# THE HOMOTOPY TYPE OF GAUGE THEORETIC MODULI SPACES

Ralph L. Cohen and R. James Milgram

Dept. of Mathematics
Stanford University

## §1. Introduction

In recent years Gauge theory has been perhaps the most important technique in the study of differentiable structures on four dimensional manifolds. In particular the study of the moduli spaces of (anti) self dual connections on a principal bundle over a Riemannian four dimensional manifold has yielded the definition, by Donaldson, of polynomial invariants that have been highly successful at distinguishing smooth structures on homeomorphic manifolds. Thus studying the homotopy type of these moduli spaces and variants of them has been, and continues to be of fundamental importance in Algebraic Topology.

In this article we will survey various results about the algebraic topology of these moduli spaces. We will not discuss their applications to the study of four manifolds. We refer the reader to [DK] for such a discussion. We will limit ourselves here to a discussion about the various examples of moduli spaces that arise in gauge theory, where and how they arise, and what is known about their homotopy types.

This paper is organized as follows. In section one we give the basic background material for studying the topology of spaces of connections and describe some calculations of their cohomology groups. In section two we study the moduli space of $SU(2)$ - monopoles. These are "time invariant" self dual connections on $\mathbb{R}^4$ whose topology is given by spaces of rational functions. We then describe the work of [CCMM] and others concerning the homotopy type of these spaces. In section three we study a circle action on the space of connections on $S^4$ and consider the corresponding moduli space of $S^1$ - invariant self dual connections on $S^4$, or "hyperbolic monopoles". We then describe work of [CJR] concerning the homotopy type of these spaces, and how they are related to the full moduli space of self dual connections on $S^4$. The topology of these full moduli spaces of instantons is taken up in sections four and five. In section four the basic geometry of these spaces is studied, and the work of [BHMM] proving the Atiyah - Jones conjecture is outlined. This conjecture was proved by relating these moduli spaces to certain moduli spaces of holomorphic vector bundles, which were then given certain natural combinatorial stratifications defined in terms of spaces of Toeplitz matrices. These stratifications in turn induced spectral sequences

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converging to the homology of these moduli spaces, which was then analyzed. In section five the work of [Mil2] describing the topology of these Toeplitz varieties is outlined, and their applications to the rational homotopy type of the original space of instantons on $S^4$ is studied. Finally in section six work of [MM1, MM2] is reviewed, in which the homotopy type of higher rank monopole spaces are studied in terms of spaces of holomorphic maps from the Riemann sphere to flag manifolds.

## §1. Background Material

In this section we recall the basic constructions of Yang - Mills theory and discuss the homotopy type of spaces of connections. We will limit ourselves to the study of principal $SU(n)$ - bundles, however most of what we say applies equally well with very little modification to any principal bundle. There are now many good references for details of these constructions. We recommend [AB, DK, FU, L] .

Let $M^m$ be a closed,oriented, smooth Riemannian $m$ - dimensional manifold and let

$$P \longrightarrow M$$

be a principal $SU(n)$ - bundle. Let $\zeta(P) \to M$ be the complex $n$ - dimensional vector bundle induced from $P$ via the standard representation of $SU(n)$ on $\mathbb{C}^n$. Let $\mathcal{A}(P)$ be the space of connections on the principal bundle $P$, or equivalently on the vector bundle $\zeta(P)$. One way of thinking of a connection $A \in \mathcal{A}(P)$ is as covariant derivative

$$D_A : \Omega^0(M; \zeta(P)) \longrightarrow \Omega^1(M; \zeta(P))$$

where $\Omega^q(M; \zeta(P))$ denotes the space of differential $q$ - forms with coefficients in the bundle $\zeta(P)$. A covariant derivative $D_A$ is a $\mathbb{C}$ - linear map that satisfies Leibniz' rule:

$$D_A(f\sigma) = df \otimes \sigma + fD_A\sigma$$

for any smooth function $f \in C^\infty(M)$ and zero form $\sigma \in \Omega^0(M; \zeta(P))$.

