HOLOMORPHIC K - THEORY, ALGEBRAIC CO-CYCLES, AND LOOP GROUPS

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ABSTRACT. In this paper we study the "holomorphic K -theory" of a projective variety. This K - theory is defined in terms of the homotopy type of spaces of holomorphic maps from the variety to Grassmannians and loop groups. This theory has been introduced in various places such as [12], [9], and a related theory was considered in [11]. This theory is built out of studying algebraic bundles over a variety up to "algebraic equivalence". In this paper we will give calculations of this theory for "flag like varieties" which include projective spaces, Grassmannians, flag manifolds, and more general homogeneous spaces, and also give a complete calculation for symmetric products of projective spaces. Using the algebraic geometric definition of the Chern character studied by the authors in [6], we will show that there is a rational isomorphism of graded rings between holomorphic K - theory and the appropriate "morphic cohomology" groups, defined in [7] in terms of algebraic co-cycles in the variety. In so doing we describe a geometric model for rational morphic cohomology groups in terms of the homotopy type of the space of algebraic maps from the variety to the "symmetrized loop group" $\Omega U(n)/\Sigma_n$ where the symmetric group Σ_n acts on $U(n)$ via conjugation. This is equivalent to studying algebraic maps to the quotient of the infinite Grassmannians $BU(k)$ by a similar symmetric group action. We then use the Chern character isomorphism to prove a conjecture of Friedlander and Walker stating that if one localizes holomorphic K - theory by inverting the Bott class, then rationally this is isomorphic to topological K - theory. Finally this will allows us to produce explicit obstructions to periodicity in holomorphic K - theory, and show that these obstructions vanish for generalized flag manifolds.

INTRODUCTION

The study of the topology of holomorphic mapping spaces $Hol(X, Y)$, where X and Y are complex manifolds has been of interest to topologists and geometers for many years. In particular when Y is a Grassmannian or a loop group, the space of holomorphic maps

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yields parameter spaces for certain moduli spaces of holomorphic bundles (see [21], [1], [4]). In this paper we study the K -theoretic properties of such holomorphic mapping spaces.

More specifically, let X be any projective variety, let $Gr_n(\mathbb{C}^M)$ denote the Grassmannian of n - planes in $\mathbb{C}^M$ (with its usual structure as a smooth projective variety), and let $\Omega U(n)$ denote the loop group of the unitary group $U(n)$, with its structure as an infinite dimensional smooth algebraic variety ([21]). We let

$$Hol(X; Gr_n(\mathbb{C}^M)) \quad \text{and} \quad Hol(X; \Omega U(n))$$

denote the spaces of algebraic maps between these varieties (topologized as subspaces of the corresponding spaces of continuous maps, with the compact open topologies). We use this notation because if X is smooth these spaces of algebraic maps are precisely the same as holomorphic maps between the underlying complex manifolds. The *holomorphic K -theory* space $K_{hol}(X)$ is defined to be the Quillen - Segal group completion of the union of these mapping spaces, which we write as

$$K_{hol}(X) = Hol(X; \mathbb{Z} \times BU)^+ = Hol(X; \Omega U)^+.$$

This group completion process will be described carefully below. The holomorphic K - groups will be defined to be the homotopy groups

$$K^{-q}_{hol}(X) = \pi_q(K_{hol}(X)).$$

A variant of this construction was first incidentally introduced in [12], and subsequently developed in [16] where one obtains various connective spectra associated to an algebraic variety X, using spaces of algebraic cycles. The case of Grassmannians is the one treated here. A theory related to holomorphic K - theory was also studied by Karoubi in [11] and the construction we use here coincides with the definition of "semi-topological K - theory" studied by Friedlander and Walker in [9]. Indeed their terminology reflects the fact that for a smooth projective variety X holomorphic K - theory sits between algebraic K - theory of the associated scheme and the topological K - theory of its underlying topological space. More precisely, using Morel and Voevodsky's description algebraic K theory of a smooth variety X (via their work on $\mathbb{A}^1$ - homotopy theory [19]), Friedlander and Walker showed that there are natural transformations

$$K_{alg}(X) \xrightarrow{\alpha} K_{hol}(X) \xrightarrow{\beta} K_{top}(X)$$

so that the map $\beta : K_{hol}(X) \to K_{top}(X)$ is the map induced by including the holomorphic mapping space $Hol(X; \mathbb{Z} \times BU)$ in the topological mapping space $Map(X; \mathbb{Z} \times BU)$, and where the composition $\beta \circ \alpha : K_{alg}(X) \to K_{top}(X)$ is the usual transformation from algebraic

K -theory to topological K -theory induced by forgetting the algebraic stucture of a vector bundle.

In this paper we calculate the holomorphic K -theory of a large class of varieties, including "flag - like varieties", a class that includes Grassmannians, flag manifolds and more general homogeneous spaces. We also give a complete calculation of the holomorphic K -theory of arbitrary symmetric products of projective spaces. Since the *algebraic* K -theory of such symmetric product spaces is not in general known, these calculations should be of interest in their own right. We then study the Chern character for holomorphic K - theory, using the algebraic geometric description of the Chern character constructed by the authors in [6]. The target of the Chern character transformation is the "morphic cohomology" $L^*H^*(X) \otimes \mathbb{Q}$, defined in terms of algebraic co-cycles in X [7]. We then prove the following.

Theorem 1. *For any projective variety (or appropriate colimit of project varieties) X, the Chern character is a natural transformation*

$$ch : K_{hol}^{-q}(X) \otimes \mathbb{Q} \longrightarrow \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

which is an isomorphism for every $q \geq 0$. Furthermore it preserves a natural multiplicative structure, so that it is an isomorphism of graded rings.

In the proof of this theorem we develop techniques which will yield the following interesting descriptions of morphic cohomology that don't involve the use of higher Chow varieties.

Consider the following quotient spaces by appropriate actions of the symmetric groups:

$$\begin{aligned}
\Omega U/\Sigma &= \varinjlim{}_n \Omega U(n)/\Sigma_n, \\
BU/\Sigma &= \varinjlim{}_{n,m} Gr_m(\mathbb{C}^{nm})/\Sigma_m \\
SP^\infty(\mathbb{CP}^\infty) &= \varinjlim{}_n (\prod_n \mathbb{CP}^\infty)/\Sigma_n
\end{aligned}$$

Theorem 2. *If X is any projective variety, then the Quillen - Segal group completion of the following spaces of algebraic maps*

$$Mor(X, \Omega U/\Sigma)^+ \quad Mor(X, BU/\Sigma)^+, \quad \textit{and} \quad Mor(X, SP^\infty(\mathbb{CP}^\infty))^+$$

are all rationally homotopy equivalent. Moreover their k^{th} - rational homotopy groups (which we call π_k) are isomorphic to the rational morphic cohomology groups

$$\pi_k \cong \oplus_{p=1}^{\infty} L^p H^{2p-k}(X) \otimes \mathbb{Q}.$$

Among other things, the relation between morphic cohomology and the morphism space into the "symmetrized" loop group allows, using loop group machinery, a geometric description of these cohomology groups in terms of a certain moduli space of algebraic bundles with symmetric group action.

We then use Theorem 1 to prove the following result about "Bott periodic holomorphic K - theory". $K^*_{hol}(X)$ is a module over $K^*_{hol}(point)$ in the usual way, and since $K^*_{hol}(point) = K^*_{top}(point)$, we have a "Bott class" $b \in K^{-2}_{hol}(point)$. The module structure then defines a transformation

$$b_* : K^{-q}_{hol}(X) \to K^{-q-2}_{hol}(X).$$

If $K^*_{hol}(X)[\frac{1}{b}]$ denotes the localization of $K^*_{hol}(X)$ obtained by inverting this operator, we will then prove the following rational version of a conjecture of Friedlander and Walker [9].

Theorem 3. *The map $\beta : K^*_{hol}(X)[\frac{1}{b}] \otimes \mathbb{Q} \to K^*_{top}(X) \otimes \mathbb{Q}$ is an isomorphism.*

Finally we describe a necessary conditions for the holomorphic K - theory of a smooth variety to be Bott periodic (i.e $K_{hol}(X) \cong K_{hol}(X)[\frac{1}{b}]$) in terms of the Hodge filtration of its cohomology. We will show that generalized flag varieties satisfy this condition and their holomorphic K - theory is Bott periodic. We also give examples of varieties for which these conditions fail and hence whose holomorphic K - theory is not Bott periodic.

This paper is organized as follows. In section 1 we give the definition of holomorphic K - theory in terms of loop groups and Grassmannians, and prove that the holomorphic K - theory space, $K_{hol}(X)$ is an infinite loop space. In section 2 we give a proof of a result of Friedlander and Walker that $K^0_{hol}(X)$ is the Grothendieck group of the monoid of algebraic bundles over X modulo a notion of "algebraic equivalence". We prove this theorem here for the sake of completeness, and also because our proof allows us to compute the holomorphic K - theory of flag - like varieties, which we also do in section 2. In section 3 we identify the equivariant homotopy type of the holomorphic K - theory space $K_{hol}(\prod_n \mathbb{P}^1)$, where the group action is induced by the permutation action of the symmetric group Σ_n. This will allow us to compute the holomorphic K - theory of symmetric products of projective spaces, $K_{hol}(SP^m(\mathbb{P}^n))$. In section 4 we recall the Chern character defined in [6] we prove that is an isomorphism of rational graded rings(Theorem 1). In section 5 we prove Theorem

2 giving alternative descriptions of morphic cohomology. Finally in section 6 rational maps in the holomorphic K - theory spaces $K_{hol}(X)$ are studied, and they are used, together with the Chern character isomorphism, to prove Theorem 3 regarding Bott periodic holomorphic K - theory.

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1. The Holomorphic K-theory space

In this section we define the holomorphic K-theory space $K_{hol}(X)$ for a projective variety X and show that it is an infinite loop space.

For the purposes of this paper we let $\Omega U(n)$ denote the group of based *algebraic* loops in the unitary group $U(n)$. That is, an element of $\Omega U(n)$ is a map $\gamma : S^1 \to U(n)$ such that $\gamma(1) = 1$ and γ has finite Fourier series expansion. Namely, γ can be written in the form

$$\gamma(z) = \sum_{k=-N}^{k=N} A_k z^k$$

for some N, where the A_k's are $n \times n$ matrices. It is well known that the inclusion of the group of algebraic loops into the space of all smooth (or continuous) loops is a homotopy equivalence of infinite dimensional complex manifolds [21].

Let X be a projective variety. It was shown by Valli in [23] that the holomorphic mapping space $Hol(X, \Omega U(n))$ has a $\mathcal{C}_2$ - operad structure in the sense of May [17]. Here $\mathcal{C}_2$ is the little 2-dimensional cube operad. This in particular implies that the Quillen - Segal group completion, which we denote with the superscript $+$ (after Quillen's $+$ - construction), $Hol(X, \Omega U(n))^+$ has the structure of a two - fold loop space. (Recall that up to homotopy, the Quillen - Segal group completion of a topological monoid A is the loop space of the classifying space, ΩBA.) By taking the limit over n, we define the holomorphic K-theory space to be the group completion of the holomorphic mapping space.

Definition 1.

$$K_{hol}(X) = Hol(X, \Omega U)^+.$$

If $A \subset X$ is a subvariety, we then define the *relative* holomorphic K-theory

$$K_{hol}(X, A)$$

to be the homotopy fiber of the natural restriction map, $K_{hol}(X) \to K_{hol}(A)$.

