Math 215A Supplementary Material

Let X be a topological space and $\mathcal{U} = \{U_i\}$ a collection of subsets whose interiors cover X. The singular chain complex $C_*(X)$ has a natural subcomplex of **$\mathcal{U}$-small chains**

$$C_*^{\mathcal{U}}(X) \tag{1}$$

generated by singular simplices whose image lies in at least one of the U_i. In these notes, we will finish the proof of the following key proposition:

Proposition 1 (Subdivision). *The natural inclusion*

$$\iota : C_*^{\mathcal{U}}(X) \longrightarrow C_*(X) \tag{2}$$

is a **chain homotopy equivalence**, *i.e. there is a natural map $\rho : C_*(X) \to C_*^{\mathcal{U}}(X)$ such that both $\rho \circ \iota$ and $\iota \circ \rho$ are chain homotopic to the identity map.*

Recall that two maps of chain complexes $f, g : A_* \to B_*$ are **chain homotopic** if there is a **chain homotopy** between them, i.e. a map $H : A_* \to B_{*1}$ satisfying $\partial_B \circ H + H \circ \partial_A = g - f$. As a simple consequence of this, f and g are the same map on homology. In particular,

Corollary 1. *The map*

$$\iota_* : H_*^{\mathcal{U}}(X) \longrightarrow H_*(X) \tag{3}$$

is an isomorphism.

Formal structure of construction. Let us recall how we began. First, call a chain $\alpha \in C_*(X)$ **small** if it's a part of of the subcomplex $C_*^{\mathcal{U}}(X)$. The proof will proceed by constructing two natural maps: a **chain map**

$$b_{(n)}^X : C_n(X) \longrightarrow C_n(X) \text{ for all } X,\ n \tag{4}$$

and a chain homotopy

$$R_{(n)}^X : C_n(X) \longrightarrow C_{n+1}(X) \text{ for all } X,\ n \tag{5}$$

between b^X and id, so $\partial R^X + R^X \partial = b^X - id$. The letter b stands for **barycentric subdivision**, and intuitively is the operation of taking a singular simplex, and subdividing it into a sum of smaller simplices. It will follow from the construction that

Property 1. *For any singular simplex $\sigma \in C_*(X)$, there exists a $K_\sigma \in \mathbb{N}$ such that $(b^X)^{K_\sigma}(\sigma)$ is small.*

Exercise 1. Show that if R^X is a chain homotopy between b^X and id, then $\sum_{i=0}^{K-1} R^X (b^X)^i$ is a chain homotopy between $(b^X)^K$ and id.

Thus, this will allow us to define the homotopy inverse

$$\rho : C_*^{\mathcal{U}}(X) \longrightarrow C_*(X) \tag{6}$$

on generators as

$$\sigma \longmapsto (b^X)^{K_\sigma}(\sigma) \text{ for some choice of } K_\sigma \text{ as in Property ??.} \tag{7}$$

The chain homotopy between $\iota \circ \rho$ and $id_{C_*^{\mathfrak{U}}(X)}$ is

$$\begin{aligned} \bar{R} : C_n^{\mathfrak{U}}(X) &\longrightarrow C_{n+1}^{\mathfrak{U}}(X) \\ \sigma &\longmapsto \sum_{i=0}^{K_\sigma - 1} R^X(b^X)^i\sigma,\ K_\sigma \text{ as before.} \end{aligned} \tag{8}$$

Since R^X and b^X both send small chains to small chains,

$$\bar{R} : C_n^{\mathfrak{U}}(X) \longrightarrow C_{n+1}^{\mathfrak{U}}(X) \tag{9}$$

also gives a chain homotopy between $\rho \circ \iota$ and $id_{C_*(X)}$.

A preliminary construction. Recall that $\Delta_k = \{\sum_{i=0}^k t_i e_i | \sum_{i=0}^k t_i = 1\} \subset \mathbb{R}^{k+1}$. Thus, points on a simplex can be thought of as coordinates $(t_0, \ldots, t_k)$ with entries summing to one.

Now, let V be a convex set in $\mathbb{R}^d$, and $p \in V$ a point. For a singular n-simplex $\alpha : \Delta_n \to V$, define a singular $n+1$ simplex

$$\mathrm{Cone}_p(\alpha) : \Delta_{n+1} \longrightarrow V \tag{10}$$

as

$$\mathrm{Cone}_p(\alpha)(t_0, \ldots, t_{n+1}) = \begin{cases} t_0 p + (1-t_0)\alpha(\frac{t_1}{1-t_0}, \ldots, \frac{t_{n+1}}{1-t_0}) & t_0 < 1 \\ p & t_0 = 1 \text{ (and thus all other } t_i = 0). \end{cases} \tag{11}$$

One should think of this singular simplex geometrically as the convex hull of the vertex p and the singular n simplex opposite it, α. Linearly extend this construction to obtain a map on singular chains

$$\mathrm{Cone}_p : C_n(V) \longrightarrow C_{n+1}(V). \tag{12}$$

As a map on chains, the cone satisfies the following property.

