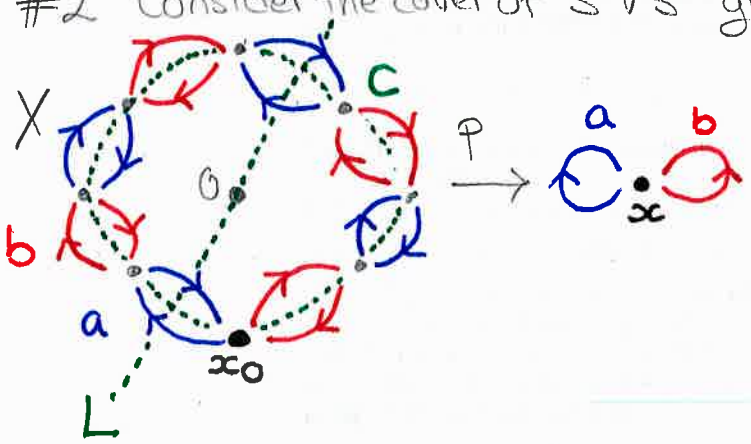
MATH 215A HW3 (SOLUTIONS)

#1 If $\pi_1(X)$ is finite, $f_*(\pi_1(X)) \leq \pi_1(S^1) \cong \mathbb{Z}$ is also finite, hence trivial. Therefore $f: X \to S^1$ can be lifted to the universal cover $p: \mathbb{R} \to S^1$, in other words, we can factor f as $X \xrightarrow{\tilde{f}} \mathbb{R} \xrightarrow{p} S^1$. Since $\mathbb{R}$ is contractible, f is null homotopic.

#2 Consider the cover of $S^1 \vee S^1$ given by the 2-oriented graph X shown below



Let $\sigma: X \to X$ be the homeomorphism obtained by rotating X about the centre O by a $\pi/2$ clockwise angle.

Let $\tau: X \to X$ be the homeomorphism obtained by composing the reflection about the line L with the reflection about the circle C.

Both σ and τ are deck transformations. Moreover, any vertex of X can be taken to any other vertex of X by repeatedly applying σ or τ to X. Hence $p: X \to S^1 \vee S^1$ is normal, hence $N := p_*(\pi_1(X, x_0)) \leq \pi_1(S^1 \vee S^1, x)$ is a normal subgroup. Van Kampen's theorem shows that $\pi_1(X, x_0)$ is free on the generators $\{a^2, ab^2a^{-1}, (ab)a^2(ab)^{-1}, (aba)b^2(aba)^{-1}, (ab)^2a^2(ab)^{-2}, (ababa)b^2(ababa)^{-1}, (ab)^3a^2(ab)^{-3}, (ab)^4, b^2\}$, as shown below:

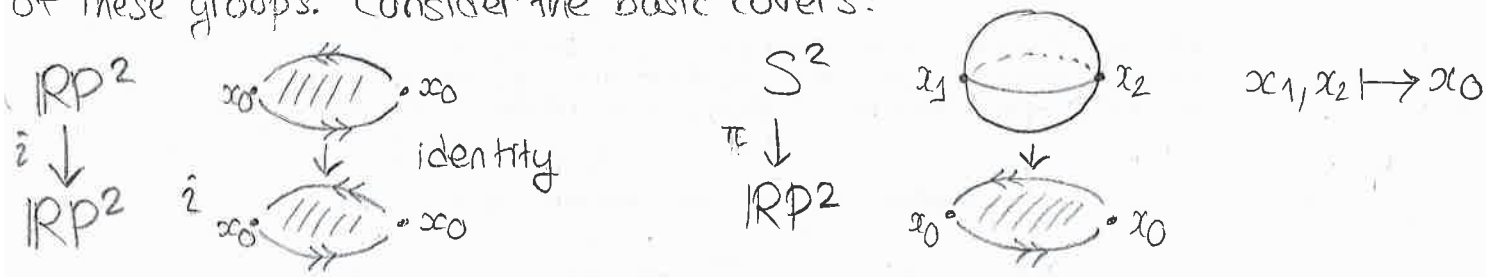
← Free generators of $\pi_1(X, x_0)$

Therefore a^2, b^2 and $(ab)^4$ are in N, so that N contains the normal subgroup of $\pi_1(S^1 \vee S^1, x)$ generated by a^2, b^2 and $(ab)^4$. On the other hand, conjugates of these three words have been shown to generate N, so we also have the other inclusion. It follows that $N = p_*(\pi_1(X, x_0))$ is precisely the normal subgroup of $\pi_1(S^1 \vee S^1, x) \cong \mathbb{F}\langle a, b\rangle$ generated by a^2, b^2 and $(ab)^4$, as desired.