Before we go on we point out certain basic properties of $K_{hol}(X)$.

1. By the geometry of loop groups studied in [21] (more specifically the "Grassmannian model of a loop group") one knows that every element of the algebraic loop group $\Omega U(n)$ lies in a finite dimensional Grassmannian. When one takes the limit over n, it was observed in [4] that one has the holomorphic diffeomorphism $\mathbb{Z} \times BU \cong \Omega U$, where here BU is given the complex structure as a limit of Grassmannians, and ΩU denotes the limit of the *algebraic* loop groups $\Omega U(n)$. Thus we could have replaced ΩU by $\mathbb{Z} \times BU$ in the definition of $K_{hol}(X)$. That is, we have an equivalent definition:

Definition 2.

$$K_{hol}(X) = Hol(X, \mathbb{Z} \times BU)^+.$$

This definition has the conceptual advantage that

$$\pi_0\left(Hol(X, BU(n))\right) = lim_{m\to\infty}\pi_0\left(Hol(X, Gr_n(\mathbb{C}^m))\right)$$

where $Gr_n(\mathbb{C}^m)$ is the Grassmannian of dimension n linear subspaces of $\mathbb{C}^m$. Moreover this set corresponds to equivalence classes of rank n holomorphic bundles over X that are embedded (holomorphically) in an m - dimensional trivial bundle.

2. It is necessary to take the group completion in our definition of $K_{hol}(X)$. For example, the results of [4] imply that

$$Hol_*(\mathbb{P}^1, \Omega U) \cong \coprod_{k=0}^{\infty} BU(k)$$

where Hol_* denotes basepoint preserving holomorphic maps. Thus this holomorphic mapping space is not an infinite loop space without group completing. In fact after we group complete we obtain

$$K_{hol}(\mathbb{P}^1, *) \cong \mathbb{Z} \times BU$$

and so we have the "periodicity" result

$$K_{hol}(\mathbb{P}^1, *) \cong K_{hol}(*).$$

A more general form of "holomorphic Bott periodicity" is contained in D. Rowland's Ph.D thesis [22] where it is shown that

$$K_{hol}(X \times \mathbb{P}^1, X) \cong K_{hol}(X)$$

for any smooth projective variety X. A more general projective bundle theorem was proved in [9].

We now observe that the two fold loop space mentioned above for holomorphic K-theory can actually be extended to an infinite loop structure.

Proposition 4. *The space $K_{hol}(X) = Hol(X, \mathbb{Z} \times BU)^+$ is an infinite loop space.*

Proof. Let $\mathcal{L}_*$ be the linear isometries operad. That is, $\mathcal{L}_m$ is the space of linear (complex) isometric embeddings of $\oplus_m \mathbb{C}^\infty$ into $\mathbb{C}^\infty$. These spaces are contractible, and the usual operad action

$$\mathcal{L}_m \times_{\Sigma_m} (Gr_n(\mathbb{C}^\infty))^m \longrightarrow Gr_{nm}(\mathbb{C}^\infty)$$

give holomorphic maps for each $\alpha \in \mathcal{L}_m$. It is then simple to verify that this endows the holomorphic mapping space

$$\amalg_n Hol(X, Gr_n(\mathbb{C}^\infty))$$

with the structure of a $\mathcal{L}_*$ - operad space. Since this is an E_∞ operad in the sense of May [17], this implies that the group completion, $K_{hol}(X) = (\amalg_n Hol(X, Gr_n(\mathbb{C}^\infty))^+$ has the structure of an infinite loop space. $\square$

As is usual, we define the (negative) *holomorphic K - groups* to be the homotopy groups of this infinite loop space:

Definition 3. For $q \geq 0$,

$$K_{hol}^{-q}(X) = \pi_q(K_{hol}(X)) = \pi_q(Hol(X, \Omega U)^+).$$

Remarks.

a. Notice that as usual, the holomorphic K-theory is a ring. Namely, the spectrum (in the sense of stable homotopy theory) corresponding to the infinite loop space $K_{hol}(X)$, is in fact a ring spectrum. The ring structure is induced by tensor product operation on Grassmannians,

$$Gr_n(\mathbb{C}^m)^{\otimes k} \to Gr_{nk}(\mathbb{C}^{mk}).$$

We leave it to the reader to check the details that this structure does indeed induce a ring structure on the holomorphic K-theory. Indeed, this parallels the well-known fact that whenever $\mathbf{E}$ is a ring spectrum and X is an arbitrary space, then $\mathrm{Map}(X, \mathbf{E})$ has a natural structure of ring spectrum, where $\mathrm{Map}(-,-)$ denotes the space of continuous maps, with the appropriate compact-open, compactly generated topology; cf. [18].

b. A variant of this construction was first incidentally introduced in [12], and subsequently developed in [16] where one obtains various connective spectra associated to an algebraic variety X, using spaces of algebraic cycles. The case of Grassmannians is, up to π_0 considerations, the one treated here. A theory related to holomorphic K - theory was also studied by Karoubi in [11], and the definition given here coincides with the notion of "semi - topological K- theory" introduced and studied by Friedlander and Walker in [9].

2. The holomorphic K - theory of flag varieties and a general description of $K^0_{hol}(X)$.

The main goal of this section is to prove the following theorem which yields an effective calculation of $K^0_{hol}(X)$, when X is a flag variety.

Theorem 5. *Let X be a generalized flag variety. That is, X is a homogeneous space of the form $X = G/P$ where G is a complex algebraic group and $P < G$ is a parabolic subgroup. Then the natural map from holomorphic K - theory to topological K - theory,*

$$\beta : K^0_{hol}(X) \longrightarrow K^0_{top}(X)$$

is an isomorphism

The proof of this theorem involves a comparison of holomorphic K - theory with algebraic K - theory. As a consequence of this comparison we will recover Friedlander and Walker's description of $K^0_{hol}(X)$ for any smooth projective variety X in terms of "algebraic equivalence classes" of algebraic bundles [9]. We begin by defining this notion of algebraic equivalence.

Definition 4. Let X be a projective variety (not necessarily smooth), and $E_0 \to X$ and $E_1 \to X$ algebraic bundles. We say that E_0 and E_1 are *algebraically equivalent* if there exists a constructible, connected algebraic curve T and an algebraic bundle E over $X \times T$, so that the restrictions of E to $X \times \{t_o\}$ and $X \times \{t_1\}$ are E_0 and E_1 respectively, for some $t_0, t_1 \in T$. Here a *constructible* curve means a finite union of irreducible algebraic (not necessarily complete) curves in some projective space.

Notice that two algebraically equivalent bundles are *topologically* isomorphic, but *not* necessarily isomorphic as algebraic bundles.

Theorem 6. *For any smooth projective algebraic variety X, the group $K^0_{hol}(X)$ is isomorphic to the Grothendieck group completion of the monoid of algebraic equivalence classes of algebraic bundles over X.*

The description of $Mor(X, BU(n))$ given in [4] provides our first step in understanding $K^0_{hol}(X)$.

As in [4], if X is a projective variety then we call an algebraic bundle $E \to X$ *embeddable*, if there exists an algebraic embedding of E into a trivial bundle: $E \hookrightarrow X \times \mathbb{C}^N$ for some large N. Let $\phi : E \hookrightarrow X \times \mathbb{C}^N$ be such an embedding. We identify an embedding ϕ with the composition $\phi : E \hookrightarrow X \times \mathbb{C}^N \hookrightarrow X \times \mathbb{C}^{N+M}$, where $\mathbb{C}^N$ is included in $\mathbb{C}^{N+M}$ as the first N coordinates. We think of such an equivalence class of embeddings as an embedding $E \hookrightarrow X \times \mathbb{C}^\infty$. We refer to the pair (E, ϕ) as an *embedded* algebraic bundle.

Let X be any projective variety, and let E be a rank k holomorphic bundle over X that is holomorphically embeddable in a trivial bundle, define $Hol_E(X, BU(k))$ to be the space of holomorphic maps $\gamma : X \to BU(k)$ such that $\gamma^*(\xi_k) \cong E$, where $\xi_k \to BU(k)$ is the universal holomorphic bundle. This is topologized as a subspace of the continuous mapping space, which is endowed with the compact - open topology.

Let $Aut(E)$ be the gauge group of holomorphic bundle automorphisms of E. The following lemma identifies the homotopy type of $Hol_E(X, BU(k))$ in terms of $Aut(E)$.

Lemma 7. *$Hol_E(X, BU(k))$ is naturally homotopy equivalent to the classifying space*

$$Hol_E(X, BU(k)) \simeq B(Aut(E)).$$

Proof. As was described in [4], elements in $Hol_E(X, BU(k))$ are in bijective correspondence to isomorphism classes of rank k embedded holomorphic bundles, (ζ, ϕ). By modifying the embedding ϕ via an isomorphism between ζ and E, we see that $Hol_E(X, BU(k)$ is homeomorphic to the space of holomorphic embeddings $\psi : E \hookrightarrow X \times \mathbb{C}^\infty$, modulo the action of the holomorphic automorphism group, $Aut(E)$. The space of holomorphic embeddings of E in an infinite dimensional trivial bundle is easily seen to be contractible [4], and the action of $Aut(E)$ is clearly free, with local sections. Again, see [4] for details. The lemma follows. □

Corollary 8. *The space $Hol_E(X; BU(k))$ is connected.*

Now as above, we say that two embedded algebraic bundles (E_0, ϕ_0) and (E_1, ϕ_1), are *path equivalent* if there is topologically embedded bundle (E, ϕ), over $X \times I$, which gives a path equivalence between E_0 and E_1, and over each slice $X \times \{t\}$ is an embedded algebraic bundle. Finally, notice that the set of (algebraic) isomorphism classes of embedded algebraic bundles forms an abelian monoid.

Lemma 9. *For any projective algebraic variety X, the group $K^0_{hol}(X)$ is isomorphic to the Grothendieck group completion of the monoid of path equivalence classes of embedded algebraic bundles over X.*

Proof. Recall that

$$K^0_{hol}(X) = \pi_0\left(\coprod_n Mor(X, BU(n))\right)^+.$$

But the set of path components of the Quillen - Segal group completion of a topological E_∞ space is the Grothendieck group completion of the discrete monoid of path components of the original E_∞ - space. Now as observed above the morphism space $Hol(X, BU(n))$ is given by configurations of isomorphism classes of embedded algebraic bundles, (E, ϕ). Thus $\pi_0(Hol(X, BU(n))$ is the set of path equivalence classes of such pairs; i.e the set of path equivalence classes of embedded algebraic bundles of rank n. We may therefore conclude that $K^0_{hol}(X)$ is the Grothendieck group completion of the monoid of path equivalence classes of embeddable algebraic bundles. □

We now strengthen this result as follows.