Given two connections $(D_A)_1$ and $(D_A)_2$ on $\zeta(P)$ and a function $f \in C^\infty(M)$ one can take the convex combination

$$f(D_A)_1 + (1 - f)(D_A)_2$$

and obtain a new connection. From this it is not difficult to see that the space of connections on $\zeta$ is affine modelled on the vector space of one forms $\Omega^1(M; ad(P))$, where $ad(P)$ is the vector bundle

$$ad(P) : P \times_{SU(n)} \mathfrak{su}(n) \longrightarrow P/SU(n) \cong M$$

where $\mathfrak{su}(n)$ is the Lie algebra equipped with the adjoint action of the Lie group $SU(n)$. Notice that $ad(P)$ can be viewed as a bundle whose fiber at a point $x \in M$ is the space of skew hermitian endomorphisms of the fiber of $\zeta(P)$ at $x$.

Now let $X$ be a vector field on $M$ and $D_A$ a connection on the vector bundle $\zeta(P) \to M$. The covariant derivative in the direction of $X$, which we denote by $(D_A)_X$ is an operator on the space of sections of $\zeta(P)$,

$$(D_A)_X : \Omega^0(M; \zeta(P)) \longrightarrow \Omega^0(M; \zeta(P))$$

defined by

$$(D_A)_X(\phi) = \langle D_A(\phi); X \rangle.$$

One can then define the curvature $F_A \in \Omega^2(M; ad(P))$ by defining its action on a pair of vector fields $X$ and $Y$ to be

$$F_A(X,Y) = (D_A)_X(D_A)_Y - (D_A)_Y(D_A)_X - (D_A)_{[X,Y]}.$$

To interpret this formula, notice that apriori $F_A(X,Y)$ is a second order differential operator on the space of sections of $\zeta(P)$. However, a direct calculation shows that for $f \in C^\infty(M)$ and $\phi \in \Omega^0(M; \zeta(P))$ then

$$F_A(X,Y)(f\phi) = fF_A(X,Y)(\phi)$$

and hence that $F_A(X,Y)$ is in fact a zero order operator on $\Omega^0(M; \zeta(P))$. But a zero order operator on the space of sections of $\zeta(P)$ is a section of the bundle of endomorphisms $End(\zeta)$. By construction this is a section of the bundle of skew hermitian endomorphisms, i.e $ad(P)$. Thus $F_A$ assigns to any pair of vector fields $X$ and $Y$ a section of $ad(P)$. Moreover it is straightforward to check that this assignment is tensorial in $X$ and $Y$, (i.e $F_A(fX, Y) = F_A(X, fY) = fF_A(X,Y)$). Thus $F_A$ is an element of $\Omega^2(M; ad(P))$. The curvature measures the lack of commutativity in second order partial covariant derivatives.

Let $A \in \mathcal{A}(P)$ be a connection. The Yang - Mills functional applied to $A$, $\mathcal{YM}(A)$ is the $L^2$ norm of the curvature,

$$\mathcal{YM}(A) = \frac{1}{2}\int_M \|F_A\|^2 \, d(vol).$$

We view $\mathcal{YM}$ as a mapping $\mathcal{YM} : \mathcal{A}(P) \longrightarrow \mathbb{R}$. From the mathematical point of view (as opposed to the physical) Yang - Mills theory is the study of the critical point behavior of $\mathcal{YM}$. We begin by investigating the group of symmetries of $\mathcal{A}$ that $\mathcal{YM}$ preserves. This is known as the *gauge group*, or group of gauge transformations of $P$.

**Definition.** *The gauge group $\mathcal{G}(P)$ of the principal bundle $P$ is the group of bundle automorphisms of $P \longrightarrow M$. That is, an element $\phi \in \mathcal{G}(P)$ is a bundle isomorphism of $P$ with itself lying over the identity:*

$$\begin{array}{ccc} P & \xrightarrow[\cong]{\phi} & P \\ \downarrow & & \downarrow \\ M & \xrightarrow{=} & M. \end{array}$$

*Equivalently, $\mathcal{G}(P)$ is the group $\mathcal{G}(P) = Aut_{SU(n)}(P)$ of $SU(n)$ - equivariant diffeomorphisms of the space $P$.*

The gauge group $\mathcal{G}(P)$ acts on the space of connections $\mathcal{A}(P)$ by a standard pull - back construction. The gauge group of the trivial bundle and its corresponding action on the space of connections are particularly important. The following is a straightforward exercise.