#3 Observe that any word $w \in \mathbb{F}\langle a, b\rangle / a^2, b^2$ is either $(ab)^n, (ba)^n$ for some $n \in \mathbb{Z}$ or conjugate to either a or b. It follows that up to conjugacy the only subgroups of $\mathbb{F}\langle a, b\rangle / a^2, b^2 \cong \pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2)$ are:

$\langle\rangle, \langle a\rangle, \langle b\rangle, \langle a, b(ab)^n\rangle, \langle b, a(ba)^n\rangle, \langle (ab)^n\rangle, \langle a, b\rangle$.

where we have also used that $b\cdot(ab)^n\cdot b^{-1}=(ba)^n$, so $\langle (ab)^n\rangle \sim \langle (ba)^n\rangle$.
Since unbased connected covers of $\mathbb{RP}^2\vee\mathbb{RP}^2$ are in 1-1 correspondence with the conjugacy classes of subgroups of $\pi_1(\mathbb{RP}^2\vee\mathbb{RP}^2)$, it will suffice to produce connected covers $p: X\to \mathbb{RP}^1\vee\mathbb{RP}^1$ with $p_*(\pi_1(X))\leq \pi_1(\mathbb{RP}^2\vee\mathbb{RP}^2)$ equal to each of these groups. Consider the basic covers:



If we let i_1 and π_1 be the compositions of i and π with the inclusion of $\mathbb{RP}^2$ into the first copy of the wedge $\mathbb{RP}^2\vee\mathbb{RP}^2$ and i_2, π_2 be the compositions of i and π with the inclusion of $\mathbb{RP}^2$ into the second copy, we have explicit covers given by: $\langle a,b\rangle \longleftrightarrow i_1\vee i_2$ (here and below wedge of maps will be used),

$1 \leftrightarrow \cdots\vee\pi_1\vee\pi_2\vee\pi_1\vee\pi_2\vee\cdots$, $\langle a\rangle \leftrightarrow i_1\vee\pi_2\vee\pi_1\vee\pi_2\vee\cdots$, $\langle b\rangle \leftrightarrow i_2\vee\pi_1\vee\pi_2\vee\pi_1\vee\cdots$

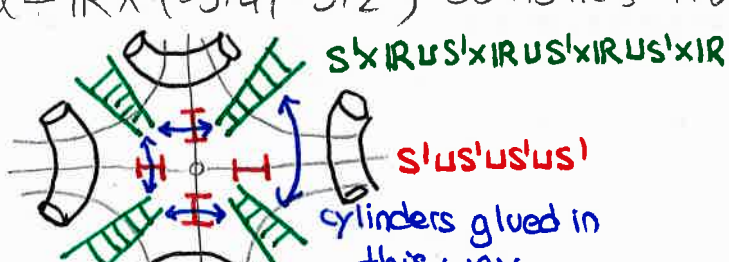
$\langle (ab)^n\rangle \leftrightarrow \underbrace{\cdots\vee\pi_1\vee\pi_2\vee\pi_1\vee\pi_2\vee\cdots}_{\text{cyclic wedge of } 2n \text{ spheres}}$, $\langle a, b(ab)^{2n-1}\rangle \leftrightarrow i_1\vee\underbrace{\pi_2\vee\pi_1\vee\cdots\vee\pi_1\vee\pi_2}_{\text{wedge of } 2n-1 \text{ spheres}}\vee i_1$

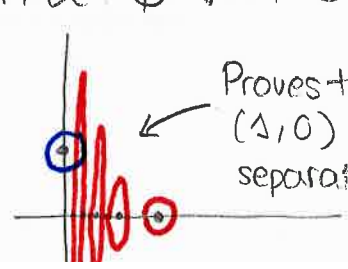
$\langle a, b(ab)^{2n}\rangle \leftrightarrow i_1\vee\underbrace{\pi_2\vee\pi_1\vee\cdots\vee\pi_1}_{\text{wedge of } 2n \text{ spheres}}\vee i_2$, $\langle b, a(ab)^{2n-1}\rangle \leftrightarrow i_2\vee\underbrace{\pi_1\vee\pi_2\vee\cdots\vee\pi_2\vee\pi_1}_{\text{wedge of } 2n-1 \text{ spheres}}\vee i_2$