Lemma 10. *Two embedded bundles (E_0, ϕ_0) and (E_1, ϕ_1) are path equivalent if and only if they are algebraically equivalent.*

Proof. Let $f : X \times I \to BU(n)$ be the (continuous) classifying map for the topological bundle E over $X \times I$, which gives the path equivalence between E_0 and E_1, and denote f_0 and f_1 the restrictions of f to $X \times \{0\}$ and $X \times \{1\}$. Since $X \times I$ is compact, the image of f is contained in some Grassmannian $Gr_n(\mathbb{C}^m) \subset BU(n)$. It follows that f_0 and f_1 lie in the same path component of $Hol(X, Gr_n(\mathbb{C}^m))$. Since $Hol(X, Gr_n(\mathbb{C}^m))$ is a disjoint union of constructible subsets of the Chow monoid $\mathcal{C}_{dimX}(X \times Gr_n(\mathbb{C}^m))$, then f_0 and f_1 lie in the same connected component of a constructible subset in some projective space. Using the fact that any two points in an irreducible algebraic variety Y lie in some irreducible algebraic curve $C \subset Y$ (see [20, p. 56]), one concludes that any two points in a connected constructible subset of projective space lie in a connected constructible curve. Let T be a connected constructible curve contained in $Hol(X, Gr_n(\mathbb{C}^m))$ and containing f_0 and f_1. Under the canonical identification $Hol(T, Hol(X, Gr_n(\mathbb{C}^m))) \cong Hol(X \times T, Gr_n(\mathbb{C}^m))$, one identifies the inclusion

$$i : T \hookrightarrow Hol(X, Gr_n(\mathbb{C}^m))$$

with an algebraic morphism $\bar{i} : X \times T \to Gr_n(\mathbb{C}^m)$. This map classifies the desired embedded bundle E over $X \times T$.

The converse is clear. $\square$

The above two lemmas imply the following.

Proposition 11. *For any projective algebraic variety X, (not necessarily smooth), the group $K^0_{hol}(X)$ is isomorphic to the Grothendieck group completion of the monoid of algebraic equivalence classes of embedded algebraic bundles over X.*

Notice that Theorem 6 implies that we can remove the "embedded" condition in the statement of this proposition. We will show how that can be done later in this section.

Recall from the last section that the forgetful map from the category of colimits of projective varieties to the category of topological spaces, induces a map of morphism spaces,

$$Hol(X; \mathbb{Z} \times BU) \to Map(X; \mathbb{Z} \times BU)$$

which induces a natural transformation

$$\beta : K_{hol}(X) \to K_{top}(X).$$

Corollary 12. *Let X be a colimit of projective varieties. Then the induced map β : $K^0_{hol}(X) \to K^0_{top}(X)$ is induced by sending the class of an embedded bundle to its underlying topological isomorphism type:*

$$\beta : K^0_{hol}(X) \to K^0_{top}(X)$$
$$[E, \phi] \to [E]$$

In order to approach Theorem 1 we need to understand the relationship between algebraic K - theory, $K^0_{alg}(X)$, and holomorphic K - theory, $K^0_{hol}(X)$ for X a smooth projective variety. For such a variety $K^0_{alg}(X)$ is the Grothendieck group of the exact category of algebraic bundles over X. Roughly speaking the relationship between algebraic and holomorphic K -theories for a smooth variety is the passage from isomorphism classes of holomorphic bundles to algebraic equivalence classes of holomorphic bundles. This relationship was made precise in [9] using the Morel - Voevodsky description of algebraic K - theory of a smooth scheme in terms of an appropriate morphism space. In particular, recall that

$$K^0_{alg}(X) = Mor_{\mathcal{H}((Sm/\mathbb{C})_{Nis}}(X, R\Omega B(\sqcup_{n \geq 0} BGL_n(\mathbb{C}))).$$

where $\mathcal{H}((Sm/\mathbb{C})_{Nis})$ is the homotopy category of smooth schemes over $\mathbb{C}$, using the Nisnevich topology. See [19] for details. In particular a morphism of projective varieties, $f : X \to Gr_n(\mathbb{C}^M)$ induces an element in the above morphism space and hence a class $\langle f\rangle \in K^0_{alg}(X)$. It also induces a class $[f] \in \pi_0(Hol(X;\mathbb{Z}\times BU)^+ = K^0_{hol}(X)$. As seen in [9] this correspondence extends to give a forgetful map from the morphisms in the homotopy category $\mathcal{H}((Sm/\mathbb{C})_{Nis})$ to homotopy classes of morphisms in the category of colimits of projective varieties. This defines a natural transformation

$$\alpha : K^0_{alg}(X) \to K^0_{hol}(X)$$

for X a colimit of smooth projective varieties.

Lemma 13. *For X a smooth projective variety the transformation*

$$\alpha : K^0_{alg}(X) \to K^0_{hol}(X)$$

is surjective.

Proof. As observed above, the set of path components of the Quillen - Segal group completion of a topological monoid Y is the Grothendieck - group completion of the discrete monoid of path components:

$$\pi_0(Y^+) = (\pi_0(Y))^+.$$

Therefore we have that $K^0_{hol}(X)$ is the Grothendieck group completion of $\pi_0(Hol(X,\mathbb{Z}\times BU)$. Thus every element $\gamma \in K^0_{hol}(X)$ can be written as

$$\gamma = [f] - [g]$$

where f and g are holomorphic maps from X to some Grassmannian. By the above observations $\gamma = \alpha(\langle f\rangle - \langle g\rangle)$. □

This lemma and Proposition 11 allow us to prove the following interesting splitting property of $K^0_{hol}(X)$ which is not immediate from its definition.

Theorem 14. *Let*

$$0 \to [F,\phi_F] \to [E,\phi_E] \to [G,\phi_G] \to 0$$

be a short exact sequence of embedded holomorphic bundles over a smooth projective variety X. Then in $K^0_{hol}(X)$ we have the relation

$$[E,\phi_E] = [F,\phi_F] + [G,\phi_G].$$

Proof. This follows from Lemma 13 and the fact that short exact sequences split in $K^0_{alg}(X)$. ☐

Lemma 13 will also allow us to prove Theorem 5 which we now proceed to do. We begin with a definition.

Definition 5. We say that a smooth projective variety X is *flag - like* if the following properties hold on its K - theory:

1. the usual forgetful map
$$\psi : K^0_{alg}(X) \to K^0_{top}(X)$$
is an isomorphism, and

2. $K^0_{alg}(X)$ is generated (as an abelian group) by embeddable holomorphic bundles.

Remark: We call such varieties “flag - like” because generalized flag varieties (homogeneous spaces G/P as in the statement of Theorem 5) satisfy these conditions. We now state a strengthening of Theorem 5 which we prove.

Theorem 15. *Suppose X is a flag - like smooth projective variety. Then the homomorphisms*
$$\alpha : K^0_{alg}(X) \to K^0_{hol}(X)$$
and
$$\beta : K^0_{hol}(X) \to K^0_{top}(X)$$
are isomorphisms of rings.

Proof. Let X be a flag - like smooth projective variety. Since every embeddable holomorphic bundle is represented by a holomorphic map $f : X \to Gr_n(\mathbb{C}^M)$, for some Grassmannian, then property (2) implies that $K^0_{alg}(X)$ is generated by classes $\langle f \rangle$, where f is such a holomorphic map. But then $\psi(\langle f \rangle) \in K^0_{top}(X)$ clearly is the class represented by f in $\pi_0(Map(X, \mathbb{Z} \times BU) = K^0_{top}(X)$. But this means that the map $\psi : K^0_{alg}(X) \to K^0_{top}(X)$ is given by the composition
$$\beta \circ \alpha : K^0_{alg}(X) \to K^0_{hol}(X) \to K^0_{top}(X).$$
But since ψ is an isomorphism this means α is injective. But we already saw in corollary 13 that α is surjective. Thus α, and therefore β, are isomorphisms. Clearly from their descriptions, α and β preserve tensor products, and hence are ring isomorphisms. ☐

We now use this result to prove Theorem 6. Let X be a smooth, projective variety and let $e : X \hookrightarrow \mathbb{CP}^n$ be a projective embedding. We begin by describing a construction which will allow *any* holomorophic bundle E over X to be viewed as representing an element of $K^0_{hol}(X)$ (i.e E does not necessarily have to be embeddable).

So let $E \to X$ be a holomorphic bundle over X. Recall that by tensoring E with a line bundle of sufficiently negative Chern class, it will become embeddable. (This is dual to the statement that tensoring a holomorphic bundle over a smooth projective variety with a line bundle with sufficiently large Chern class produces holomorphic bundle that is generated by global sections.) So for sufficiently large k, the bundle $E \otimes O(-k)$ is embeddable. Here $O(-k)$ is the k - fold tensor product of the canonical line bundle $O(-1)$ over $\mathbb{CP}^n$, which, by abuse of notation, we identify with its restriction to X. Now choose a holomorphic embedding

$$\phi : E \otimes O(-k) \hookrightarrow X \times \mathbb{C}^N.$$

Then the pair $(E \otimes O(-k), \phi)$ determines an element of $K^0_{hol}(X)$.

Now from Theorem 15 we know that $K^0_{alg}(\mathbb{CP}^n) \cong K^0_{hol}(\mathbb{CP}^n) \cong K^0_{top}(\mathbb{CP}^n)$ as rings. But since $O(-k) \otimes O(k) = 1 \in K^0_{top}(\mathbb{CP}^n)$, this means that if

$$\iota_k = \otimes_k \iota : O(-k) = \otimes_k O(-1) \hookrightarrow \otimes_k \mathbb{C}^{n+1}$$

is the canonical embedding, then the pair $(O(-k), \iota_k)$ represents an invertible class in $K^0_{hol}(\mathbb{CP}^n)$. We denote its inverse by $O(-k)^{-1} \in K^0_{hol}(\mathbb{CP}^n)$, and, as before, we use the same notation to denote its restriction to $K^0_{hol}(X)$.

Write $A(E) = (E \otimes O(-k), \phi) \otimes O(-k)^{-1}_e \in K^0_{hol}(X)$.

Proposition 16. *The assignment to the holomorphic bundle E the class*

$$A(E) = [E \otimes O(-k), \phi] \otimes O(-k)^{-1} \in K^0_{hol}(X)$$

is well defined, and only depends on the (holomorphic) isomorphism class of E.

Proof. We first verify that given any holomorphic bundle $E \to X$, that $A(E)$ is a well defined element of $K^0_{hol}(X)$. That is, we need to show that this class is independent of the choices made in its definition. More specifically, we need to show that

$$[E \otimes O(-k), \phi] \otimes O(-k)^{-1} = [E \otimes O(-q), \psi] \otimes O(-q)^{-1}$$

for any appropriate choices of k, q, ϕ, and ψ. We do this in two steps.

Case 1: $\mathbf{k = q}$. In this case it suffices to show that $(E \otimes O(-k), \phi)$ and $(E \otimes O(-k), \psi)$ lie in the same path component of the morphism space $Hol(X, BU(n))$, where n is the rank of E. Now using the notation of Lemma 7 we see that these two elements both lie in $Hol_{E\otimes O(-k)}(X, BU(n))$, which, as proved in Corollary 8 there is path connected.

Case 2: General Case: Suppose without loss of generality that $q > k$. Then clearly the classes $(E \otimes O(-k), \phi) \otimes O(-k)^{-1}$ and $(E \otimes O(-k) \otimes O(-(q-k)), \phi \otimes \iota_{q-k}) \otimes O(-k)^{-1} \otimes O(-(q-k))^{-1}$ represent the same element of $K^0_{hol}(X)$. But this latter class is $(E \otimes O(-q), \phi \otimes \iota_{q-k}) \otimes O(-q)^{-1}$ which we know by case 1 represents the same K - theory class as $(E \otimes O(-q), \psi) \otimes O(-q)^{-1}$.

Thus $A(E)$ is a well defined class in $K^0_{hol}(X)$. Clearly the above arguments also verify that $A(E)$ only depends on the isomorphism type of E. □

Notice that this argument implies that K^0_{hol} encodes *all* holomorphic bundles (not just embeddable ones). We will use this to complete the proof of theorem 6.

