Math 215B

Take-home Midterm Exam February 6, 2020

Instructions. You are welcome to use the results from the books or class. If you have any questions about the exam, you may e-mail me or ask me in person. Discussion of the problems with anyone else is not permitted. Please send your solutions in a pdf file to me via email (rlc@stanford.edu) by 5:00 pm, Tuesday, February 11.

Good luck!

Name ________________________________

1. (12) ________________

2. (30) ________________

3. (24) ________________

4. (12 plus 10 extra credit) ________________

5. (22) ________________

Total (100 plus 10 extra credit) ________________
1. Let $\pi : \xi \to X$ be a $k$-dimensional vector bundle over a space $X$ of the homotopy type of a CW-complex. Let $\pi_q : Fr_q(\xi) \to X$ be the associated $q$-frame bundle. This bundle consists of those $q$-tuples of vectors $(v_1, \ldots v_q) \in \xi^\times q$ for which

- $\pi(v_1) = \cdots = \pi(v_q)$ (in other words, the vectors $v_1, \ldots v_k$ all live in the same fiber of $\pi : \xi \to X$).
- The vectors $v_1, \ldots, v_q$ are linearly independent.

Then $\pi_q(v_1, \ldots, v_q) = \pi(v_1) = \cdots = \pi(v_q) \in X$.

Show that there is a continuous section $\sigma : X \to Fr_q(\xi)$ (i.e a map $\sigma$ such that $\pi_q \circ \sigma = id_X$) if and only if there is a vector bundle isomorphism

$\xi \xrightarrow{\cong} \zeta \times \mathbb{R}^q$

$\pi \downarrow$ $\downarrow \tilde{p}$

$X \xrightarrow{=} X$

where $p : \zeta \to X$ is a $(k-q)$-dimensional vector bundle and $\tilde{p}$ is the composition $\tilde{p} : \zeta \times \mathbb{R}^q \xrightarrow{project} \zeta \xrightarrow{p} X$.

Notice in particular that if $q = k$, the one has a section of $Fr_k(\xi) \to X$ if and only if $\xi \to X$ is isomorphic to the trivial $k$-dimensional bundle.

2. For any space $X$ let $Vect^d(X)$ denote the set of isomorphism classes of $d$-dimensional vector bundles over $X$.

(a) Compute $Vect^d(S^1)$. Justify your answer.

(b) Compute the fundamental group of the Grassmannian, $\pi_1(Gr_d(\mathbb{R}^\infty))$.

(c) Let $X$ be a simply-connected space. Prove that any one-dimensional vector bundle over $X$ is trivial.

3. Let $T$ be a closed, connected, orientable surface (two-dimensional manifold).

(a) Show that there are infinitely many nonisomorphic complex line bundles over $T$. 

2
(b) Let \( x_0 \in T \). Show that the restriction of any complex line bundle \( p : E \to T \) to the “punctured surface”, \( T - x_0 \), is trivial.

4. A Lie group is a \( C^\infty \) differentiable manifold together with a group structure so that both the multiplication map \( G \times G \to G \) and the inversion map \( G \to G \) given by \( g \to g^{-1} \) are \( C^\infty \) maps.

(a) Show that a compact Lie group \( G \) has a trivial tangent bundle.

**Hint:** Show that the tangent bundle \( TG \) is isomorphic to the trivial bundle \( G \times T_{id}G \) where \( T_{id}G \) is the tangent space at the identity element of \( G \). Make use of the left and right translation maps \( L_g : G \to G \) given by \( L_g(h) = gh \) and \( R_g : G \to G \) given by \( R_g(h) = hg \). These are defined for any \( g \in G \) and are differentiable maps.

(b) **Extra Credit:** Show that for \( G \) a compact Lie group, and \( K < H < G \) compact sub-Lie groups (i.e subgroups that are also submanifolds), then the projection map

\[
p : G/K \to G/H
\]

is a locally trivial fiber bundle with fiber \( H/K \).

5. Let \( M^n \) be a closed differentiable manifold, and let \( e_0 : M^n \to \mathbb{R}^N \) and \( e_1 : M^n \to \mathbb{R}^N \) be two immersions of \( M^n \). We say that \( e_0 \) and \( e_1 \) are isotopic if there is a one-parameter family of immersions connecting \( e_0 \) and \( e_1 \). That is, \( e_0 \) and \( e_1 \) are isotopic if there is a continuous map \( H : M^n \times [0, 1] \to \mathbb{R}^N \) so that

(a) \( H(x, 0) = e_0(x) \) and \( H(x, 1) = e_1(x) \) for all \( x \in M^n \)

(b) The map \( H_t : M^n \to \mathbb{R}^N \) defined by \( H_t(x) = H(x, t) \) is a differentiable immersion for every \( t \in [0, 1] \).

Smale’s theorem about “turning a sphere inside out” says that any two immersions \( S^2 \to \mathbb{R}^3 \) are isotopic.

(a) Show, however, that there are infinitely many distinct isotopy classes of immersions \( S^1 \to \mathbb{R}^2 \). You may use Smale’s theorem saying that the space
of immersions $M \looparrowright \mathbb{R}^N$ is weakly homotopy equivalent to the space of bundle monomorphisms $TM \to T\mathbb{R}^N$.

(b) Describe a representative of each isotopy class you find.