**Proposition 1.1.** *Let $P$ be the trivial bundle $M \times SU(n) \longrightarrow M$. Then the gauge group $\mathcal{G}(P)$ is given by the function space from $M$ to $SU(n)$,*

$$\mathcal{G}(P) \cong C^\infty(M; SU(n)).$$

*Furthermore if $\phi : M \longrightarrow SU(n)$ and $A \in \Omega^1(M; \mathfrak{su}(n))$ are identified with elements of $\mathcal{G}(P)$ and $\mathcal{A}(P)$ respectively, then the induced action of $\phi$ on $A$ is given by*

$$\phi^*(A) = \phi^{-1} A \phi + \phi^{-1} d\phi.$$

It is not difficult to see that in general the gauge group $\mathcal{G}(P)$ does not act freely on the space of connections $\mathcal{A}(P)$. However there is an important subgroup $\mathcal{G}_0(P) < \mathcal{G}(P)$ that does. This is the group of *based* gauge transformations. To define this group, let $x_0 \in M$ be a fixed basepoint, and let $P_{x_0}$ be the fiber of $P$ at $x_0$.

**Definition.** *The based gauge group $\mathcal{G}_0(P)$ is a subgroup of the group of bundle automorphisms $\mathcal{G}(P)$ which pointwise fix the fiber $P_x$. That is,*

$$\mathcal{G}_0(P) = \{\phi \in \mathcal{G}(P) : \textit{if}\, v \in P_{x_0} \textit{then}\, \phi(v) = v\}.$$

The following is standard. See [AB] for example.

**Theorem 1.2.** *The based gauge group $\mathcal{G}_0(P)$ acts freely on the space of connections $\mathcal{A}(P)$.*

Let $\mathcal{B}(P)$ and $\mathcal{B}_0(P)$ be the orbit spaces of connections on $P$ up to gauge and based gauge equivalence respectively,

$$\mathcal{B}(P) = \mathcal{A}(P)/\mathcal{G}(P) \quad \mathcal{B}_0(P) = \mathcal{A}(P)/\mathcal{G}_0(P).$$

Now it is straightforward to check directly that the Yang - Mills functional is invariant under gauge transformations. Thus it yields maps

$$\mathcal{YM} : \mathcal{B}(P) \longrightarrow \mathbb{R} \quad \text{and} \quad \mathcal{YM} : \mathcal{B}_0(P) \longrightarrow \mathbb{R}.$$

It is therefore important to understand the homotopy types of these orbit spaces. Because of the freeness of the action of $\mathcal{G}_0(P)$, the homotopy type of the orbit space $\mathcal{B}_0(P)$ is easier to understand, and so we will concentrate on it for the remainder of this section.

Since the space of connections $\mathcal{A}(P)$ is affine, it is contractible. Thus by the above theorem $\mathcal{B}_0(P) = \mathcal{A}(P)/\mathcal{G}_0(P)$ is the classifying space of the based gauge group,

$$\mathcal{B}_0(P) = B\mathcal{G}_0(P).$$

But the classifying spaces of the gauge groups are relatively easy to understand. The following description of their homotopy types was originally proved in [G].