Proof. Let X be a smooth projective variety and let $\mathcal{H}_X$ denote the Grothendieck group of monoid of algebraic equivalence classes of holomorphic bundles over X. We show that the correspondence A described in the above theorem induces an isomorphism

$$A : \mathcal{H}_X \xrightarrow{\cong} K^0_{hol}(X).$$

We first show that A is well defined. That is, we need to know if E_0 and E_1 are algebraically equivalent, then $A(E_0) = A(E_1)$. So let $E \to X \times T$ be an algebraic equivalence. Since T is a curve in projective space, we can find a projective embedding of the product, $e : X \times T \hookrightarrow \mathbb{CP}^n$. Now for sufficiently large k, $E \otimes O(-k)$ is embeddable, and given an embedding ϕ_E, the pair $(E \otimes O(-k), \phi_E)$ defines an algebraic equivalence between the embedded bundles $(E_0 \otimes O(-k), \phi_0)$ and $(E_1 \otimes O(-k), \phi_1)$, where the ϕ_i are the appropriate restrictions of the embedding ϕ. Thus

$$[E_0 \otimes O(-k), \phi_0] = [E_1 \otimes O(-k), \phi_1] \in K^0_{hol}(X).$$

Thus

$$[E_0 \otimes O(-k), \phi_0] \otimes O(-k)^{-1} = [E_1 \otimes O(-k), \phi_1] \otimes O(-k)^{-1} \in K^0_{hol}(X).$$

But these classes are $A(E_0)$ and $A(E_1)$. Thus $A : \mathcal{H}_X \to K^0_{hol}(X)$ is well defined.

Notice also that A is surjective. This is because, as was seen in the proof of the last theorem, if (E, ϕ) is and embedded holomorphic bundle, then $A(E) = [E, \phi] \in K^0_{hol}(X)$.

The essential point here being that the choice of the embedding ϕ does not affect the holomorphic K - theory class, since the space of such choices is connected.

Finally notice that A is injective. This is follows from two the two facts:

1. The classes $[O(-k)^{-1}]$ are units in the ring structure of $K^0_{hol}(X)$, and
2. If bundles of the form $E_0 \otimes O(-k)$ and $E_1 \otimes O(-k)$ are algebraically equivalent then the bundles E_0 and E_1 are algebraically equivalent.

$\square$

3. The equivariant homotopy type of $K_{hol}(\prod_n \mathbb{P}^1)$ and the holomorphic K-theory of symmetric products of projective spaces

The goal of this section is to completely identify the holomorphic K - theory of symmetric products of projective spaces, $K_{hol}(SP^n(\mathbb{P}^m))$. Since the algebraic K -theory of these spaces is not in general known, this will give us new information about algebraic bundles over these symmetric product spaces. These spaces are particularly important in this paper since, as we will point out below, symmetric products of projective spaces are representing spaces for morphic cohomology.

Our approach to this question is to study the equivariant homotopy type of $K_{hol}(\prod_n \mathbb{P}^1)$, where the symmetric group Σ_n acts on the holomorphic K - theory space $K_{hol}(\prod_n \mathbb{P}^1) = Hol(\prod_n \mathbb{P}^1; \mathbb{Z} \times BU)^+$ by permuting the coordinates of $\prod_n \mathbb{P}^1$. It acts on the topological K -theory space $K_{top}(\prod_n \mathbb{P}^1) = Map(\prod_n \mathbb{P}^1; \mathbb{Z} \times BU)$ in the same way. The main result of this section is t the following.

Theorem 17. *The natural map $\beta : K_{hol}(\prod_n \mathbb{P}^1) \to K_{top}(\prod_n \mathbb{P}^1)$ is a Σ_n - equivariant homotopy equivalence.*

Before we begin the proof of this theorem we observe the following consequences:

Corollary 18. *Let $G < \Sigma_n$ be a subgroup. Then the induced map on the K - theories of the orbit spaces,*

$$\alpha : K_{hol}(\prod_n \mathbb{P}^1/G) \to K_{top}(\prod_n \mathbb{P}^1/G)$$

is a homotopy equivalence.

Proof. By Theorem 17, $\alpha : K_{hol}(\prod_n \mathbb{P}^1) \to K_{top}(\prod_n \mathbb{P}^1)$ is a Σ_n - equivariant homotopy equivalence. Therefore it induces a homotopy equivalence on the fixed point sets,

$$\alpha : K_{hol}(\textstyle\prod_n \mathbb{P}^1)^G \xrightarrow{\simeq} K_{top}(\textstyle\prod_n \mathbb{P}^1)^G.$$

But these fixed point sets are $K_{hol}(\prod_n \mathbb{P}^1/G)$ and $K_{top}(\prod_n \mathbb{P}^1/G)$ respectively. □

Corollary 19. *$\alpha : K_{hol}(\mathbb{CP}^n) \to K_{top}(\mathbb{CP}^n)$ is a homotopy equivalence.*

Proof. Let $G = \Sigma_n$ in the above corollary. $\prod_n(\mathbb{P}^1)/\Sigma_n = SP^n(\mathbb{P}^1) \cong \mathbb{CP}^n$. □

The following example will be important because as seen earlier, symmetric products of projective spaces form representing spaces for morphic cohomology.

Corollary 20. *Let r and k be any positive integers. Then*

$$\alpha : K_{hol}(SP^r(\mathbb{CP}^k)) \to K_{top}(SP^r(\mathbb{CP}^k))$$

is a homotopy equivalence.

Proof. Let G be the wreath product $G = \Sigma_r \int \Sigma_k$ viewed as a subgroup of the symmetric group Σ_{rk}. The obtain an identification of orbit spaces

$$\left(\prod_{rk} \mathbb{P}^1\right) / \left(\Sigma_r \int \Sigma_k\right) = SP^r(SP^k(\mathbb{P}^1)) \cong SP^r(\mathbb{CP}^k).$$

Finally, apply the above corollary when $n = rk$. □

Observe that this corollary gives a complete calculation of the holomorphic K- theory of symmetric products of projective spaces, since their topological K - theory is known.

In order to begin the proof of Theorem 17 we need to expand our notion of holomorphic K - theory to include unions of varieties. So let A and B be subvarieties of $\mathbb{CP}^n$, then define $Hol(A \cup B, \mathbb{Z} \times BU)$ to be the space of those continuous maps on $A \cup B$ that are holomorphic when restricted to A and B. This space still has the action of the little isometry operad and so we can take a group completion and define $K_{hol}(A \cup B) = Hol(A \cup B, \mathbb{Z} \times BU)^+$. If $A \cup B$ is connected, then we can define the reduced holomorphic K - theory as before, $\tilde{K}_{hol}(A \cup B) =$ the homotopy fiber of the restriction map $K_{hol}(A \cup B) \to K_{hol}(x_0)$, where $x_0 \in A \cap B$. With this we can now define the holomorphic K - theory of a smash product of varieties.

Definition 6. Let X and Y be connected projective projective varieties with basepoints x_0 and y_0 respectively. We define $\tilde{K}_{hol}(X \wedge Y)$ to be the homotopy fiber of the restriction map

$$\rho : \tilde{K}_{hol}(X \times Y) \to \tilde{K}_{hol}(X \vee Y)$$

where $X \vee Y = \{(x, y_0)\} \cup \{(x_0, y)\} \subset X \times Y$.

Recall that in topological K -theory, the Bott periodicity theorem can be viewed as saying the Bott map $\beta : \tilde{K}_{top}(X) \to \tilde{K}_{top}(X \wedge S^2)$ is a homotopy equivalence for any space X. In [22] Rowland studies the holomorphic analogue of this result. She studies the Bott map $\beta : \tilde{K}_{hol}(X) \to \tilde{K}_{hol}(X \wedge \mathbb{P}^1)$ and, using the index of a family of $\bar{\partial}$ operators, defines a map

$\bar{\partial} : \tilde{K}_{hol}(X \wedge \mathbb{P}^1) \to \tilde{K}_{hol}(X)$. Using a refinement of Atiyah's proof of Bott periodicity [1], she proves the following.

Theorem 21. *Given any smooth projective variety X, the Bott map*

$$\beta : \tilde{K}_{hol}(X) \to \tilde{K}_{hol}(X \wedge \mathbb{P}^1)$$

is a homotopy equivalence of infinite loop spaces. Moreover its homotopy inverse is given by the map

$$\bar{\partial} : \tilde{K}_{hol}(X \wedge \mathbb{P}^1) \to \tilde{K}_{hol}(X).$$

The fact that the Bott map β is an isomorphism also follows from the "projective bundle theorem" of Lawson and Walker [9] which was proven independently, using different techniques. This result in the case when $X = S^0$ was proved in [4]. The statement in this case is

$$\tilde{K}_{hol}(\mathbb{P}^1) \simeq \tilde{K}_{hol}(S^0) = \mathbb{Z} \times BU = \tilde{K}_{top}(S^0) \simeq \tilde{K}_{top}(S^2).$$

Combining this with Theorem 21 (iterated several times) we get the following:

Corollary 22. *For a positive integer k, let $\bigwedge_k \mathbb{P}^1 = (\mathbb{P}^1)^{(k)}$ be the k -fold smash product of $\mathbb{P}^1$. Then we have homotopy equivalences*

$$\tilde{K}_{hol}((\mathbb{P}^1)^{(k)}) \simeq \mathbb{Z} \times BU \simeq \tilde{K}_{top}(S^{2k}),$$

We will use this result to prove Theorem 17. We actually will prove a splitting result for $K_{hol}(\prod_n \mathbb{P}^1)$ which we now state.

Let $\mathcal{S}_n$ denote the category whose objects are (unordered) subsets of $\{1, \cdots, n\}$. Morphisms are inclusions. Notice that the cardinality of the set of objects,

$$|Ob(\mathcal{S}_n)| = 2^n.$$

Notice also that the set of objects $Ob(\mathcal{S}_n)$ has an action of the symmetric group Σ_n induced by the permutation action of Σ_n on $\{1, \cdots, n\}$.

Let X be a space with a basepoint $x_0 \in X$. For $\theta \in Ob(\mathcal{S}_n)$, define

$$\prod_\theta X = X^\theta \subset X^n$$

by $X^\theta = \{(x_1, \cdots, x_n)$ such that if j is not an element of θ, then $x_j = x_0 \in X\}$. Notice that if θ is a subset of $\{1, \cdots n\}$ of cardinality k, then $X^\theta \cong X^k$. The smash product

$\bigwedge_\theta X = X^{(\theta)}$ is defined similarly. The following is the splitting theorem that will allow us to prove Theorem 17.

Theorem 23. *Let X be a smooth projective variety (or a union of smooth projective varieties). Then there is a natural Σ_n - equivariant homotopy equivalence*

$$J : \tilde{K}_{hol}(X^n) \longrightarrow \prod_{\theta \in Ob(\mathcal{S}_n)} \tilde{K}_{hol}(X^{(\theta)}).$$

where the action of Σ_n on the right hand side is induced by the permutation action of Σ_n on the objects $Ob(\mathcal{S}_n)$.

Proof. In order to prove this theorem we begin by recalling the equivariant stable splitting theorem of a product proved in [2]. An alternate proof of this can be found in [3].

Given a space X with a basepoint $x_0 \in X$, let $\Sigma^\infty(X)$ denote the suspension spectrum of X. We refer the reader to [14] for a discussion of the appropriate category of equivariant spectra.

