1 Orienting Tangent Bundles

Every chart \((U_i, \phi_i)\) on \(M\) induces a trivialization \(TU_i \rightarrow U_i \times \mathbb{R}^n\) of the tangent bundle \(f(x, v) = (x, d\phi_i(v))\). The clutching function for the tangent bundle \(d\phi_i \circ d\phi_j^{-1} = d(\phi_i \circ \phi_j^{-1})\) is equivalent to the Jacobian of the transition function for the manifold \(M\). Hence, orientability of \(M\) is equivalent to orientability on the bundle \(TM\). (Notice that as a manifold, \(TM\) is always orientable. But we are talking about bundle orientation here, not manifold orientation.)

2 Moving Off of Itself

(a) Identify \(\nu_N\) with the tubular neighborhood \(\eta\). By theorem 8.6, we can find a section \(e\) of \(\eta \rightarrow N^n\) arbitrarily close to the identity such that \(e(N^n) \pitchfork N^n\). But since \(2n < m\), \(e(N^n) \pitchfork N^n\) implies \(e(N^n) \cap N^n = \emptyset\).

(b) Suppose \(\exists\) a section \(e\) such that \(e(\mathbb{RP}^1)\) is moved off of the zero section \(\mathbb{RP}^1\), then the mod-2 intersection number of \(\mathbb{RP}^1\) inside \(\mathbb{RP}^2\) must be zero by 8.12. However, we know that this is not the case since any two embeddings of \(\mathbb{RP}^1\) in \(\mathbb{RP}^2\) intersect at a point and have nonzero intersection number.

3 Intersection Product

Since \(N = [z_0, z_1, 0], K = [0, z_1, z_2]\), we know that \(N \cap K = [0, z_1, 0]\) is just a point. Hence, \([N \cap K] = [N] \cdot [K] = \pm 1\). But \(N, K\) are related by an orientation-preserving diffeomorphism. Hence \([N] \cdot [K] = 1\). Consider now the following chain of maps:

\[ H_2(\mathbb{CP}^2) \times H_2(\mathbb{CP}^2) \xrightarrow{D \times D} H^2(\mathbb{CP}^2) \times H^2(\mathbb{CP}^2) \xrightarrow{\sim} H^4(\mathbb{CP}^2) \xrightarrow{D} H_0(\mathbb{CP}^2) \]  

(1)

Recall that \(H^*(\mathbb{CP}^2) = \mathbb{Z}[x]/(x^3)\) with \(x \in H^2(\mathbb{CP}^2)\). Since we end at a nontrivial element \(1 \in H_0(\mathbb{CP}^2)\), the cup product must output the class \(x^2\) in \(H^4(\mathbb{CP}^2)\). Therefore, the cohomologies are \((\pm x, \pm x)\). By Poincare duality again, this means \([N], [K]\) each represents a generator of \(H^2(\mathbb{CP}^2)\).
4 Shriek Map

Let \( C_M([N]) = [N] \sim [M] \) be the Poincare duality map from cohomology to homology and let \( D_M \) be the inverse of \( C_M \). Then \( \Delta! = C_M \circ \Delta^* \circ D_M \times M \) and \( \Delta! \circ C_M \times M = C_M \circ \Delta^* \). By Kunneth formula, \( [M \times M] = [M] \times [M] \) and \( C_M \times M(\alpha \times \beta) = (\alpha \times \beta) \sim [M \times M] = (\alpha \sim [M]) \times (\beta \sim [M]) \).

Thus letting \( a = \alpha \sim [M], b = \beta \sim [M] \), we see \( \Delta! \circ C_M \times M(\alpha \times \beta) = \Delta!(a \times b) \). Plugging this into the identity \( \Delta! \circ C_M \times M = C_M \circ \Delta^* \) we find that up to signs,

\[
\Delta!(a \times b) = C_M \circ \Delta^*(\alpha \times \beta) = (\alpha \sim \beta) \sim [M] = (D_M(a) \sim D_M(b)) \sim [M] = a \cdot b
\]

(2)