**Theorem 1.3.** *Let $SU(n) \longrightarrow ESU(n) \longrightarrow BSU(n)$ be a universal principal bundle for the Lie group $SU(n)$. Fix a basepoint $y_0 \in BSU(n)$. Then there are homotopy equivalences*

$$B\mathcal{G}(P) \simeq Map^P(M, BSU(n)) \quad \text{and} \quad \mathcal{B}_0(P) \simeq B\mathcal{G}_0(P) \simeq Map_0^P(M, BSU(n))$$

*where $Map(M, BSU(n))$ is the space of all continuous maps from $M$ to $BSU(n)$ and $Map_0(M, BSU(n))$ is the space of those maps that preserve the basepoints. The superscript $P$ denotes the path component of these mapping spaces consisting of the homotopy class of maps that classify the principal $SU(n)$ - bundle $P$ up to isomorphism.*

*Proof.* Consider the space of all $SU(n)$ - equivariant maps from $P$ to $ESU(n)$,

$$Map^{SU(n)}(P, ESU(n)).$$

The gauge group $\mathcal{G}(P) \cong Aut^{SU(n)}(P)$ acts freely on the left of this space by composition. It is easy to see that $Map^{SU(n)}(P, ESU(n))$ is contractible, and its orbit space is given by the space of maps from the $SU(n)$ - orbit space of $P$ $(= M)$ to the $SU(n)$ - orbit space of $ESU(n)$ (= BSU(n)),

$$Map^{SU(n)}(P, ESU(n))/\mathcal{G}(P) \cong Map^P(M, BSU(n)).$$

This proves that $Map(M, BSU(n)) = B\mathcal{G}(P)$. Similarly $Map_0^{SU(n)}(P, ESU(n))$, the space of $SU(n)$ - equivariant maps that send the fiber $P_{x_0}$ to the fiber $ESU(n)_{y_0}$, is a contractible space with a free $\mathcal{G}_0(P)$ action, whose orbit space is

$$Map_0^P(M, BSU(n)).$$

Hence $Map_0^P(M, BSU(n)) = B\mathcal{G}_0(P)$.

We observe that the homotopy equivalence

$$\mathcal{B}_0(P) \simeq Map^P(M, BSU(n))$$

can be viewed in the following way. Its adjoint is a map

$$\mu : \mathcal{B}_0(P) \times M \longrightarrow BSU(n) \tag{1.4}$$

which is defined up to homotopy by the principal $SU(n)$ - bundle it classifies. This bundle is given by

$$\mathcal{A} \times_{\mathcal{G}_0(P)} P \longrightarrow \mathcal{B}_0(P) \times M.$$

Notice that the map $\mu$ induces pairings in homology

$$\mu_* : H_r(M) \otimes H_{q-r}(\mathcal{B}_0(P)) \longrightarrow H_q(BSU(n)).$$

These pairings can be viewed as homomorphisms

$$\mu_* : H_r(M) \longrightarrow H^{q-r}(\mathcal{B}_0(P); H_q(BSU(n)). \tag{1.5}$$

A particularly important example of this homomorhism is when $r = 2$ and $q = 4$. Since $H_4(BSU(n)) \cong \mathbb{Z}$ we have a homomorphism

$$\mu_* : H_2(M) \longrightarrow H^2(\mathcal{B}_0(P)).$$

This homomorphism is of particular importance when $M$ is a simply connected four dimensional manifold. It is an essential ingredient in Donaldson's description of his polynomial invariants.

The Yang - Mills functional is easily seen to be invariant under the action of the gauge group, and so defines a map

$$\mathcal{YM} : \mathcal{B}(P) \longrightarrow \mathbb{R}$$

We now discuss the variational equations for $\mathcal{YM}$, that is the equations determining when a connection $A$ on $P$ is a critical point of $\mathcal{YM}$ We will then restrict to the important case when $M$ is a compact 4 - dimensional manifold and identify the space of absolute minima of $\mathcal{YM}$. We refer the reader to [AB] for details and background for the arguments presented below.

Given a Riemannian metric on an $n$ -manifold $M^n$ and a vector bundle $\zeta \longrightarrow M$, let

$$* : \Omega^p(M; \zeta) \longrightarrow \Omega^{n-p}(M; \zeta)$$

be the Hodge star operator. Locally the operator can be described as follows. Let $I = (i_1, \cdots, i_p)$ be an increasing sequence of $p$ - integers between 1 and $n$ and let $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Then

$$*dx_I = dx_{n-I}$$

where $n - I$ is the ordered sequence of the $n - p$ integers that do not appear in $I$. The metric on $M$ and on the bundle $\zeta$ is then used to extend this definition to globally defined forms.