Since $\langle a, b(ab)^{2n}\rangle$ is conjugate to $\langle b, a(ba)^{2n}\rangle$, we are done

#4 Let $x\in X$ and choose a neighborhood $\mathcal{U}$ of x such that $\mathcal{U}\cap g(\mathcal{U})\neq\emptyset$ only for $g=g_1,g_2,\dots,g_n\in G$. Since G acts freely on X, $g_k x\neq x$ $\forall k=1,\dots,n$. Choose neighborhoods $\mathcal{U}_k$ and V_k of x and $g_k x$ such that $\mathcal{U}_k\cap V_k=\emptyset$. Set $W_k=\mathcal{U}_k\cap g_k^{-1}V_k$ and $W=\cap_{k=1}^n W_k$, a neighborhood of x. Then $W\cap gW=\emptyset$ for all $g\in G$, so indeed $G\circlearrowright X$ is a covering space action.

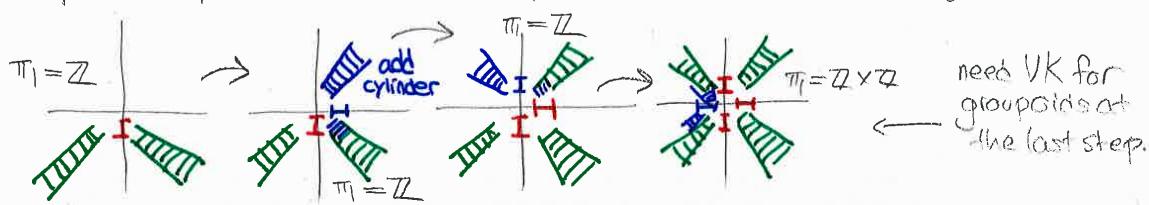
#5 For $(x,y)\in\mathbb{R}^2$ such that $x\neq 0$, $\mathcal{U}=(3x/4, 3x/2)\times\mathbb{R}$ satisfies $n\mathcal{U}\cap\mathcal{U}=\emptyset$ $\forall n\neq 0$. Similarly, for $(x,y)\in\mathbb{R}^2$ such that $y\neq 0$, $\mathcal{U}=\mathbb{R}\times(3y/4, 3y/2)$ satisfies $n\mathcal{U}\cap\mathcal{U}=\emptyset$ $\forall n\neq 0$. Observe:



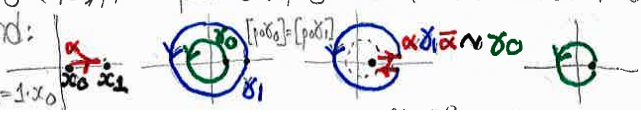


Proves that the orbits of $(1,0)$ and $(0,1)$ cannot be separated by open neighborhood so $X/\mathbb{Z}$ is not Hausdorff

Repeated application of Van kampen shows that $\pi_1(X|\mathbb{Z}) \cong \mathbb{Z}\times\mathbb{Z}$.



Alternatively, we know that $\pi_1(X|\mathbb{Z})/p_*(\pi_1(X)) \cong G$, so $\pi_1(X|\mathbb{Z})/\mathbb{Z} \simeq \mathbb{Z}$. If we also knew that $\pi_1(X|\mathbb{Z})$ is abelian this implies that $\pi_1(X|\mathbb{Z}) \simeq \mathbb{Z}\times\mathbb{Z}$. To see this, let $x_0 = (1,0)$, take paths α, γ_0 and γ_1 in X as shown, note $(p\circ\alpha, p\circ\gamma_0)$ or $(p\circ\alpha, p\circ\gamma_1)$ each generate $\pi_1(X|\mathbb{Z})$, and:



$\Rightarrow [p\circ\alpha]\cdot[p\circ\gamma_1]\cdot[p\circ\alpha]^{-1} = [p\circ\gamma_0] \Rightarrow \pi_1(X|\mathbb{Z})$ abelian