Math 215A
Take-home Midterm Exam

Instructions. You are welcome to use the results from the book or class. If you have any questions about the exam, you may e-mail or ask me in person. Discussion of the problems with anyone else before the deadline is not permitted. Please submit your solutions as a pdf file. Please email your completed exam to both Prof. Cohen and Francois-Simon. It is due by 5:00 pm on Thursday, November 1. Good luck!

Name ________________________________

1. (14) __________________
2. (14) __________________
3. (12) __________________
4. (12) __________________
5. (14) __________________
6. (18) __________________
7. (16) __________________

Total (100)__________________________
1. (14 points total) We have seen that singular homology is a *functorial* assignment, that is, given a map \( f : X \to Y \) of topological spaces, there is an induced map \( f_* : H_i(X) \to H_i(Y) \) on homology groups. In some cases, if the map \( f : X \to Y \) is particularly nice, there also exists a map \( f^! : H_i(Y) \to H_i(X) \), called a *wrong-way* or *transfer* map.

Let \( p : \tilde{X} \to X \) be a \( k \)-sheeted covering map, for some finite \( k \). Construct a map of chain complexes

\[
C_i(X) \to C_i(\tilde{X})
\]

and show that it is a chain map, giving rise to an induced map on homology

\[
p^! : H_i(X) \to H_i(\tilde{X}).
\]

that has the property that the composition

\[
p_* \circ p^! : H_i(X) \to H_i(\tilde{X}) \to H_i(X)
\]

is multiplication by \( k \).

2. (14 points total)

(a) (7 points) Let \( X \subset \mathbb{R}^3 \) be the union of \( n \) lines through the origin. Compute \( \pi_1(\mathbb{R}^3 - X) \).

(b) (7 points) Let \( \tilde{X} \) and \( \tilde{Y} \) be simply-connected covering spaces of the path-connected, locally path-connected spaces \( X \) and \( Y \). Show that if \( X \) is homotopy equivalent to \( Y \) then \( \tilde{X} \) is homotopy equivalent to \( \tilde{Y} \).

3. (12 points)

(a) Find a way of identifying pairs of faces of \( \Delta^3 \) to produce a \( \Delta \)-complex structure on \( S^3 \) having a single 3-simplex, and compute the simplicial homology of this \( \Delta \)-complex.

(b) Compute the simplicial homology groups of the \( \Delta \)-complex \( X \) obtained from \( \Delta^n \) by identifying all faces of the same dimension. Thus \( X \) has a single \( k \)-simplex for each \( k \leq n \).
4. (12 points) Spaces not distinguished by homology. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

5. (14 points total) The topological group $SO(3)$ is defined as the space of real 3 x 3 matrices $A$ with that are orthogonal (meaning $A^T = A^{-1}$) and have determinant 1. The topology on $SO(3)$ is the subspace topology, coming from the inclusion of $SO(3) \subset \mathbb{R}^9$, the space of all 3 x 3 matrices.

Let $\mathbb{H}$ denote the group of quaternions (recall that these are numbers of the form $a + b \cdot i + c \cdot j + d \cdot k$, for $a, b, c, d \in \mathbb{R}$, with non-commutative multiplication rules determined by $i^2 = j^2 = k^2 = -1, ij = -ji = k$).

(a) (6 points) A quaternion is called pure if it has 0 real part, i.e., it is of the form $b \cdot i + c \cdot j + d \cdot k$. Thinking of $\mathbb{R}^3$ as the subspace of pure quaternions in $\mathbb{H}$, any quaternion $q \in \mathbb{H}$ induces a map

$$A_q : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$x \longmapsto qxq^{-1}.$$  

Show that when restricted to the unit quaternions (those with norm 1 using the usual Euclidean norm in $\mathbb{R}^4$), such a correspondence gives a (continuous) map

$$\phi : S^3 \longrightarrow SO(3).$$

(b) (8 points) Prove that the map $\phi$ is a covering map, and use it to calculate $\pi_1(SO(3))$.

6. (18 points total) Let $X$ be the quotient space of $S^2$ obtained by identifying the north and south poles to a single point.

(a) (6 points) Put a cell complex structure on $X$ and use this to compute $\pi_1(X)$.

(b) (6 points) Put a $\Delta$-complex structure on $X$ and use this to compute $H_*^\Delta(X)$, its simplicial homology for this structure.

(c) (6 points) Compute the singular homology of $X$ directly, using excision.
7. (16 points total)

(a) (6 points) Given a group $G$ and a normal subgroup $N$, show that there exists a normal covering space $\tilde{X} \rightarrow X$ with $\pi_1(X) \cong G$, $\pi_1(\tilde{X}) \cong N$, and deck transformation group $G(\tilde{X}) \cong G/N$.

(b) (10 points) For a path-connected, locally path-connected, and semilocally simply-connected space $X$, call a path-connected covering space $\tilde{X} \rightarrow X$ abelian if it is normal and has abelian deck transformation group. Show that $X$ has an abelian covering space that is a covering space of every other abelian covering space of $X$, and that such a “universal” abelian covering space is unique up to isomorphism. Describe this covering space explicitly for $X = S^1 \vee S^1$. 