**Theorem 1.6.** *A connection $A$ on $P$ is a critical point of the Yang - Mills functional $\mathcal{YM}$ if and only if it satisfies the Yang - Mills equations:*

$$D_A(*F_A) = 0.$$

*where, as above, $D_A$ is the covariant derivative induced by $A$, and $F_A$ is the curvature 2 - form.*

We end this chapter by discussing perhaps the most important case; when $P \longrightarrow M$ is a principal $SU(n)$ - bundle over a closed, simply connected four dimensional manifold $M$.

The well known Bianchi identities for connections assert that any connection $A$ satisfies $D_A F_A = 0$. Comparing this with the Yang - Mills equations, we see that a distinguished class of solutions to the Yang - Mills equations are the *self dual* and *anti- self dual* connections; that is, connections that satisfy

$$F_A = *F_A \quad \text{and} \quad F_A = - * F_A$$

respectively. (Note that dimension four is necessary for the curvature to satisfy self duality.) The following is a well known result, derived by a direct comparison between the formulae for the Yang - Mills functional $\mathcal{YM}$ and for the Pontryagin class

$$\begin{aligned} p_1(P) &= \frac{-1}{4\pi^2} \int_M trace(F_A \wedge F_A)\, d(vol) \\ &= \frac{-1}{4\pi^2} \int_M \|(F_A)_+\|^2 - \|(F_A)_-\|^2 \, d(vol) \end{aligned}$$

where $(F_A)_+$ and $(F_A)_-$ are the projections of the curvature form $F_A$ onto to the positive and negative eigenspaces of the Hodge star operator respectively.

**Theorem 1.7.** *Let $P \longrightarrow M$ be a principal $SU(n)$ - bundle over a closed, oriented four dimensional manifold $M$. Suppose the Pontryagin class of the bundle is given by*

$$p_1(P) = k \in H^4(M^4) \cong \mathbb{Z}.$$

*Then a connection $A$ is a global minimum of the Yang - Mills functional $\mathcal{YM}$ if and only if*

$$F_A = \begin{cases} *F_A & \text{if } k \text{ is negative} \\ - * F_A & \text{if } k \text{ is positive} \\ 0 & \text{if } k = 0. \end{cases}$$

*Such connections are called self dual, anti - self dual, and flat respectively.*

Let $M$ be a simply connected, oriented, closed four - manifold and let $\mathcal{B}_k(M) = \mathcal{B}_0(P_k)$, where $P_k$ is the $SU(n)$ - bundle of Pontryagin class $p_1 = k \in H^4(M) \cong \mathbb{Z}$. Let

$$\mathcal{M}_k \hookrightarrow \mathcal{B}_k$$

be the subspace of self dual connections if $k$ is negative or anti - self dual connections if $k$ is positive. Notice that if $k = 0$, the relevant space of connections are the flat connections. It is well known that gauge equivalence classses of flat connections are in bijective correspondence with conjugacy classes of representations of the fundamental group, $\pi_1(M) \longrightarrow SU(n)$. However since we are assuming that $M$ is simply connected, there is only the trivial representation, and hence $\mathcal{M}_0$ is the one point space.

From an algebraic topological point of view, perhaps the main question in gauge theory is to understand the homotopy type of the pair $(\mathcal{B}_k, \mathcal{M}_k)$ and certain variations of it.

We now examine the homotopy type of $\mathcal{B}_k$ somewhat more closely. Again we retain the assumption that $M$ is a simply connected, four - manifold.