Lemma 1 (Cone formula).

$$\partial \mathrm{Cone}_p(\sigma) = \sigma - \mathrm{Cone}_p(\partial\alpha).$$

This is clear from looking at a picture of the cone over a simplex—in words, the boundary of the cone of a simplex is the original simplex plus the cones over the boundary of the original simplex.

Constructing the maps b^X from the simplex via naturality. Our desire for a **natural** construction means that the following squares should commute, whenever we have a map $f : X \to Y$:

$$\begin{array}{ccc} C_n(X) & \xrightarrow{b^X} & C_n(X) \\ \downarrow f_\sharp & & \downarrow f_\sharp \\ C_n(Y) & \xrightarrow{b^Y} & C_n(Y) \end{array}$$

$$
\begin{tikzcd}
C_n(X) \arrow[r, "R^X"] \arrow[d, "f_\sharp"] & C_{n+1}(X) \arrow[d, "f_\sharp"] \\
C_n(Y) \arrow[r, "R^Y"] & C_{n+1}(Y).
\end{tikzcd}
$$

This plus our desire for the maps to be linear, means that b^X and R^X are completely determined by their effect on the identity singular simplex

$$\eta_n = \Delta_n \xrightarrow{id} \Delta_n$$

How are they determined? Suppose we've defined $b^{\Delta_n}(\eta_n)$ and $R^{\Delta_n}(\eta_n)$. Given a generator

$$\sigma : \Delta_n \longrightarrow X \tag{13}$$

of $C_*(X)$, define

$$b^X(\sigma) := \sigma_\sharp(b_n^\Delta(\eta_n)), \tag{14}$$

and similarly

$$R^X(\sigma) := \sigma_\sharp(R_n^\Delta(\eta_n)), \tag{15}$$

where $\sigma_\sharp : C_*(\Delta_n) \to C_*(X)$ is the map on chains induced by the generator σ.

Constructing b^X. Let's define b_n^X by induction on n. First, for $n = 0$, set $b^{\Delta_0}(\eta_0) = \eta_0$, which by (**??**) gives a construction of b_0^X for all X. Now, say b_{n-1}^X has been defined. Inductively define

$$b_n^{\Delta_n}(\eta_n) := \mathrm{Cone}_z(b_{n-1}^{\Delta_n}(\partial\eta_n)) \tag{16}$$

where $z = z_n$ is the so-called **barycenter** (or "center of mass") of the simplex Δ_{n+1}, the point $\sum_{k=0}^{n+1} \frac{1}{n+1} e_i$. It's illustrative to draw what is happening in low dimensions. In dimension 0, b_n does nothing, and in dimension 1, this operation takes the identity 1 simplex, an interval, takes its boundary, the two endpoints, and takes the cone on this boundary from the barycenter, a point in the middle of the simplex. The result is a pair of simplices "subdividing" the interval. In dimension 2, one first subdivides the faces of the 2 simplex into halves, takes the barycenter of the 2 simplex, and takes the cone from this barycenter to all the subdivided simplices on the boundary (so the result is a sum of smaller simplices) — see the picture at the top of Hatcher, p. 120 for a picture.

Finally, one defines b_n^X using (**??**).

Lemma 2. *$b_n^X : C_n(X) \longrightarrow C_n(X)$ is a chain map.*

PROOF. By induction—the case $n = 0$ is clear. Now suppose it's true for all X for $n-1$ chains. Then on a generator $\sigma : \Delta_n \longrightarrow X$,

$$
\begin{aligned}
\partial_n b_n^X(\sigma) &= \partial_n \sigma_\sharp b_n^{\Delta_n}(\eta_n) \\
&= \sigma_\sharp \partial_n b_n^{\Delta_n}(\eta_n) \text{ ($\sigma_\sharp$ is a chain map)} \\
&= \sigma_\sharp \mathrm{Cone}_z^{\Delta_n}(b_{n-1}^{\Delta_n}(\partial\eta_n)) \\
&= \sigma_\sharp(b_{n-1}^{\Delta_n}(\partial\eta_n) - \mathrm{Cone}_z(\partial b_{n-1}^{\Delta_n}(\partial\eta_n)) \text{ (cone formula)} \\
&= \sigma_\sharp(b_{n-1}^{\Delta_n}(\partial\eta_n) - \mathrm{Cone}_z(b_{n-1}^{\Delta_n}(\partial\circ\partial\eta_n)) \text{ (induction hypothesis)} \\
&= \sigma_\sharp(b_{n-1}^{\Delta_n}(\partial\eta_n)) \; (\partial^2 = 0) \\
&= b_{n-1}^X \partial\sigma.
\end{aligned}
\tag{17}
$$

