1 Poincare Duality

We first note the following relation between cup and cap products:

\[ \langle \phi \smile \psi; \sigma \rangle = \langle \sigma \smile \phi; \psi \rangle \rightarrow \langle \sigma \cap \cdot; [M^n] \rangle = \langle \cdot \cap [M^n]; \sigma \rangle \]  \tag{1}

Now consider the sequence of maps

\[ H^{n-k}(M; F) \xrightarrow{h} \text{Hom}(H_{n-k}(M; F), F) \xrightarrow{D^*} \text{Hom}(H^{k}(M; F), F), \]  \tag{2}

where definitions of \( h \) and \( D^* \) are explained in Hatcher’s book. By the relation between cup and cap products,

\[ D^* \circ h(\sigma) = D^*(\langle \sigma; \cdot \rangle) = \langle \sigma; [M^n] \cap \cdot \rangle = \pm \langle \sigma \cap \cdot; [M^n] \rangle. \]  \tag{3}

Hence showing that the pairing is nonsingular is equivalent to showing that \( D^* \circ h \) is an isomorphism. But since \( F \) is a field, \( h \) is an isomorphism by UCT, and \( D^* \) is an isomorphism because it is the hom dual of the isomorphism map that appears in Poincare duality.

2 Compactly Supported Cohomology

We derive the result by the following chain of isomorphisms

\[ H^*_c(X; G) \simeq \varinjlim H^*(X, X - K_r; G) \simeq \varinjlim H^*(X \cup \infty, X \cup \infty - K_r; G) \]
\[ \simeq H^*(X \cup \infty, G) = \tilde{H}^*(X \cup \infty; G) \]  \tag{4}

where the first and fourth isomorphisms are by definition, the second isomorphism is by excision of \( \infty \) \(^1\) and the third isomorphism is by contractibility of \( X \cup \infty - K_r \) for sufficiently large \( r \).

\(^1\)Since \( X \cup \infty - K_r \) is contractible, the interior of \( X \cup \infty - K_r \) contains the closure of \( \infty \) for any sequence of \( K_r \). Excision hence applies.
3 Covering Space Smooth Structure

We first note that the covering space of a manifold is still a manifold (i.e. the covering space inherits second-countability+Hausdorff$^2$). To get a smooth structure, we simply take a set of charts $(\phi_i, U_i)$ for $X$ with $\{U_i\}$ evenly covered, and lift them to charts $\{\tilde{U}_{i,a}\}$ in $\tilde{X}$ with $a$ labeling the sheets of the covering space. Smoothness of transition maps follows immediately and $\pi$ is an immersion because it is a local diffeomorphism with the smooth structure defined above.

4 Submanifold Structure

(a) View $\text{Mat}_{n,n}(\mathbb{R})$ as $\mathbb{R}^{n^2}$. Define $\det : \mathbb{R}^{n^2} \to \mathbb{R}$. $\det$ is a polynomial (in particular continuous). Therefore $\det^{-1}(U)$ is open whenever $U$ is open. In particular, $\det^{-1}(\mathbb{R} - \{0\}) = GL_n(\mathbb{R})$ is open. But open subsets of $\mathbb{R}^k$ are automatically submanifolds of the same dimension. Therefore, $GL_n(\mathbb{R})$ is a smooth submanifold of dimension $n^2$.

(b) $SL_n(\mathbb{R}) = \det^{-1}(1)$. For an arbitrary $A \in SL_n(\mathbb{R})$, we can compute the directional derivative

$$\nabla_A A = \lim_{\epsilon \to 0} \frac{\det(A + \epsilon X) - \det A}{\epsilon} = \det A \lim_{\epsilon \to 0} \frac{1 + n\epsilon + O(\epsilon^2) - 1}{\epsilon} = n \det A = n. \quad (5)$$

Hence, for all $A$, the map $\det_* : T\mathbb{R}^{n^2} \to T\mathbb{R}$ (differential/pushforward of $\det$) is surjective. Hence 1 is a regular value, and by the regular value theorem, $SL_n(\mathbb{R})$ is a submanifold of codimension 1 in $\mathbb{R}^{n^2}$.