By Poincare duality one knows that $M$ has the homotopy type of a $CW$ complex of the form

$$M \simeq \left(\bigvee_g S^2\right) \cup_h D^4$$

where $h \in \pi_3(\bigvee_g S^2)$. By the classical Milnor - Whitehead theorem the attaching map $h$ is determined by the unimodular symmetric rank $g$ bilinear form given by the cup product pairing

$$\begin{aligned} H^2(M) \times H^2(M) &\longrightarrow \mathbb{Z} \\ (x, y) &\longrightarrow \langle x \cup y; [M] \rangle. \end{aligned}$$

Thus the homotopy type of $\mathcal{B}_k \simeq Map_0^k(M, BSU(n))$ is determined by this form. To understand the homology $H_*(\mathcal{B}_k)$ we may use the homotopy cofibration sequence induced by the above cell decomposition

$$\bigvee_g S^2 \hookrightarrow M \longrightarrow S^4$$

to obtain a homotopy fibration sequence

$$\Omega^4 BSU(n) \longrightarrow Map_0(M, BSU(n)) \longrightarrow \prod_g \Omega^2 BSU(n)$$

where here $\Omega X$ denotes the space of basepoint preserving loops of $X$. Equivalently there is a fibration sequence

$$\Omega^3 SU(n) \longrightarrow Map_0(M, BSU(n)) \longrightarrow \prod_g \Omega SU(n). \tag{1.8}$$

The following describes the homotopy type of $\mathcal{B}_k$ when the Lie group is $SU(2)$.

**Proposition 1.9.** *The map*

$$\mathcal{B}_k \simeq Map_0^k(M, BSU(2)) \longrightarrow \prod_g \Omega SU(2)$$

*is a rational homotopy equivalence. In particular we have the following calculation of the rational homology*

$$H_*(\mathcal{B}_k; \mathbb{Q}) \cong \otimes_{i=1}^g T(\sigma_i)$$

*where $T(\sigma_i)$ is the graded tensor algebra (over $\mathbb{Q}$) on a class $\sigma_i$ of dimension 2.*

*Proof.* Consider the fibration sequence 1.8. Since $\pi_q SU(2) \cong \pi_q S^3$ is torsion for $q > 3$, we have that the map

$$\Omega^3 SU(2) \longrightarrow Map_0(M; BSU(2))$$

induces an isomorphism on $\pi_0$, and that each component of $\Omega^3 SU(2)$ has the rational homotopy type of a point. Hence each component of $Map_0(M; BSU(2))$ maps to $\prod_g \Omega SU(2)$ via a rational homotopy equivalence. The proposition follows since $\mathcal{B}_k \simeq Map_0^k(M; BSU(2))$

Notice that 1.9 implies that the homomorphism

$$\mu_* : H_2(M; \mathbb{Q}) \longrightarrow H^2(\mathcal{B}_k; \mathbb{Q})$$

given in (1.5) is an isomorphism in this setting.

We close this section by citing two recent results by [Ma, Mi] on the cohomology groups $H^*(\mathcal{B}_q)$ when the Lie group is $SU(2)$. They both come about by analyzing spectral sequences induced by the homotopy fibration (1.8).

First notice that fibration (1.8) is a principal fibration, classified by the map

$$h^* : \prod_g \Omega^2 BSU(2) = Map_0(\bigvee_g S^2, BSU(2)) \longrightarrow Map_0(S^3, BSU(2))$$

where $h^*$ is the map induced on function spaces by the attaching map

$$h : S^3 \longrightarrow \bigvee_g S^2.$$

Notice that in cohomology $h^*$ gives $H^*(Map_0(\bigvee_g S^2, BSU(2)) = \otimes_g H^*(\Omega S^3)$ the structure of a module over the cohomology ring

$$H^*(Map_0(S^3; BSU(2))) = H^*(\Omega^2 S^3).$$

**Theorem 1.10.** *Let $k$ be any field. Then the Eilenberg - Moore in spectral sequence for the cohomology with coefficients in $k$ for principal fibration (1.8) collapses. Hence we have an isomorphism*

$$H^*(\mathcal{B}_q; k) \cong H^*(Map_0^q(M; BSU(2); k) \cong Tor_{H^*(\Omega^2 S^3; k)}\left(k; \otimes_g H^*(\Omega S^3; k)\right).$$

Notice that this gives $H^*(\mathcal{B}_q; k)$ as an algebraic functor of the quadratic form determined by the attaching map $h$, since on the one hand this form determines (and is determined by) the homotopy class of $h$, and on the other hand, $h^*$ determines the relevent module structure used in the definition of the $Tor$ group in this calculation.