Theorem 24. *There is a natural Σ_n equivariant homotopy equivalence of suspension spectra*

$$J : \Sigma^\infty(X^n) \xrightarrow{\simeq} \Sigma^\infty(\bigvee_{\theta \in Ob(\mathcal{S}_n)}(X^{(\theta)})).$$

As a corollary of this splitting theorem we get the following splitting of topological K - theory spaces.

Corollary 25. *There is a Σ_n -equivariant homotopy equivalence of topological K - theory spaces,*

$$J^* : \prod_{\theta \in Ob\mathcal{S}_n} \tilde{K}_{top}(X^{(\theta)}) \to \tilde{K}_{top}(X^n).$$

Proof. Given to spectra E and F, let $sMap(E, F)$ be the spectrum consisting of spectrum maps from E to F. We again refer the reader to [14] for a discussion of the appropriate category of spectra. If Ω^∞ is the zero space functor from spectra to infinite loop spaces, then $\Omega^\infty(sMap(E,F)) = Map_\infty(\Omega^\infty(E), \Omega^\infty(F))$, where Map_∞ refers to the space of infinite loop maps.

Let bu denote the connective topological K - theory spectrum, whose zero space is $\mathbb{Z} \times BU$. Now Theorem 24 yields a Σ_n equivariant homotopy equivalence of the mapping spectra,

$$J^* : sMap(\Sigma^\infty(\bigvee_{\theta \in Ob(\mathcal{S}_n)}(X^{(\theta)}), bu) \xrightarrow{\simeq} sMap(\Sigma^\infty(X^n), bu),$$

and therefore of infinite loop mapping spaces,

$$J^* : Map_\infty(\Omega^\infty\Sigma^\infty(\bigvee_{\theta \in Ob(\mathcal{S}_n)}(X^{(\theta)}), \mathbb{Z} \times BU) \xrightarrow{\simeq} Map_\infty(\Omega^\infty\Sigma^\infty(X^n), \mathbb{Z} \times BU).$$

But since $\Omega^\infty\Sigma^\infty(Y)$ is, in an appropriate sense, the *free* infinite loop space generated by a space Y, then given any other infinite loop space W, the space of infinite loop maps, $Map_\infty(\Omega^\infty\Sigma^\infty(Y), W)$ is equal to the space of (ordinary) maps $Map(Y, W)$. Thus we have a Σ_n - equivariant homotopy equivalence of mapping spaces,

$$J^* : Map((\bigvee_{\theta \in Ob(\mathcal{S}_n)}(X^{(\theta)}), \mathbb{Z} \times BU) \xrightarrow{\simeq} Map(X^n, \mathbb{Z} \times BU).$$

$\square$

Notice that Theorem 23 is the holomorphic version of Corollary 25 . In order to prove this result, we need to develop a holomorphic version of the arguments used in proving Theorem 25. For this we consider the notion of "holomorphic stable homotopy equivalence", as follows. Suppose that X is a smooth projective variety (or union of varieties) and E is a spectrum whose zero space is a smooth projective variety (or a union of such), define $sHol(\Sigma^\infty(X), E)$ to be the subspace of $Map_\infty(\Omega^\infty\Sigma^\infty(X), \Omega^\infty(E))$ consisting of those infinite loop maps $\phi : \Omega^\infty\Sigma^\infty(X) \to \Omega^\infty(E)$ so that the composition

$$X \hookrightarrow \Omega^\infty\Sigma^\infty(X) \xrightarrow{\phi} \Omega^\infty(E)$$

is holomorphic. Notice, for example, that $sHol(\Sigma^\infty(X); bu) = Hol(X, \mathbb{Z} \times BU)$.

Now suppose X and Y are both smooth projective varieties, (or unions of such).

Definition 7. A map of suspension spectra, $\psi : \Sigma^\infty(X) \to \Sigma^\infty(Y)$ is called a *holomorphic stable homotopy equivalence*, if the following two conditions are satisfied.

1. ψ is a homotopy equivalence of spectra.

2. If E is any spectrum whose zero space is a smooth projective variety (or a union of such), then the induced map on mapping spectra, $\psi^* : sMap(\Sigma^\infty(Y), E) \to sMap(\Sigma^\infty(X), E)$ restricts to a map

$$\psi^* s : Hol(\Sigma^\infty(Y), E) \to sHol(\Sigma^\infty(X), E)$$

which is a homotopy equivalence.

With this notion we can complete the proof of Theorem 23. This requires a proof of Theorem 24 that will respect holomorphic stable homotopy equivalences. The version of this theorem given in [3] will do this. We now recall that proof and refer to [3] for details.

Let X be a connected space with basepoint $x_0 \in X$. Let X_+ denote X with a disjoint basepoint, and let $X \vee S^0$ denote the wedge of X with the two point space S^0. Topologically X_+ and $X \vee S^0$ are the same spaces, but their basepoints are in different connected components. However their suspension spectra $\Sigma^\infty(X_+)$ and $\Sigma^\infty(X \vee S^0)$ are stably homotopy equivalent spectra with units (i.e via a stable homotopy equivalence $j : \Sigma^\infty(X_+) \simeq \Sigma^\infty(X \vee S^0)$ that respects the obvious unit maps $\Sigma^\infty(S^0) \to \Sigma^\infty(X_+)$ and $\Sigma^\infty(S^0) \to \Sigma^\infty(X \vee S^0)$.) Moreover it is clear that if X is a smooth projective variety then $\Sigma^\infty(X_+)$ and $\Sigma^\infty(X \vee S^0)$ are holomorphically stably homotopy equivalent in the above sense. Now by taking smash products n -times of this equivalence, we get a Σ_n - equivariant holomorphic stable homotopy equivalence,

$$J_n : \Sigma^\infty((X_+)^{(n)}) = (\Sigma^\infty((X_+))^{(n)} \xrightarrow{j^{(n)}} (\Sigma^\infty(X \vee S^0))^{(n)} = \Sigma^\infty((X \vee S^0)^{(n)}).$$

Now notice that the n - fold smash product $(X_+)^{(n)}$ is naturally (and Σ_n equivariantly) homeomorphic to the cartesian product $(X^n)_+$. Notice also that the n fold iterated smash product of $X \vee S^0$ is Σ_n - equivariantly homeomorphic to the wedge of the smash products,

$$(X \vee S^0)^{(n)} = \left(\bigvee_{\theta \in Ob(\mathcal{S}_n)} X^{(\theta)} \right) \vee S^0.$$

Thus J_n gives a Σ_n - equivariant stable homotopy equivalence,

$$J_n : \Sigma^\infty(((X^n)_+) \xrightarrow{\simeq} \Sigma^\infty(\left(\bigvee_{\theta \in Ob(\mathcal{S}_n)} X^{(\theta)}\right) \vee S^0)).$$

which gives a proof of Theorem 24. Moreover when X is a smooth projective variety (or a union of such) this equivariant stable homotopy equivalence is a holomorphic one. In particular, given any such X, this implies there is a Σ_n equivariant homotopy equivalence

$$J_n^* : sHol_*(\Sigma^\infty((X^n)_+); bu) \xrightarrow{\simeq} sHol_*(\Sigma^\infty(\left(\bigvee_{\theta \in Ob(\mathcal{S}_n)} X^{(\theta)}\right) \vee S^0)); bu).$$

where $sHol_*$ refers to those maps of spectra that preserve the units. If we remove the units from each of these mapping spectra we conclude that we have a Σ_n equivariant homotopy equivalence

$$J_n^* : sHol(\Sigma^\infty(X^n); bu) \xrightarrow{\simeq} sHol(\Sigma^\infty \left(\bigvee_{\theta \in Ob(\mathcal{S}_n)} X^{(\theta)}\right); bu).$$

But these spaces are precisely $Hol(X^n, \mathbb{Z} \times BU)$ and $Hol(\bigvee_{\theta \in Ob(\mathcal{S}_n)} X^{(\theta)}; \mathbb{Z} \times BU) = \prod_{\theta \in Ob(\mathcal{S}_n)} Hol(X^{(\theta)}; \mathbb{Z} \times BU)$ respectively. Theorem 23 now follows. □

We are now in a position to prove Theorem 17.

Proof. By theorems 23 and 25 we have the following homotopy commutative diagram:

$$\begin{array}{ccc}
\tilde{K}_{hol}((\mathbb{P}^1)^n) & \xrightarrow[\simeq]{J_n^*} & \prod_{\theta \in Ob(\mathcal{S}_n)} \tilde{K}_{hol}((\mathbb{P}^1)^{(\theta)}) \\
\beta \downarrow & & \downarrow \beta \\
\tilde{K}_{top}((\mathbb{P}^1)^n) & \xrightarrow[\simeq]{J_n^*} & \prod_{\theta \in Ob(\mathcal{S}_n)} \tilde{K}_{top}((\mathbb{P}^1)^{(\theta)}).
\end{array}$$

Notice that all the maps in this diagram are Σ_n equivariant, and by the results of theorems 23 and 25 the horizontal maps are Σ_n -equivariant homotopy equivalences. Furthermore, by Corollary 22 the maps $\tilde{K}_{hol}((\mathbb{P}^1)^{(\theta)}) \to \tilde{K}_{top}((\mathbb{P}^1)^{(\theta)})$ are homotopy equivalences. Now since the Σ_n action on $\prod_{\theta \in Ob(\mathcal{S}_n)} \tilde{K}_{hol}((\mathbb{P}^1)^{(\theta)})$ and on $\prod_{\theta \in Ob(\mathcal{S}_n)} \tilde{K}_{top}((\mathbb{P}^1)^{(\theta)})$ is given by permuting the factors according to the action of Σ_n on $Ob(\mathcal{S}_n)$, this implies that the right hand vertical map in this diagram, $\beta : \prod_{\theta \in Ob(\mathcal{S}_n)} \tilde{K}_{hol}((\mathbb{P}^1)^{(\theta)}) \to \prod_{\theta \in Ob(\mathcal{S}_n)} \tilde{K}_{top}((\mathbb{P}^1)^{(\theta)})$ is a Σ_n -equivariant homotopy equivalence. Hence the left hand vertical map

$$\beta : \tilde{K}_{hol}((\mathbb{P}^1)^n) \to \tilde{K}_{top}((\mathbb{P}^1)^n)$$

is also a Σ_n - equivariant homotopy equivalence. This is the statement of Theorem 17. $\square$

4. The Chern character for holomorphic K - theory

In this section we study the Chern character for holomorphic K - theory that was defined by the authors in [6]. The values of this Chern character are in the rational Friedlander - Lawson "morphic cohomology groups", $L^*H^*(X) \otimes \mathbb{Q}$. Our goal is to show that the Chern character is gives an isomorphism

$$ch : K_{hol}^{-q}(X) \otimes \mathbb{Q} \cong \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}.$$

Recall the following basic results about the Chern character proved in [6].

Theorem 26. *There is a natural transformation of functors from the category of colimits of projective varieties to algebras over the rational numbers,*

$$ch : K_{hol}^{-*}(X) \otimes \mathbb{Q} \longrightarrow \bigoplus_{k=0}^{\infty} L^k H^{2k-*}(X) \otimes \mathbb{Q}$$

that satisfies the following properties.

