1 Aspherical Space

When $Y$ consists only of 0-cells, connectedness of $Y$ guarantees that every map from $Y \to X$ is nullhomotopic. We now proceed by induction. Suppose all maps from the k-skeleton to $X$ are nullhomotopic. Take one attachment map $f: (D^{k+1}, \partial D^{k+1}) \to Y \to X$. By the induction hypothesis, there exists a homotopy $\tilde{f}_t: \partial D^{k+1} \to Y \to X$ such that $\tilde{f}_0 = f|_{\partial D^k}$ and $\tilde{f}_1 = x_0$. For each $t$, by the homotopy extension property for $(D^{k+1}, \partial D^{k+1})$, $\tilde{f}_t$ extends to a homotopy $\tilde{F}_t$ on $D^k$ with $\tilde{F}_1|_{\partial D^{k+1}} = x_0$. Hence the image of $\tilde{F}_1$ is now a $S^{k+1}$ inside $X$. Since $\pi_{k+1}(X,x_0) = 0$, we can homotope $\tilde{F}_1$ to the constant map $x_0$. This completes the induction step.

If in addition, $X$ has the homotopy type of a finite CW complex, then CW approximation tells us there is a finite CW complex $Z \sim X$ ($\sim$ here means homotopy equivalence). In particular, we can find $f: X \to Z, g: Z \to X$ such that $f \circ g \sim id_Z$ and $g \circ f \sim id_X$. Since $g$ is nullhomotopic (by what we have proven), $id_X$ is nullhomotopic and $X$ is contractible.

2 Loop Space and Suspension

Recall the definitions:

\[
\Sigma X = X \times S^1 / (X \times \{1\} \cup x_0 \times S^1) \quad \Omega Y = \{f: (S^1, \{1\}) \to (Y, y_0)\} \tag{1}
\]

(a) For any map $f: \Sigma X \to Y$, we can define the map $Gf: X \to \Omega Y$ via $Gf(x)(s) = f([x,s])$. The basepoints for $\Sigma X, \Omega Y, X, Y$ are $[x_0, s], \epsilon_0, x_0, y_0$ respectively. To check that this is a well-defined map of homotopy classes, we consider a basepoint preserving homotopy $f_t: \Sigma X \to Y$ with $f_t([x_0, s]) = y_0$. By definition, $Gf_t(x_0)(s) = f_t([x_0, s]) = y_0$ for all $s$. Therefore $Gf_t(x_0) = \epsilon_0$ is the basepoint of $\Omega Y$ and $G$ is a map between homotopy classes. The inverse can be constructed easily.

Now consider the special case where $X = S^{n-1}$. The map $G: [\Sigma S^{n-1}, Y] \to [S^{n-1}, \Omega Y]$ between homotopy classes preserve the group structures. Thus we have the chain of group isomorphisms

\[
\pi_{n-1}(\Omega Y, \epsilon_0) = [S^{n-1}, \Omega Y] \xrightarrow{G} [\Sigma X, Y] \xrightarrow{G^{-1}} [\Sigma X, Y] = [S^n, Y] = \pi_n(Y, y_0). \tag{2}
\]

(b) Pick a basepoint $e_0 \in EG$ such that $p(e_0) = b_0 \in BG$. Since $EG$ is contractible, we can find a homotopy $p \circ H_t: EG \to EG$ such that $p \circ H_{-1}(e) = b_0, p \circ H_1(e) = b$. Now consider the path
space $P(BG) = \{ \gamma : I \rightarrow BG | \gamma(0) = b_0 \}$. The homotopy $p \circ H_t$ induces a map $F : EG \rightarrow P(BG)$ via $F(e)(t) = p \circ H_t(e)$. Now notice that the map $f : \Sigma G \rightarrow BG$ appearing in corollary 4.10 is just $f(g,t) = F((b_0,g),t)$ where $(b_0,g)$ represents an element in the fiber $p^{-1}(b_0)$. Via the correspondence in part (a), $f$ induces a map $\bar{F} : EG \rightarrow P(BG)$ via $\bar{F}(e)(t) = p \circ H_t(e)$. We have therefore completed the commutative diagram:

$$
\begin{array}{ccc}
G & \rightarrow & EG \\
\downarrow f & & \downarrow F \\
\Omega BG & \rightarrow & P(BG) \\
\end{array}
\begin{array}{c}
p \\
id \\
p \\
\end{array}
BG
$$

Since $EG, P(BG)$ are both contractible, $\pi_n(EG) = \pi_n(P(BG)) = 0$ for all $n$. Hence, when we extend both rows in the diagram via the long exact sequence of homotopy groups, the map $\bar{f} : \pi_n(G) \rightarrow \pi_n(\Omega BG)$ is sandwiched between four isomorphisms. By the five lemma, $\bar{f}$ must also be an isomorphism.

### 3 Joining Constructions

(a) We do this by induction. The base case $x_0 * x_1$ is clearly just a segment, the convex hull of two points. Suppose $x_0 \ldots * x_{k-1} = \Delta^{k-1}$, then by definition $\Delta^{k-1} * x_k$ is just the cone $C \Delta^{k-1}$ which is homeomorphic to $\Delta^k$.

(b) Present the convex hull as $\Delta^k = \{(t_1, \ldots, t_k) | \sum_{i=0}^k t_i \leq 1, t_i \geq 0, \forall t \}$ (this is not the standard presentation because we drop the $t_0$'s coordinate and change the sum condition to an inequality). Consider the map $\phi : \Delta^k \times G^{k+1} \rightarrow G^{*(k+1)}$ given by

$$
\phi(\vec{t}, \vec{g}) = [(g_0, t_1, \ldots, g_{k-1}, t_k, g_k)]
$$

where on the RHS, $[\ldots]$ is the quotient under the equivalence relation on $G^{*(k+1)}$. Denote by $\mu$ the action of $G$ which is identity on $\Delta^k$ and diagonal on $G^{k+1}$. Then

$$
\phi(\mu \vec{g}(\vec{t}, \vec{g})) = \phi(\vec{t}, \tilde{g} \vec{g}) = [(\tilde{g} g_0, t_1, \ldots, \tilde{g} g_{k-1}, t_k, \tilde{g} g_k)] = \tilde{g} \phi(\vec{t}, \vec{g}).
$$

This shows $G$-equivariance. In $\Delta^k \times G^{k+1}$, $t_i \neq 0$ for all $i$ and the equivalence relations imposed by the join construction on $t_i = \{0,1\}$ do not matter. Thus $\phi$ is a homeomorphism onto its image.