Now this $Tor$ - group may be difficult to compute in general. When $M$ is a $Spin$ - manifold (i.e the second Stiefel - Whitney class $w_2(M) = 0$), the following result of [Ma, Mi] gives a more explicit calculation of the mod 2 cohomology.

**Theorem 1.12.** *If $M$ is a $Spin$ - manifold, the Serre spectral sequence in mod 2 - cohomology for the fibration (1.8) collapses. Hence we have an isomorphism of $\mathbb{Z}_2$ - vector spaces*

$$H^*(\mathcal{B}_q; \mathbb{Z}_2) = \otimes_g H^*(\Omega S^3; \mathbb{Z}_2) \otimes H^*(\Omega^3 S^3; \mathbb{Z}_2).$$

**Remark.** This theorem says that for simply connected, $Spin$ four manifolds the $\mathbb{Z}_2$ - vector spaces $H^*(\mathcal{B}_q; \mathbb{Z}_2)$ only depends on the rank of $H_2(M)$ ( $= g$ in our notation). However the isomorphism in 1.12 is *not* a ring isomorphism. Actually $H^*(\mathcal{B}_q; \mathbb{Z}_2)$ is a certain twisted tensor product of the rings $\otimes_g H^*(\Omega S^3; \mathbb{Z}_2)$ and $H^*(\Omega^3 S^3; \mathbb{Z}_2)$. The twisting is determined by the attaching map $h$, or equivalently the quadratic form on $H^2(M; \mathbb{Z}_2)$ it induces.

## §2. Monopoles, Rational Functions, and Braids

In this section we consider connections on trivial principal bundles over four dimensional manifolds of the form $M^4 = \mathbb{R}^3 \times X$, where $X$ is either the real line or the circle group. Viewing $X$ as a one dimensional topological group, the corresponding space of connections is acted upon by $X$, and we will consider those connections which are invariant under this action. The resulting moduli spaces of self dual invariant connections are known as spaces of "monopoles". In this section we will discuss these moduli spaces in the examples when $\mathbb{R}^3$ has both the Euclidean metric and $X = \mathbb{R}$ and when $\mathbb{R}^3$ has a hyperbolic metric and $X = S^1$.

By a theorem of Taubes [T1] in both of these cases, the ambient manifold of based gauge equivalence classes of invariant connections is homotopy equivalent to the space of basepoint preserving, smooth self maps of the sphere $S^2$, $\Omega^2 S^2$. Via theorems of Atiyah and Donaldson, the relevant moduli spaces are given up to homeomorphism by the space of based *holomorphic* maps of the Riemann sphere to itself, or equivalently, the space of rational functions of one complex variable. Thus the resulting natural homotopy theoretic problem is to understand how much of the homotopy type of the entire space of smooth maps from $S^2$ to itself is detected by the space of of holomorphic maps. The answer to this question was recently worked out by F. Cohen, R. Cohen, B. Mann, and R.J. Milgram in [CCMM]. In this work a curious relationship between the space of monopoles and Artin's braid groups was deduced. We will discuss this work in this chapter. We refer the reader to [CCMM] for details and to [AH] for a more geometric description of the space of monopoles.