1. *The Chern character is compatible with the Chern character for topological K - theory. That is, the following diagram commutes:*

$$\begin{array}{ccc} K_{hol}^{-q}(X) \otimes \mathbb{Q} & \xrightarrow{\beta_*} & K_{top}^{-q}(X) \otimes \mathbb{Q} \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q} & \xrightarrow[\phi_*]{} & \bigoplus_{k=0}^{\infty} H^{2k-q}(X;\mathbb{Q}) \end{array}$$

   where ϕ_* is the natural transformation from morphic cohomology to singular cohomology as defined in [7]

2. *Let $ch_k : K_{hol}^{-q}(X) \otimes \mathbb{Q} \to L^k H^{2k-q}(X) \otimes \mathbb{Q}$ be the projection of ch onto the k^{th} factor. Also let $c_k : K_{hol}^{-q}(X) \to L^k H^{2k-q}(X)$ be the k^{th} Chern class defined in* [12, §6] *(see* [16, §4] *for details). Then there is a polynomial relation between natural transformations*

$$c_k = a_k ch_k + p(ch_1, \cdots, ch_{k-1})$$

   where a_k is a nonzero rational number, and $p(ch_1, \cdots, ch_{k-1})$ is some polynomial in the first $k-1$ Chern characters.

As mentioned above the goal of this section is to prove the following theorem regarding the Chern character.

Theorem 27. *For every $q \geq 0$, the Chern character for holomorphic K - theory*

$$ch : K^{-q}_{hol}(X) \otimes \mathbb{Q} \to \bigoplus_{k \geq 0} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

is an isomorphism.

Proof. Recall from [7] that the suspension theorem in morphic cohomology implies that morphic cohomology can be represented by morphisms into spaces of zero cycles in projective spaces. Since zero cycles are given by points in symmetric products this can be interpreted in the following way. Let $SP^\infty(\mathbb{P}^\infty)$ be the infinite symmetric product of the infinite projective space. Given a projective variety X, let $Mor(X, \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))$ denote the colimit of the the algebraic morphism spaces $Mor(X; SP^n(\mathbb{P}^m))$.

Lemma 28. *Let X be a colimit of projective varieties. Then*

$$\pi_q(Mor(X; (\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))^+) \cong \bigoplus_{k \geq 0} L^k H^{2k-q}(X).$$

Similarly, $\mathbb{Z} \times BU$ represents holomorphic K - theory in the sense that

$$\pi_q(Mor(X; \mathbb{Z} \times BU)^+) \cong K^{-q}_{hol}(X). \tag{4.1}$$

Thus to prove Theorem 27 we will describe a relationship between the representing spaces $\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)$ and $\mathbb{Z} \times BU$.

Using the identification in Lemma 28, let

$$\iota \in \bigoplus_{k=1}^{\infty} L^k H^{2k}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))$$

correspond to the class in $\pi_0((Mor(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))^+)$ represented by the identity map $id : \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty) \to \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)$.

Lemma 29. *There exists a unique class $\tau \in K^0_{hol}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$ with Chern character $ch(\tau) = \iota \in \bigoplus_{k=0}^{\infty} L^k H^{2k}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$.*

Proof. By Corollary 20 in section 3, we know that for every k and n, $K_{hol}(SP^k(\mathbb{P}^n)) \to K_{top}(SP^k(\mathbb{P}^n))$ is a homotopy equivalence. It follows that by taking limits we have that

$K_{hol}(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty)) \to K_{top}(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty))$ is a homotopy equivalence. But we also know from [15] that the natural map

$$\phi : \bigoplus_{k=0}^{\infty} L^k H^{2k}(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty))\otimes\mathbb{Q} \to \bigoplus_{k=0}^{\infty} H^{2k}(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty);\mathbb{Q})$$

is an isomorphism. This is true because for the following reasons.

1. Since products $\prod_n(\mathbb{P}^1)$ have "algebraic cell decompositions" in the sense of [15], its morphic cohomology and singular cohomology coincide,

$$\phi : L^k H^p(\textstyle\prod_n(\mathbb{P}^1)) \xrightarrow{\cong} H^p(\textstyle\prod_n \mathbb{P}^1).$$

2. Since both morphic cohomology and singular cohomology admit transfer maps ([7]) there is a natural identification of $L^kH^p(SP^r(\mathbb{P}^m)\otimes\mathbb{Q}$ and $H^p(SP^r(\mathbb{P}^m);\mathbb{Q})$ with the $\Sigma_r \int \Sigma_m$ invariants in $L^kH^p(\prod_{rm}\mathbb{P}^1)\otimes\mathbb{Q}$ and $H^p(\prod_{rm}\mathbb{P}^1;\mathbb{Q})$ respectively. Since the natural transformation $\phi : L^kH^p(\prod_{rm}\mathbb{P}^1) \to H^p(\prod_{rm}\mathbb{P}^1)$ is equivariant, then we get an induced isomorphism on the invariants,

$$\phi : L^k H^p(SP^r(\mathbb{P}^m))\otimes\mathbb{Q} \xrightarrow{\cong} H^p(SP^r(\mathbb{P}^m);\mathbb{Q}).$$

3. By taking limits over r and m we conclude that

$$\phi : L^k H^p(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty))\otimes\mathbb{Q} \longrightarrow H^p(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty);\mathbb{Q})$$

   is an isomorphism.

Using this isomorphism and the compatibility of the Chern character maps in holomorphic and topological K - theories, to prove this theorem it is sufficient to prove that there exists a unique class $\tau \in K^0_{top}((\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty))\otimes\mathbb{Q}$ with (topological) Chern character

$$ch(\tau) = \iota \in [\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty);\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty)]\otimes\mathbb{Q} \cong \oplus_{k=0}^{\infty} H^{2k}(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty);\mathbb{Q})$$

where $\iota \in [\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty);\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty)]$ is the class represented by the identity map. But this follows because the Chern character in topological K - theory, $ch : \tilde{K}^0_{top}(X)\otimes\mathbb{Q} \to \oplus_{k=1}^{\infty} H^{2k}(X;\mathbb{Q})$ is an isomorphism. □

We now show how the element $\tau \in K^0_{hol}(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty))\otimes\mathbb{Q}$ defined in the above lemma will yield an inverse to the Chern character transformation.

Theorem 30. *The element $\tau \in K^0_{top}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$ defines natural transformations*

$$\tau_* : \bigoplus_{k \geq 0} L^k H^{2k-q}(X) \otimes \mathbb{Q} \longrightarrow K^{-q}_{hol}(X) \otimes \mathbb{Q}$$

such that the composition

$$ch \circ \tau_* : \bigoplus_{k \geq 0} L^k H^{2k-q}(X) \otimes \mathbb{Q} \to K^{-q}_{hol}(X) \otimes \mathbb{Q} \to \bigoplus_{k \geq 0} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

is equal to the identity.

Proof. The set of path components of the Quillen - Segal group completion of a topological monoid is the Grothendieck group completion of the discrete monoid of path components. If we use the notation $\hat{\mathcal{M}}$ to mean the Grothendieck group of a discrete monoid $\mathcal{M}$, this says that

$$K^0_{hol}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) = \pi_0(Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times BU)^+) \cong (\pi_0(Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times BU)))\hat{}\,,$$

and hence

$$K^0_{hol}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q} \cong (\pi_0(Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times BU)_{\mathbb{Q}}))\hat{}\,,$$

where the subscript $\mathbb{Q}$ denotes the holomorphic mapping space localized at the rationals. This means that τ can be represented as a difference of classes,

$$\tau = [\tau_1] - [\tau_2]$$

where $\tau_i \in Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times BU)_{\mathbb{Q}}$.

Now consider the composition pairing

$$Hol(X; \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \times Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times BU) \to Hol(X; \mathbb{Z} \times BU) \to Hol(X; \mathbb{Z} \times BU)^+.$$

which localizes to a pairing

$$Hol(X; \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}} \times Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times BU)_{\mathbb{Q}} \to Hol(X; \mathbb{Z} \times BU)_{\mathbb{Q}} \to Hol(X; \mathbb{Z} \times BU)^+_{\mathbb{Q}}.$$

Using this pairing, τ_1 and τ_2 each define transformations

$$\tau_i : Hol(X; \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}} \to Hol(X; \mathbb{Z} \times BU)^+_{\mathbb{Q}}.$$

Using the fact that $Hol(X; \mathbb{Z} \times BU)^+_{\mathbb{Q}}$ is an infinite loop space, then the subtraction map is well defined up to homotopy,

$$\tau_1 - \tau_2 : Hol(X; \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}} \to Hol(X; \mathbb{Z} \times BU)^+_{\mathbb{Q}}.$$

We need the following intermediate result about this construction.

Lemma 31. *For any projective variety (or colimit of varieties) X, the map*

$$\tau_1 - \tau_2 : Hol(X; \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_\mathbb{Q} \to Hol(X; \mathbb{Z} \times BU)^+_\mathbb{Q}.$$

is a map of H - spaces.

Proof. Since the construction of these maps was done at the representing space level, it is sufficient to verify the claim in the case when X is a point. That is, we need to verify that the compositions

$$\begin{aligned} \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)_\mathbb{Q} \times \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)_\mathbb{Q} &\xrightarrow{(\tau_1-\tau_2)\times(\tau_1-\tau_2)} (\mathbb{Z} \times BU)_\mathbb{Q} \times (\mathbb{Z} \times BU)_\mathbb{Q} \\ &\xrightarrow{\mu} (\mathbb{Z} \times BU)_\mathbb{Q} \end{aligned} \tag{4.2}$$

and

$$\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)_\mathbb{Q} \times \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)_\mathbb{Q} \xrightarrow{\nu} \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)_\mathbb{Q} \xrightarrow{(\tau_1-\tau_2)} (\mathbb{Z} \times BU)_\mathbb{Q} \tag{4.3}$$

represent the same elements of $K^0_{hol}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty) \times \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$, where μ and ν are the monoid multiplications in $\mathbb{Z} \times BU$ and $\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)$ respectively. But by Corollary 20 of the last section, this is the same as $K^0_{top}((\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \times (\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))) \otimes \mathbb{Q}$. Now in the topological category, we know that the class $\tau \in K^0_{hol}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$ is the inverse to the Chern character and hence induces a rational equivalence of H - spaces

$$\tau : (\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_\mathbb{Q} \xrightarrow{\simeq} (\mathbb{Z} \times BU)_\mathbb{Q}.$$

This implies that the compositions 4.2 and 4.3 repesent the same elements of $K^0_{top}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$, and hence the same elements in $K^0_{hol}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$. $\square$

Thus the map

$$\tau_1 - \tau_2 : Hol(X, \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_\mathbb{Q} \to Hol(X, \mathbb{Z} \times BU)^+_\mathbb{Q}$$

is an H - map from a $\mathcal{C}_\infty$ operad spaces (as described in §1), to an infinite loop space. But any such rational H - map extends in a unique manner up to homotopy, to a map of H - spaces of their group completions

$$\tau_1 - \tau_2 : Hol(X.\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))^+_\mathbb{Q} \to Hol(X, \mathbb{Z} \times BU)^+_\mathbb{Q}.$$

This map is natural in the category of colimits of projective varieties X. Since any H - map between rational infinite loop spaces is homotopic to an infinite loop map, this then defines a natural transformation of rational infinite loop spaces,

$$\tau = \tau_1 - \tau_2 : Hol(X.\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))^+_\mathbb{Q} \to Hol(X, \mathbb{Z} \times BU)^+_\mathbb{Q}. \tag{4.4}$$

