets S of virtual cardinal $-k$, satisfying $S \geq S_k$.*

Theorem 2.2 now follows from propositions 2.7 and 2.8 and standard Morse theory arguments, using the that the energy functional on $\Omega U(n)$ satisfies the Palais - Smale condition.

Now that we have theorem 2, we have identified the homotopy type of the holomorphic mapping space, $Hol_k(\mathbb{P}^1, BU(n))$ in terms of the Mitchell filtration, $F_{k,n}$. By taking the limit over the rank n and using proposition 1.4, we know, as asserted in theorem 1 of the introduction, that

$$Hol_k(\mathbb{P}^1, BU) \simeq BU(k).$$

What remains to complete the proof of theorem 1, is to prove the following.

Theorem 2.9. *The Bott map $\beta : BU \to \Omega U$ induces a homotopy equivalence of holomorphic mapping spaces,*

$$Hol_k(\mathbb{P}^1, BU) \underset{\simeq}{\xrightarrow{\beta}} Hol_k(\mathbb{P}^1, \Omega U).$$

Proof. As was observed earlier using the Grassmannian model, the polynomial loop group, $\Omega_{pol}U(n)$ consists of the union of finite dimensional Grassmannian manifolds. Moreover, as was studied in [9], if one takes the limit over the rank, then one has that

$$\Omega_{pol}U \cong \mathbb{Z} \times BU$$

and that the inclusion of polynomial loops into smooth loops,

$$\Omega_{pol}U \hookrightarrow \Omega U$$

is the Bott homotopy equivalence

$$\beta : \mathbb{Z} \times BU \xrightarrow{\simeq} \Omega U.$$

Since, via the Grassmannian models we see that the inclusion of polynomial loops into smooth loops is a holomorphic map between (infinite dimensional) complex manifolds, this gives an induced map on holomorphic mapping spaces,

$$\beta : Hol_k(\mathbb{P}^1, BU) \to Hol_k(\mathbb{P}^1, \Omega U).$$

We shall now prove that this map is a homotopy equivalence. To do this, recall the theorem of Atiyah [2] and Donaldson [3] stating that the holomorphic mapping space $Hol_k(\mathbb{P}^1, \Omega U(n))$ is diffeomorphic to the moduli space of (based) gauge equivalence classes of anti-self dual connections on a principal $U(n)$ bundle over S^4 of Chern class $c_2 = -k$:

$$Hol_k(\mathbb{P}^1, \Omega U(n)) \cong \mathcal{M}_k^{U(n)}(S^4).$$

These diffeomorphisms clearly preserve the stabilization processes over the rank (see [10]) and therefore defines diffeomorphism

$$Hol_k(\mathbb{P}^1, \Omega U) \cong \mathcal{M}_k^U(S^4).$$

Now the homotopy type of these moduli spaces were studied by Kirwan [5] and by Sanders [10] where it was shown that

$$\mathcal{M}_k^U(S^4) \simeq BU(k).$$

In [10] it was shown that this homotopy equivalence can be realized by symbol of the Dirac operator coupled to an anti - self dual connection.

Thus by this result and theorem 2.7, the map $\beta : Hol_k(\mathbb{P}^1, BU) \to Hol_k(\mathbb{P}^1, \Omega U)$ is a map between spaces that are both homotopy equivalent to $BU(k)$. Furthermore, by including holomorphic maps into smooth maps, we know that the following diagram commutes:

$$\begin{array}{ccc} Hol_k(\mathbb{P}^1, BU) & \xrightarrow{\beta} & Hol_k(\mathbb{P}^1, \Omega U) \\ \cap\big\downarrow & & \big\downarrow\cap \\ C^\infty(S^2, BU) & \xrightarrow[\beta]{} & C^\infty(S^2, \Omega U) \end{array}$$

where in the bottom row C^∞ is meant to denote basepoint preserving smooth maps. Now by Bott periodicity the map on the bottom row is a homotopy equivalence, where each of the spaces are homotopy equivalent to BU. Thus up to homotopy, the map β in the top row can be viewed as a self map of $BU(k)$ that makes the following diagram homotopy commute:

$$\begin{tikzcd}
BU(k) \arrow[r, "\beta"] \arrow[d, "\cap"'] & BU(k) \arrow[d, "\cap"] \\
BU \arrow[r, "="'] & BU.
\end{tikzcd}$$

Since in homology, $H_*(BU(k)) \to H_*(BU)$ is injective, any such map β must induce an isomorphism in homology and hence be a homotopy equivalence.

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