#1. If \( \pi_1(X) \) is finite, \( f_* (\pi_1(X)) \leq \pi_1(S^4) \cong \mathbb{Z} \) is also finite, hence trivial. Therefore \( f : X \to S^4 \) can be lifted to the universal cover \( p : \mathbb{R} \to S^4 \); in other words, we can factor \( f \) as \( X \xrightarrow{\phi} \mathbb{R}^2 \xrightarrow{p} S^4 \). Since \( \mathbb{R} \) is contractible, \( f \) is null homotopic.

#2. Consider the cover of \( S^4 \cup S^4 \) given by the 2-oriented graph \( X \) shown below.

Let \( \sigma : X \to X \) be the homeomorphism obtained by rotating \( X \) about the centre \( O \) by a \( \pi/2 \) clockwise angle.

Let \( \tau : X \to X \) be the homeomorphism obtained by composing the reflection about the line \( L \) with the reflection about the circle \( C \).

Both \( \sigma \) and \( \tau \) are deck transformations. Moreover, any vertex of \( X \) can be taken to any other vertex of \( X \) by repeatedly applying \( \sigma \) or \( \tau \) to \( X \). Hence \( p : X \to S^4 \cup S^4 \) is normal, hence \( N := p_* (\pi_1(X,x_0)) \leq \pi_1(S^4 \cup S^4) \) is a normal subgroup. Van Kampen's theorem shows that \( \pi_1(X,x_0) \) is free on the generators \( \{a^2, ab^2a^{-1}, (ab)a^2(ab)^{-1}, (aba)b^2(abab)^{-1}, (ab)^2a^2(ab)^{-2}, (ababa)b^2(abab)^{-1}, \} \), as shown below:

Therefore \( a^2, b^2 \) and \( (ab)^4 \) are in \( N \), so that \( N \) contains the normal subgroup of \( \pi_1(S^4 \cup S^4, x) \) generated by \( a^2, b^2 \) and \( (ab)^4 \). On the other hand, conjugates of these three words have been shown to generate \( N \), so we also have the other inclusion. It follows that \( N = p_* (\pi_1(X,x_0)) \) is precisely the normal subgroup of \( \pi_1(S^4 \cup S^4, x) \) generated by \( a^2, b^2 \) and \( (ab)^4 \), as desired.

#3. Observe that any word \( w \in L \langle a,b \rangle / L \langle a^2, b^2 \rangle \) is either \( (ab)^n \), \( (ba)^n \) for some \( n \in \mathbb{Z} \) or conjugate to either \( a \) or \( b \). It follows that up to conjugacy the only subgroups of \( L \langle a,b \rangle / L \langle a^2, b^2 \rangle \cong \pi_4(\mathbb{R}P^2 \vee \mathbb{R}P^2) \) are:

\[ \langle a^2 \rangle, \langle b^2 \rangle, \langle a, b(ab)^n \rangle, \langle b, a(ba)^n \rangle, \langle (ab)^n \rangle, \langle a/b \rangle. \]
where we have also used that $b \cdot (ab)^n \cdot b^{-1} = (ba)^n$, so $<ab^n> \sim <(ba)^n>$.
Since unbased connected covers of $\mathbb{RP}^2 \vee \mathbb{RP}^2$ are in 1-1 correspondence with the conjugacy classes of subgroups of $\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2)$, it will suffice to produce connected covers $p: X \to \mathbb{RP}^2 \vee \mathbb{RP}^2$ with $p_\ast(\pi_1(X)) \leq \pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2)$ equal to each of these groups. Consider the basic covers:

If we let $i_1$ and $i_2$ be the compositions of $i$ and $\pi$ with the inclusion of $\mathbb{RP}^2$ into the first copy of the wedge $\mathbb{RP}^2 \vee \mathbb{RP}^2$ and $i_2 \circ \pi_2$ be the composition of $i$ and $\pi$ with the inclusion of $\mathbb{RP}^2$ into the second copy, we have explicit covers given by:

- $<a, b> \leftrightarrow i_1 \vee i_2$ (here and below wedge of maps will be used);
- $1 \leftrightarrow i_1 \vee i_2 \vee i_1 \vee i_2 \vee i_1 \vee i_2$;
- $<a> \leftrightarrow i_2 \vee i_1 \vee i_2 \vee i_1 \vee i_2$;
- $<b> \leftrightarrow i_2 \vee i_1 \vee i_2 \vee i_1 \vee i_2$;
- $<a, b(ab)^{2n-1}> \leftrightarrow \text{wedge of } 2n \text{ spheres}$;
- $<b, a(ab)> \leftrightarrow \text{wedge of } 2n \text{ spheres}$.

Since $<a, b(ab)^{2n}>$ is conjugate to $<b, a(ba)^{2n}>$, we are done.

#4 Let $x \in X$ and choose a neighborhood $U$ of $x$ such that $U \cap g(U) = \emptyset$ only for $g = g_1g_2\cdots g_n \in G$. Since $G$ acts freely on $X$, $g_0x = x \forall k = 1, \ldots, n$. Choose neighborhoods $U_k$ and $V_k$ of $x$ and $g_0x$ such that $U_k \cap V_k = \emptyset$. Set $W_k = U_k \cap g_1^{-1} V_k$ and $W = \cap_{k=1}^n W_k$, a neighborhood of $x$. Then $W \cap gW = \emptyset$ for all $g \in G$, so indeed $G \times X$ is a covering space action.

#5 For $(x, y) \in \mathbb{R}^2$ such that $x \neq 0$, $U = (3x/4, 3x/2) \times \mathbb{R}$ satisfies $nU = \emptyset \forall n \neq 0$. Similarly, for $(x, y) \in \mathbb{R}^2$ such that $y \neq 0$, $U = \mathbb{R} \times (3y/4, 3y/2)$ satisfies $nU = \emptyset \forall n \neq 0$. Observe:

Proves that the orbits of $(\delta, 0)$ and $(0, \delta)$ cannot be separated by open neighborhood so $X/\mathbb{Z}$ is not Hausdorff.
Repeated application of Van Kampen shows that \( \pi_1(X, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z} \).

\[ \pi_1 = \mathbb{Z} \quad \Rightarrow \quad \pi_1 = \mathbb{Z} \quad \Rightarrow \quad \pi_1 = \mathbb{Z} \times \mathbb{Z} \quad \Rightarrow \quad \pi_1 = \mathbb{Z} \times \mathbb{Z} \]

Alternatively, we know that \( \pi_1(X, \mathbb{Z})/\pi_1(X, \mathbb{Z}) \cong G \), so \( \pi_1(X, \mathbb{Z})/\mathbb{Z} \cong \mathbb{Z} \). If we also know that \( \pi_1(X, \mathbb{Z}) \) is abelian, this implies that \( \pi_1(X, \mathbb{Z})/\mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z} \). To see this, let \( x_0 = (1, 0) \), take paths \( \alpha, \delta_0 \) and \( \delta_1 \) in \( X \) as shown, note \( [p \delta_0/p \delta_0] \) or \( [p \delta_1/p \delta_1] \) each generate \( \pi_1(X, \mathbb{Z}) \), and:

\[ [\alpha] \cdot [p \delta_1] \cdot [p \delta_0] = [p \delta_0] \Rightarrow \pi_1(X, \mathbb{Z}) \text{ abelian} \]