So when we apply homotopy groups τ defines natural transformations

$$\tau_* : \bigoplus_{k\geq 0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q} \longrightarrow K^{-q}_{hol}(X) \otimes \mathbb{Q}. \tag{4.5}$$

Now notice that if we let $X = \mathbb{Z}\times SP^\infty(\mathbb{P}^\infty)$ in (4.4), and $\iota \in Hol(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty), \mathbb{Z}\times SP^\infty(\mathbb{P}^\infty))^+_{\mathbb{Q}}$ be the class represented by the identity map, then by definition, one has that

$$\tau(\iota) \in Hol(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty), \mathbb{Z}\times BU)^+_{\mathbb{Q}}$$

represents the class $[\tau] \in K^0_{hol}(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty))$ described in Lemma 29. Moreover this lemma tells us that $ch([\tau]) = [\iota] \in \bigoplus_{k\geq 0}^{\infty} L^k H^{2k}(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty))$. Now as in section 4, we view the Chern character as represented by an element $ch \in Hol(\mathbb{Z}\times BU; \mathbb{Z}\times SP^\infty(\mathbb{P}^\infty))^+_{\mathbb{Q}}$ which is a map of rational infinite loop spaces, then this lemma tells us that the elements

$$ch\circ\tau(\iota) \in Hol(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty); \mathbb{Z}\times SP^\infty(\mathbb{P}^\infty))^+_{\mathbb{Q}}$$

and

$$\iota \in Hol(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty); \mathbb{Z}\times SP^\infty(\mathbb{P}^\infty))^+_{\mathbb{Q}}$$

are both maps of rational infinite loop spaces and lie in the same path component of $Hol(\mathbb{Z}\times SP^\infty(\mathbb{P}^\infty); \mathbb{Z}\times SP^\infty(\mathbb{P}^\infty))^+_{\mathbb{Q}}$. But this implies the $ch\circ\tau$ and ι define the homotopic natural transformations of rational infinite loop spaces,

$$ch\circ\tau \simeq \iota : Hol(X, \mathbb{Z}\times SP^\infty(\mathbb{P}^\infty))^+_{\mathbb{Q}} \to Hol(X; \mathbb{Z}\times SP^\infty(\mathbb{P}^\infty))^+_{\mathbb{Q}}.$$

When we apply homotopy groups this means that

$$ch\circ\tau = id : \bigoplus_{k\geq 0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q} \longrightarrow \bigoplus_{k\geq 0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

which was the claim in the statement of Theorem 30. $\square$

We now can complete the proof of Theorem 27. That is we need to prove that

$$ch : K^{-q}_{hol}(X) \otimes \mathbb{Q} \longrightarrow \bigoplus_{k\geq 0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

is an isomorphism. By Theorem 30 we know that ch is surjective. We therefore need to show that it is injective. To do this we prove we will prove the following:

Lemma 32. *The composition of natural transformations*

$$\tau_*\circ ch : K^{-q}_{hol}(X) \otimes \mathbb{Q} \to \bigoplus_{k\geq 0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q} \to K^{-q}_{hol}(X) \otimes \mathbb{Q}$$

is the identity.

Proof. These transformations are induced on the representing level by maps of rational infinite loop spaces,

$$ch : (\mathbb{Z} \times BU)_{\mathbb{Q}} \to (\mathbb{Z} \times SP^{\infty}(\mathbb{P}^{\infty}))_{\mathbb{Q}}$$

and

$$\tau : (\mathbb{Z} \times SP^{\infty}(\mathbb{P}^{\infty}))_{\mathbb{Q}} \to (\mathbb{Z} \times BU)_{\mathbb{Q}}.$$

The composition

$$\tau \circ ch : (\mathbb{Z} \times BU)_{\mathbb{Q}} \to (\mathbb{Z} \times BU)_{\mathbb{Q}}$$

represents an element of rational holomorphic K - theory,

$$[\tau \circ ch] \in K^0_{hol}(\mathbb{Z} \times BU)_{\mathbb{Q}}.$$

Now the fact that τ is an inverse of the Chern character in topological K - theory tells us that

$$[\tau \circ ch] = j \in K^0_{top}(\mathbb{Z} \times BU)_{\mathbb{Q}},$$

where $j \in K^0_{top}(\mathbb{Z} \times BU) = \pi_0(Map(\mathbb{Z} \times BU; \mathbb{Z} \times BU))$ is the class represented by the identity map. But according to the results in §2, we know

$$K^0_{hol}(\mathbb{Z} \times BU)_{\mathbb{Q}} \cong K^0_{top}(\mathbb{Z} \times BU)_{\mathbb{Q}}.$$

So by the compatibility of the Chern characters in holomorphic and topological K - theories, we conclude that

$$[\tau \circ ch] = j \in K^0_{hol}(\mathbb{Z} \times BU)_{\mathbb{Q}}.$$

This implies that $\tau \circ ch : (\mathbb{Z} \times BU)_{\mathbb{Q}} \to (\mathbb{Z} \times BU)_{\mathbb{Q}}$ and the identity map $id : (\mathbb{Z} \times BU)_{\mathbb{Q}} \to (\mathbb{Z}\times BU)_{\mathbb{Q}}$ induce the same natural transformations $Hol(X; \mathbb{Z}\times BU)^+_{\mathbb{Q}} \to Hol(X; \mathbb{Z}\times BU)^+_{\mathbb{Q}}$. Applying homotopy groups implies that

$$\tau_* \circ ch : K^{-q}_{hol}(X) \otimes \mathbb{Q} \to \bigoplus_{k\geq 0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q} \to K^{-q}_{hol}(X) \otimes \mathbb{Q}$$

is the identity as claimed. □

This lemma implies that $ch : K^{-q}_{hol}(X) \otimes \mathbb{Q} \to \bigoplus_{k\geq 0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$ is injective. As remarked above this was the last remaining fact to be verified in the proof of Theorem 27. □

5. STABILITY OF RATIONAL MAPS AND BOTT PERIODIC HOLOMORPHIC K - THEORY

In this section we study the space of rational maps in the morphism spaces used to define holomorphic K - theory. We will show that the "stability property" for rational maps in the morphism space $Hol(X, \mathbb{Z} \times BU)$ amounts to the question of whether Bott perioidicity holds in $K^*_{hol}(X)$. We then use the Chern character isomorphism proved in the last section to prove a conjecture of Friedlander and Walker [9] that rationally, Bott periodic holomorphic K - theory is isomorphic to topological K - theory. (Friedlander and Walker actually conjectured that this statement is true integrally.) Along the way we prove another conjecture of Friedlander and Walker, that the rationally, total Chern class (as defined in [12])

$$c : K^{-q}_{hol}(X) \otimes \mathbb{Q} \longrightarrow \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

is an isomorphism for every $q \geq 0$. (We remark that Friedlander and Walker proved this result for $q = 0$.)

Given a projective variety Y with basepoint $y_0 \in Y$, let $Hol_{y_0}(\mathbb{P}^1, Y)$ denote the space of holomorphic (algebraic) maps $f : \mathbb{P}^1 \to Y$ satsfying the basepoint condition $f(\infty) = y_0$. We refer to this space as the space of based rational maps in Y. In [5] the "group completion" of this space of rational maps $Hol_{y_0}(\mathbb{P}^1, Y)^+$ was defined. This notion of group completion had the property that if $Hol_{y_0}(\mathbb{P}^1, Y)$ has the structure of a topological monoid, then $Hol_{y_0}(\mathbb{P}^1, Y)^+$ is the Quillen - Segal group completion. In general $Hol_{y_0}(\mathbb{P}^1, Y)^+$ was defined to be a space of limits of "chains "of rational maps, topologized using Morse theoretic considerations. We refer the reader to [5] for details. We recall also from that paper the following definition.

Definition 8. The space of rational maps in a projective variety Y is said to *stabilize*, if the group completion of the space of rational maps is homotopy equivalent to the space of continuous maps,

$$Hol_{y_0}(\mathbb{P}^1, Y)^+ \simeq \Omega^2 Y.$$

In [5] criteria for when the rational maps in a projective variety (or symplectic manifold) stabilize were discussed and analyzed. In this paper we study the implications in holomorphic K -theory of the stability of rational maps in the varieties $Hol(X, Gr_n(\mathbb{C}^M))$, where X is a smooth projective variety, and $Gr_n(\mathbb{C}^M)$is the Grassmannian of n - dimensional subspaces of $\mathbb{C}^M$. (The fact that the space of morphisms from one projective variety to another is in turn algebraic is well known. See, for example [10, 9] for discussions about the algebraic structure of morphisms between varieties.) We actually study rational maps in $Hol(X;\mathbb{Z}\times BU)$, which is a colimit of projective varieties. In fact we will study rational maps in the group completion $Hol(X;\mathbb{Z}\times BU)^+$ by which we mean the group completion of the relative morphism space,

$$Hol_*(\mathbb{P}^1; Hol(X;\mathbb{Z}\times BU)^+) = Hol(\mathbb{P}^1\times X, \infty\times X;\mathbb{Z}\times BU)^+.$$

Theorem 33. *Let X be a smooth projective variety. Then the space of rational maps in the group completed morphism space $Hol(X;\mathbb{Z}\times BU)^+$ stabilizes if and only if the holomorphic K - theory space $K_{hol}(X)$ satisfies Bott periodicity:*

$$K_{hol}(X)\simeq \Omega^2 K_{hol}(X).$$

Proof. The space of rational maps in the morphism space $Hol(X;\mathbb{Z}\times BU)^+$ stabilizes if and only if the group completion of its space of rational maps is the two fold loop space,

$$Hol_*(\mathbb{P}^1; Hol(X;\mathbb{Z}\times BU))^+ \simeq \Omega^2(Hol(X;\mathbb{Z}\times BU)^+). \tag{5.1}$$

But by definition, the left hand side is equal to $Hol(\mathbb{P}^1\times X,\infty\times X;\mathbb{Z}\times BU)^+ = K_{hol}(\mathbb{P}^1\times X;\infty\times X)$. But by Rowland's theorem [22] or by the more general "projective bundle theorem" of Friedlander and Walker[9] we know that the Bott map

$$\beta : K_{hol}(X)\to K_{hol}(\mathbb{P}^1\times X;\infty\times X)$$

is a homotopy equivalence. Combining this with property 5.1, we have that the space of rational maps in the morphism space $Hol(X;\mathbb{Z}\times BU)^+$ stabilizes if and only if the following composition is a homotopy equivalence

$$\begin{array}{rcl} B : K_{hol}(X) & \xrightarrow[\simeq]{\beta} & K_{hol}(\mathbb{P}^1\times X;\infty\times X) \\ = Hol_*(\mathbb{P}^1; Hol(X;\mathbb{Z}\times BU)^+) & \longrightarrow & \Omega^2 Hol(X;\mathbb{Z}\times BU)^+ = \Omega^2 K_{hol}(X). \end{array} \tag{5.2}$$

□

By applying homotopy groups, the Bott map (5.2) $B : K_{hol}(X)\to\Omega^2 K_{hol}(X)$ defines a homomorphism

$$B_* : K^{-q}_{hol}(X)\to K^{-q-2}_{hol}(X)$$

Let $b\in K^{-2}_{hol}(point)$ be the image under $B_*$of the unit $1\in K^0_{hol}(point)$. Clearly this class lifts the Bott class in topological K - theory, $b\in K^{-2}_{top}(point)$. Observe further that B_* :

$K^{-q}_{hol}(X) \to K^{-q-2}_{hol}(X)$ is given by multiplication by the Bott class $b \in K^{-2}_{hol}(point)$, using the module structure of $K^{*}_{hol}(X)$ over the ring $K^{*}_{hol}(point)$. The homomorphism $B_* :$ $K^{-q}_{hol}(X) \to K^{-q-2}_{hol}(X)$ was studied in [9] and it was conjectured there that if $K^{*}_{hol}(X)[\frac{1}{b}]$ denotes the localization of $K^{*}_{hol}(X)$ obtained by inverting the Bott class, then one obtains topological K - theory. We now prove the following rational version of this conjecture.

