(1). Let $X$ be an aspherical space, and $Y$ a connected, finite CW-complex. Show that every map $f : Y \to X$ is null homotopic. Show that as a result, if $X$ has the homotopy type of a finite CW-complex, $X$ is contractible.

(2) Let $X$ be a space with a basepoint $x_0 \in X$. Recall that the (reduced) suspension of $X$, $\Sigma X$, is the space

$$\Sigma X = X \times S^1 / \{ X \times \{1\} \cup x_0 \times S^1 \}$$

Here I am thinking of $S^1$ as the unit complex numbers. Let $(Y, y_0)$ be another space with basepoint. Consider the (based) “loop space”

$$\Omega Y = \text{Map}((S^1, \{1\}), (Y, y_0)).$$

This is the space of maps from $S^1$ to $Y$ that take $1 \in S^1$ to the basepoint $y_0 \in Y$, endowed with the compact - open topology.

(a). Prove that there is a bijection

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

Here the notation $[-,-]$ denotes the set of homotopy classes of basepoint preserving maps. As a special case, conclude that $\pi_n(Y, y_0) \cong \pi_{n-1}(\Omega Y, \epsilon_0)$, where $\epsilon_0 : S^1 \to Y$ is the constant map at the basepoint $y_0$.

(b). Let $G$ be a topological group, and consider the map $f : G \to \Omega BG$ defined in the proof of Corollary 4.10 in the text. Prove that $f$ induces an isomorphism in homotopy groups (in all degrees). Such a map is called a “weak homotopy equivalence”.

(3) (a). Let $\{x_0, \cdots, x_k\}$ be a collection of $k+1$ - distinct points in $\mathbb{R}^L$ for some large integer $L$. Prove that the $k$ - fold join $x_0 \ast x_1 \ast \cdots \ast x_k$ is the convex hull of these points and hence is the $k$ - dimensional simplex $\Delta^k$ with vertices $\{x_0, \cdots, x_k\}$.

(b). Prove that there is a natural $G$ - equivariant map

$$\Delta^k \times G^{k+1} \to G^{\ast(k+1)}$$

which is a homeomorphism onto its image when restricted to $\tilde{\Delta}^k \times G^{k+1}$ where $\tilde{\Delta}^k \subset \Delta^k$ is the interior. Here $G$ acts on $\Delta^k \times G^{k+1}$ trivially on the simplex $\Delta^k$ and diagonally on $G^{k+1}$. $G$ acts diagonally on the iterated join $G^{\ast(k+1)}$ as well, as described in section 4.41 of the text.