(1) Show that the Poincaré Duality theorem implies that if $F$ is a field and $M^n$ is a closed $F$-oriented manifold with fundamental class $[M^n] \in H_n(M^n; F)$, then the pairing

$$H^k(M^n; F) \times H^{n-k}(M^n; F) \rightarrow F$$

$$\phi \times \psi \rightarrow \langle \phi \cup \psi, [M^n] \rangle$$

is nonsingular for every $k = 0, \ldots, n$.

(2) Show that

$$H^*_c(\mathbb{R}^n; G) \cong \tilde{H}^*(S^n; G)$$

and more generally that

$$H^*(X; G) \cong H^*(X \cup \infty; G)$$

where $X \cup \infty$ is the one-point compactification of $X$. Here $H^*_c$ is the cohomology with compact supports.

(3) Let $\pi: \tilde{X} \rightarrow X$ be a covering space. Let $\Phi$ be a smooth structure on $X$. Prove that there is a smooth structure $\tilde{\Phi}$ on $\tilde{X}$ so that $\pi: (\tilde{X}, \tilde{\Phi}) \rightarrow (X, \Phi)$ is an immersion.

(4) Prove that the following are submanifolds of the space of $n \times n$ matrices, $\text{Mat}_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Compute their dimensions.

1. $GL_n(\mathbb{R})$
2. $SL_n(\mathbb{R})$
3. $SO(n)$

(5). (a) Let $x \in S^n$, and $[x] \in \mathbb{RP}^n$ be the corresponding element. Consider the functions $f_{i,j} : \mathbb{RP}^n \rightarrow \mathbb{R}$ defined by $f_{i,j}([x]) = x_ix_j$. Show that these functions define a diffeomorphism between $\mathbb{RP}^n$ and the submanifold of $\mathbb{R}^{(n+1)^2}$ consisting of all symmetric $(n+1) \times (n+1)$ matrices $A$ of trace 1 satisfying $AA = A$.

(b) Use the above to show that $\mathbb{RP}^n$ is compact.

(c) Prove that an $n$-dimensional vector bundle $\zeta$ has $n$ linearly independent sections if and only if $\zeta$ is trivial.