Theorem 34. *Let X be a smooth projective variety. Then the map from holomorphic K - theory to topological K - theory $\beta : K_{hol}(X) \to K_{top}(X)$ induces an isomorphism*

$$\beta_* : K^{*}_{hol}(X)[1/b] \otimes \mathbb{Q} \xrightarrow{\cong} K^{*}_{top}(X) \otimes \mathbb{Q}.$$

Proof. The proof of this theorem involves proving another conjecture of Friedlander and Walker [9] involving the total Chern class from holomorphic K - theory to morphic cohomology. Namely, recall the Chern classes

$$c_k : K^{-q}_{hol}(X) \to L^k H^{2k-q}(X)$$

defined originally in [12]. Taking the direct sum of these maps gives us the *total Chern class* map,

$$c : K^{-q}_{hol}(X) \to \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X).$$

We will need the following result, which was conjectured in [9].

Theorem 35. *The total Chern class*

$$c : K^{-q}_{hol}(X) \otimes \mathbb{Q} \to \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

is an isomorphism for all $q \geq 0$.

We note that in the case $q = 0$, this theorem was proved in [9]. The proof in general will follow quickly from our Theorem 27 stating that the total Chern character is a rational isomorphism. However we will delay its proof until after we have completed the proof of Theorem 34. So assume the validity of Theorem 35 for now.

Recall the operation in morphic cohomology $S : L^k H^q(X) \to L^{k+1} H^q(X)$ defined in [7]. In [7] it was shown that the natural map from morphic cohomology to singular cohomology

$\phi : L^k H^q(X) \to H^q(X)$ makes the following diagram commute:

(5.3)
$$\begin{array}{ccc} L^k H^q(X) & \xrightarrow{S} & L^{k+1} H^q(X) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ H^q(X) & \xrightarrow{=} & H^q(X). \end{array}$$

It also follows from the "Poincare duality theorem" proved in [8] that if X is an n - dimensional smooth variety, then $L^s H^q(X) = H^q(X)$ for $s \geq n$. Furthermore for $k < n$ $\phi : L^k H^q(X) \to H^q(X)$ factors as the composition

(5.4)
$$\phi : L^k H^q(X) \xrightarrow{S} L^{k+1} H^q(X) \xrightarrow{S} \cdots \xrightarrow{S} L^n H^q(X) = H^q(X)$$

Let $L^* H^q(X)[1/S]$ denote the localization of $L^* H^q(X)$ obtained by inverting the transformation $S : L^* H^q(X) \to L^{*+1} H^q(X)$. Specifically

$$L^* H^q(X)[1/S] = \varinjlim \{ L^* H^q(X) \xrightarrow{S} L^{*+1} H^q(X) \xrightarrow{S} \cdots \}$$

Then (5.3) and (5.4) imply we have an isomorphism with singular cohomology,

(5.5)
$$\beta_* : L^* H^q(X)[1/S] \xrightarrow{\cong} H^q(X).$$

Now it was shown in [9] that the Bott class and the S - map are compatible via the total Chern class in the sense that the following diagrams commute:

(5.6)
$$\begin{array}{ccc} K^{-q}_{hol}(X) & \xrightarrow{\cdot b} & K^{-q-2}_{hol}(X) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) & \xrightarrow[S]{} & \bigoplus_{k=0}^{\infty} L^{k+1} H^{2k-q}(X). \end{array}$$

Now since the Chern classes in holomorphic K - theory and those for topological K - theory are compatible, this means we get a commutative diagram:

$$\begin{array}{ccc} K^{-q}_{hol}(X)[1/b] & \xrightarrow{\beta} & K^{-q}_{top}(X) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ \bigoplus_{k=0}^{\infty} L^* H^{2k-q}(X)[1/S] & \xrightarrow[\phi]{} & \bigoplus_{k=0}^{\infty} H^{2k-q}(X). \end{array}$$

By (5.5) we know that the bottom horizontal map is an isomorphism. Moreover by Theorem 35 we know that the left hand vertical map becomes an isomorphism after tensoring with the rationals. Of course the right hand vertical map is also a rational isomorphsim. Hence the top horizontal map is a rational isomorphism,

$$\beta_* : K^{-q}_{hol}(X)[1/b] \otimes \mathbb{Q} \xrightarrow{\cong} K^{-q}_{top}(X) \otimes \mathbb{Q}.$$

□

Thus Theorem 34 is proven modulo the proof of Theorem 35. We end this section by proving this theorem.

Proof. We first prove that the total Chern class

$$c : K^{-q}_{hol}(X) \otimes \mathbb{Q} \longrightarrow \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

is injective. So suppose that for some $\alpha \in K^{-q}_{hol}(X) \otimes \mathbb{Q}$, we have that $c(\alpha) = 0$. So each Chern class $c_q(\alpha) = 0$ for $q \geq 0$. Now recall from section 4 that in the algebra of operations between $K^{-q}_{hol}(X) \otimes \mathbb{Q}$ and $\bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$, that the that the Chern classes and Chern character are related by a formula of the form

$$c_k = a_k ch_k + p(ch_1, \cdots, ch_{k-1}) \tag{5.7}$$

where a_k is a nonzero rational number, and $p(ch_1, \cdots, ch_{k-1})$ is some polynomial in the first $k-1$ Chern classes. So since each $c_q(\alpha) = 0$ then an inductive argument using (5.7) implies that each $ch_q(\alpha) = 0$. Thus the total Chern character $ch(\alpha) = 0$. But since the total Chern character is an isomorphism (Theorem 27), this implies that $\alpha = 0 \in K^{-q}_{hol}(X) \otimes \mathbb{Q}$. This proves that the total Chern class operation is injective.

We now prove that $c : K^{-q}_{hol}(X) \otimes \mathbb{Q} \longrightarrow \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$ is surjective. To do this we will prove that for every k and element $\gamma \in L^k H^{2k-q}(X) \otimes \mathbb{Q}$ there is a class $\alpha_k \in K^{-q}_{hol}(X)$ with $c_k(\alpha_k) = \gamma$ and $c_q(\alpha_k) = 0$ for $q \neq k$. We prove this by induction on k. So assume this statement is true for $k \leq m-1$, and we now prove it for $k = m$. Let $\gamma_m \in L^m H^{2m-q}(X) \times \mathbb{Q}$. Since the total Chern character is an isomorphism, there is an element $\alpha_m \in K^{-q}_{hol}(X) \otimes \mathbb{Q}$ with $ch_m(\alpha_m) = \gamma_m$, and $ch_q(\alpha_m) = 0$ for $q \neq m$. But formula (5.7) implies that $c_q(\alpha_m) = 0$ for $q < m$, and $c_m(\alpha_m) = \frac{1}{a_m}\gamma_m$, where a_m is a nonzero rational number. Thus the total Chern class has value $c(a_k \alpha_m) = \gamma_m$. This proves that the total Chern class is surjective, and therefore the fact that it is an isomorphism. □

In most of the calculations of $K_{hol}(X)$ done so far we have seen examples of when $K_{hol}(X) \cong K_{top}(X)$. In particular in these examples the holomorphic K - theory is periodic, $K^*_{hol}(X) \cong K^*_{hol}(X)[\frac{1}{b}]$. As we have seen from Theorem 34, these two conditions are rationally equivalent. We end by using the above results to give a necessary condition for the holomorphic K - theory to be Bott periodic, and use it to describe examples where periodicity fails, and therefore provide examples that have distinct holomorphic and topological K - theories.

Theorem 36. *Let X be a smooth projective variety. Then if $K^*_{hol}(X) \otimes \mathbb{Q} \cong K^*_{hol}(X)[\frac{1}{b}]$ (or equivalently $K^*_{hol}(X) \otimes \mathbb{Q} \cong K^*_{top}(X) \otimes \mathbb{Q}$), then in the Hodge filtration of its cohomology we have*

$$H^{k,k}(X;\mathbb{C}) \cong H^{2k}(X;\mathbb{C})$$

for every $k \geq 0$.

Proof. Consider the commutative diagram involving the total Chern character

$$\begin{array}{ccc} K^0_{hol}(X) \otimes \mathbb{C} & \xrightarrow{\beta_*} & K^0_{top}(X) \otimes \mathbb{C} \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ \bigoplus_{k \geq 0} L^k H^{2k}(X) \otimes \mathbb{C} & \xrightarrow{\phi} & \bigoplus_{k \geq 0} H^{2k}(X;\mathbb{C}) \end{array} \tag{5.8}$$

By Theorem 34, if $K^*_{hol}(X)$ is Bott periodic, then the top horizontal map $\beta : K^0_{hol}(X) \otimes \mathbb{C} \to K^0_{top}(X) \otimes \mathbb{C}$ is an isomorphism. But by theorem 27 we know that the two vertical maps in this diagram are isomorphisms. Thus if $K^*_{hol}(X)$ is Bott periodic, then the bottom horizontal map in this diagram is an isomorphism. That is,

$$\phi : L^k H^{2k}(X) \otimes \mathbb{C} \longrightarrow H^{2k}(X;\mathbb{C})$$

is an isomorphism, for every $k \geq 0$. But as is shown in [7], $L^k H^{2k}(X) \cong \mathcal{A}_k(X)$, where $\mathcal{A}_k(X)$ is the space of algebraic k - cycles in X up to algebraic (or homological) equivalence. Moreover the image of $\phi : L^k H^{2k}(X) \otimes \mathbb{C} \to H^{2k}(X;\mathbb{C})$ is the image of the natural map induced by including algebraic cycles in all cycles, $\mathcal{A}_k \otimes \mathbb{C} \to H^{2k}(X;\mathbb{C})$, which lies in Hodge filtration $H^{k,k}(X;\mathbb{C}) \subset H^{2k}(X;\mathbb{C})$. Thus $\phi : L^k H^{2k}(X) \otimes \mathbb{C} \to H^{2k}(X;\mathbb{C})$ is an isomorphism implies that the composition

$$\mathcal{A}_k(X) \otimes \mathbb{C} \to H^{k,k}(X;\mathbb{C}) \subset H^{2k}(X;\mathbb{C})$$

is an isomorphism. In particular this means that $H^{k,k}(X;\mathbb{C}) = H^{2k}(X;\mathbb{C})$. □

We end by noting that for a flag manifold X, we know by Theorem 5 that $K^0_{hol}(X) \cong K^0_{top}(X)$, and indeed $H^{p,p}(X;\mathbb{C}) \cong H^{2p}(X;\mathbb{C})$. However in general this theorem tells us that if have a variety X having nonzero $H^{p,q}(X;\mathbb{C})$ for some $p \neq q$, then $K^*_{hol}(X)$ is *not* Bott periodic, and in particular is distinct from topological K - theory. Certainly abelian varieties of dimension ≥ 2 are examples of such varieties.

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