Math 215A

Take-home Final Exam

Instructions. Please answer the following questions with full justifications of your arguments. You are welcome to use the results from the book or class. If you have any questions about the exam, you may e-mail or ask me in person. Discussion of the problems with anyone else before the deadline is not permitted. Please submit your solutions as a pdf file. Please email your completed exam to both Prof. Cohen and Francois-Simon. It is due by 5:00 pm on Thursday, December 6. Good luck!

Name ______________________________

1. (24)	__________
2. (14)	__________
3. (24)	__________
4. (24)	__________
5. (14)	__________
Total (100)	__________

1. (24 points) For this question, assume that all spaces are path connected, locally path connected, and semi-locally simply connected.

   (a) Suppose X is a space with basepoint $x_0 \in X$. Let $Y \subset X$ be a path-connected subspace that contains the basepoint. Let $p : \tilde{X} \longrightarrow X$ denote the universal cover, and suppose that the pre-image of Y

   $$\tilde{Y} = p^{-1}(Y)$$

   is path-connected. Prove that the map induced by inclusion $i_* : \pi_1(Y, x_0) \longrightarrow \pi_1(X, x_0)$ is surjective.

   (b) Is every covering space of $\mathbb{RP}^2 \times \mathbb{RP}^3$ isomorphic to a product of covering spaces $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \to \mathbb{RP}^2 \times \mathbb{RP}^3$ where $p_1 : \tilde{X}_1 \to \mathbb{RP}^2$ and $p_2 : \tilde{X}_2 \to \mathbb{RP}^3$? Why or why not?

2. (14 points) Let $X = \mathbb{RP}^2 \vee S^3$ and $Y = \mathbb{RP}^3$. Prove that the homology and cohomology groups of X and Y are isomorphic with any coefficients, but that X and Y do not have the same homotopy type.

   Hint: Show that $H^*(X; \mathbb{Z}/2)$ and $H^*(Y; \mathbb{Z}/2)$ have different ring structures.

3. (24 points)

   (a) Let M^n be a closed, path connected orientable manifold. Let $x \in U \subset M$ where U is an open neighborhood homeomorphic to $\mathbb{R}^n$. Consider the "pinch map" $p : M^n \to S^n$ defined as the composition

   $$p : M^n \xrightarrow{quotient} M^n / (M^n - U) \xrightarrow[\cong]{homeo} S^n.$$

   Show that

   $$p_* : H_n(M^n; \mathbb{Z}) \to H_n(S^n; \mathbb{Z})$$

   is an isomorphism.

   (b) One of the most famous mathematics problems of the 20th century was the *"Poincaré Conjecture"*. In 1900 Poincaré claimed that any closed, simply connected, 3-dimensional manifold is homotopy equivalent to the sphere S^3. This claim was proved in 2003 by G. Perelman.

In this problem we ask you to prove a weaker, but related statement. Namely, prove that if M^3 is a closed, simply connected manifold, then there is a map $g : M^3 \to S^3$ that induces an isomorphism in homology groups in all dimensions.

4. (24 points)

   (a) Let M be a compact, connected, oriented n dimensional manifold, and let F be a field. Prove that the pairing

$$\mu : H^k(M;F) \times H^{n-k}(M;F) \to F$$

$$\mu(\alpha, \beta) = (\alpha \cup \beta)([M])$$

   is nondegenerate. That is, the induced map

$$H^k(M;F) \longrightarrow Hom(H^{n-k}(M;F);F)$$

   is an isomorphism. Here $[M] \in H_n(M;F)$ is the fundamental class.

   (b) Let M be a compact, connected, oriented 3-dimensional manifold, with

$$\pi_1(M) = \mathbb{Z}.$$

   Determine the cohomology groups of M, and the ring structure on cohomology, all with F coefficients, where F is a field.

5. (14 points) Is $(S^2 \times S^4) \vee S^8$ homotopy equivalent to a compact closed manifold? Explain.