(c) We work instead with $O(n) = \{ M \in \text{Mat}_{n,n}(\mathbb{R}) | MM^T = I \}$. Since $SO(n)$ is just one of the two path components of $O(n)$, if we can show $O(n)$ is a submanifold, then $SO(n)$ is a submanifold of the same dimension. Now consider the map $f : \mathbb{R}^{n^2} \approx \mathbb{R}^{n^2} \to \mathbb{R}^{n(n+1)/2} \approx \text{Sym}_{n,n}(\mathbb{R})$ defined by $f(A) = AA^T$. At some arbitrary element $A$ with $f(A) = I$, we have:

$$\nabla_X f(A) = \lim_{\epsilon \to 0} \frac{(A + \epsilon X)(A + \epsilon X)^T - AA^T}{\epsilon} = \lim_{\epsilon \to 0} \frac{\epsilon(tAX^T + tXA^T)}{\epsilon} = AX^T + XA^T \quad (6)$$

By taking $X = \frac{SA}{2}$ for some symmetric $S$, $\nabla_X f(A)$ spans $\text{Sym}_{n,n}(\mathbb{R})$ for all $A \in f^{-1}(I)$. Therefore, $I$ is a regular value of $f$ and $O(n)$ has codimension equal to $\dim(\text{Sym}_{n,n}(\mathbb{R})) = \frac{n(n+1)}{2}$. Hence, $\dim SO(n) = \dim O(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$. 

$^2$If you want to know the proof, consult for example Lee’s monograph on smooth manifolds.
5 Real Projective Space

(a) Let \( x \in S^n \) label the element \([x] \in \mathbb{RP}^n\). Collect the components \( f_{ij}([x]) \) into a matrix \( f \). Then

\[
f_{ij}([x]) = x_i x_j = f_{ji}([x]) = \sum_j f_{ij}([x]) f_{jk}([x]) = x_i x_k \sum_j x_j = x_i x_k = f_{ik}([x])\]

\[
\text{tr} f = \sum_i x_i^2 = 1. \tag{7}
\]

This means \( f(\mathbb{RP}^n) \) is a subset of \( C = \{ m \in \text{Sym}_{n+1,n+1}(\mathbb{R}) | m^2 = m, \text{tr} m = 1 \} \). Now we show \( f \) is bijective. For injective, suppose \( f([y]) = f([x]) \), then diagonal components give constraints \( y_i^2 = x_i^2 \forall i \), implying \( y_i = \pm x_i \). But suppose that \( y_i = x_i, y_j = -x_j \), then \( x_i x_j \neq y_i y_j \) leading to a contradiction. Hence, \( \pm \) needs to be consistently chosen for all \( i \). However, since representatives \( x, y \) should only be taken from a half-sphere, we conclude that there is in fact a unique sign choice and \( f \) is injective. To show surjectivity, note that for any \( m \) with \( m^2 = m \) and \( \text{tr} m = 1 \), \( m = O^T \Lambda O \) where \( O \in SO(n+1) \) and \( \Lambda = \text{diag}(1,0,\ldots,0) \). By carrying out the matrix multiplication, we see that \( m = f([x]) \) where \( x_i = O_{1i} \). \( O \in SO(n+1) \) guarantees that \( \sum_i O_{1i}^2 = 1 \), and \( x \in S^n \).

This concludes surjectivity. In fact, this computation gives an explicit map \( g : C \to S^n \) such that \( \pi \circ g \) is the inverse of \( f \). By construction, \( f \circ \pi, g \) are polynomial functions of \( x \in \mathbb{R}^n \) and hence smooth. By the general theory of smooth manifolds, when we restrict the domain of \( f \circ \pi \) to a submanifold \( S^n \), we get a smooth map. Similarly, when we restrict the range of \( g \) to a submanifold \( S^n \), we get a smooth map (it is important that \( S^n \) is embedded). Restriction of the range to an immersed submanifold generally kills smoothness. Since \( \pi : S^n \to \mathbb{RP}^n \) is a covering map (hence a local diffeomorphism) and \( f \) is bijective, we conclude that \( f \) is in fact a diffeomorphism.

(b) For every \( A \in C \), \( \sum_{ij} A_{ij}^2 = \text{tr} A^2 = \text{tr} A = 1 \). Hence \( C \) has bounded Euclidean distance from the origin of \( \mathbb{R}^{(n+1)^2} \) and must be bounded. \( C \) is obviously closed since the three defining properties are preserved under taking limits. By Heine-Borel, \( C \) is compact. Since compactness is preserved under diffeomorphisms, \( \mathbb{RP}^n \) is compact.

(c) Suppose \( \zeta \xrightarrow{\cong} B \) is trivial, then there exists a bundle isomorphism \( \Phi : \mathbb{R}^n \times B \to \zeta \). Define a set of sections \( \sigma_i(x) = (e_i, x) \) where \( \{e_i\} \) is a complete basis in \( \mathbb{R}^n \). Then \( \{\sigma_i(x)\} \) inherits linear independence from \( \{e_i\} \). Conversely, if there is a set of linearly independent sections \( \sigma_i(x) : B \to \zeta \), then define \( \Phi : \mathbb{R}^n \times B \to \zeta \) via \( \Phi(v, x) = \sum_i v_i \sigma_i(x) \). Fiberwise, this is a linear isomorphism since \( \{\sigma_i(x)\} \) is linearly independent. Hence \( \Phi \) is a bundle isomorphism.