□

Constructing R^X. Once more, we proceed by induction. We now have two base cases: Define $R_{-1}^X = 0$, and $R_0^{\Delta_0}(\eta_0)$ to be the unique singular simplex $\Delta_1 \longrightarrow \Delta_0$ in $C_1(\Delta_0)$ (this gives a definition for all X using (**??**). Inductively, assume R_{n-1}^X and R_{n-2}^X have been defined such that

$$\partial R_{n-1}^X - R_{n-2}^X \circ \partial = b^X - id.$$

Let us define $R^{\Delta_n}(\eta_n)$ and extend to all of X via (**??**). We would like $R^{\Delta_n}(\eta_n) \in C_{n+1}(\Delta_n)$ to be an element β satisfying the chain homotopy relation

$$\partial\beta + R_{n-1}^{\Delta_n}(\partial\eta_n) = b^{\Delta_n}(\eta_n) - \eta_n, \tag{18}$$

i.e.

$$\partial\beta = -R_{n-1}^{\Delta_n}(\partial\eta_n) + b_{n-1}^{\Delta_n}(\eta_n) - \eta_n. \tag{19}$$

But

$$\alpha := -R_{n-1}^{\Delta_n}(\partial\eta_n) + b_{n-1}^{\Delta_n}(\eta_n) - \eta_n \in C_n(\Delta_n) \tag{20}$$

so if we can show that

$$\partial\alpha = 0, \tag{21}$$

it will follow from the fact that $H_n(\Delta_n) = 0$ that there exists such a β with $\partial\beta = \alpha$. Let us compute:

$$
\begin{aligned}
\partial\alpha &= \partial(b_{n-1}^{\Delta_n}(\eta_n) - \eta_n - R_{n-1}^{\Delta_n}(\partial\eta_n)) \\
&= b_{n-1}^{\Delta_n}(\partial\eta_n) - \partial\eta_n - \partial(R_{n-1}^{\Delta_n}(\partial\eta_n)) \text{ (b^X is a chain map)} \\
&= b_{n-1}^{\Delta_n}(\partial\eta_n) - \partial\eta_n - (b_{n-1}^{\Delta_n}\partial\eta_n - \partial\eta_n - R_{n-1}^{\Delta_n}(\partial\circ\partial\eta_n)) \text{ (inductive hypothesis)} \\
&= 0.
\end{aligned}
\tag{22}
$$

So we can pick a β with $\partial\beta = \alpha$ and give a definition of R_n^X using (**??**).

LEMMA 3. *R_n^X is a chain homotopy from b^X to id.*

PROOF. We've established this on the simplex η_n, so the rest is an exercise in naturality. Namely, given a generator $\sigma \in C_n(X)$,

$$\begin{aligned}
\partial R_n^X(\sigma) + R_{n-1}^X \circ \partial(\sigma) &= \partial \sigma_\sharp(R_n^{\Delta_n}(\eta_n)) + (R_{n-1}^X(\sigma_\sharp \partial \eta_n) \\
&= \sigma_\sharp \partial(R_n^{\Delta_n}(\eta_n)) + (\sigma_\sharp R_{n-1}^{\Delta_{n-1}}(\partial \eta_n)) \\
&= \sigma_\sharp(b^{\Delta_n}(\eta_n) - \eta_n) \\
&= b^X(\sigma) - \sigma.
\end{aligned} \tag{23}$$

□

The key observation about b^X that allows us to conclude is:

OBSERVATION 1. *$b^{\Delta_n}(\eta_n)$ is a collection of simplices with diameter less than $n/(n+1)$ of the original diameter of η_n.*

This is an explicit induction on n, which is obvious for $n = 0$ and 1, and uses the definition of the barycenter in the cone construction. Thus, iterating b^{Δ_n} on η_n, we can obtain a collection of simplicies with arbitrarily small diameter.

Finally, given an arbitrary singular simplex $\sigma : \Delta_n \to X$, note that there is an induced open cover $\mathcal{U}_\sigma := \{\sigma^{-1}(U_i) | U_i \in \mathcal{U}\}$ of the compact metric space $\mathcal{U}$. This cover has an associated non-negative **Lebesgue number**, the largest ϵ, such that any ϵ ball in Δ_n is contained in one of the elements of $\mathcal{U}_\sigma$, i.e. any ϵ ball is sent by σ to one of the U_i. (This follows from the **Lebesgue covering lemma** in metric topology: any open cover over a compact metric space possesses such a number, which can be seen by arguing by contradiction (**exercise**)).

Finally, we can iterate subdivision b^X on σ a finite number of times so that each new simplex within Δ_n now has diameter less than ϵ; this ensures that the result will be $\mathcal{U}$-small.