Let $P = \mathbb{R}^4 \times SU(2)$ be the trivial $SU(2)$ - bundle over $\mathbb{R}^4$. Clearly the vector bundle $ad(P)$ is also trivial, and so a connection $C$ on $P$ can be viewed as a one form

$$C \in \Omega^1(\mathbb{R}^4, ad(P)) = \Omega^1(\mathbb{R}^4, \mathfrak{su}(2)) \cong \Omega^1(\mathbb{R}^4, \mathbb{R}^3).$$

Throughout this section we will be thinking of the last coordinate of $\mathbb{R}^4$ as "time" and so we will write an element of $\mathbb{R}^4$ as $(v, t)$ where $v \in \mathbb{R}^3$ and $t \in \mathbb{R}$. Then a connection $C$ can be written in the form

$$C = A_1(v,t)dx_1 + A_2(v,t)dx_2 + A_3(v,t)dx_3 + \phi(v,t)dt$$

where $A_1, A_2, A_3$ and $\phi : \mathbb{R}^4 \longrightarrow \mathfrak{su}2 \cong \mathbb{R}^3$ are smooth maps.

The connection $C$ is said to be *time invariant* if each of the maps $A_i$ and $\phi$ are time invariant. That is

$$A_i(v,t) = A_i(v,0) \quad \text{and} \quad \phi(v,t) = \phi(v,0)$$

for all $t \in \mathbb{R}$. Notice in this case the maps $A_i$ and $\phi$ can be thought of as smooth maps $\mathbb{R}^3 \to \mathfrak{su}2$. In particular

$$A = A_1(v)dx_1 + A_2(v)dx_2 + A_3(v)dx_3$$

may be viewed as a connection on the trivial $SU(2)$ bundle over $\mathbb{R}^3$. Thus time invariant connections on $\mathbb{R}^4$ are given by pairs $(A, \phi)$ where

(1) $A$ is a connection on the trivial $SU(2)$ - bundle on $\mathbb{R}^3$,
(2) $\phi$ is an $\mathfrak{su}2$ valued smooth function on $\mathbb{R}^3$.

The connection $A$ is called the **gauge potential** and $\phi$ is called the **Higgs field.**

The pairs $(A, \phi)$ that we will consider will be required to satisfy several conditions (see [T1, AH]). To describe the first condition, observe that given a time invariant connection $C$, the square of the norm of the curvature, $\|F_C\|^2$, at any given point $v \in \mathbb{R}^3$ can be expressed in terms of the associated pair $(A, \phi)$. It is given by $\|F_A\|^2 + \|D_A\phi\|^2$ where $F_A$ is the curvature of the $\mathbb{R}^3$ - connection $A$, and $D_A$ is its covariant derivative, which operates on $\phi$ by considering $\phi$ as a zero form with coefficients in $\mathfrak{su}2$. Hence $D_A\phi \in \Omega^1(\mathbb{R}^3, \mathfrak{su}2)$. Thus the appropriate energy functional in this setting is the *Yang - Mills - Higgs* energy given by

$$\mathcal{U}(A, \phi) = \int_{\mathbb{R}^3} (\|F_A\|^2 + \|D_A\phi\|^2) d\,vol.$$

The first condition we will require is the finite energy condition:

$$\text{(I)} \qquad \mathcal{U}(A, \phi) < \infty.$$

We also require a condition on the behavior of $\phi$ at infinity. There are several different conditions which may be imposed, but the usual normalizing condition is that

$$\text{(II)} \qquad \lim_{|x| \to \infty} |\phi(x)| = 1$$

where we take the usual norms in $\mathbb{R}^3$. Actually the weakest convergence condition that seems to be sufficient is $1 - |\phi| \in L^6(\mathbb{R}^3)$. We do not assume any asymptotic conditions on the gauge potential $A$ (other than that implied by requiring that $\mathcal{U}(A, \phi) < \infty$). However we do impose an additional basepoint condition:

$$\text{(III)} \qquad \lim_{t \to \infty} \phi(t, 0, 0) = (1, 0, 0).$$

Let $\mathcal{A}$ be the space of pairs $(A, \phi)$ which satisfy these three conditions. $\mathcal{A}$ is a space of time - invariant connections on $\mathbb{R}^4$, but unlike previous spaces of connections we have considered, $\mathcal{A}$ is *not* an affine space, indeed it is not contractible. The three conditions impose an interesting topology on $\mathcal{A}$ which was identified by Taubes [T1] in the following manner.

Consider the following map

$$I : \Omega^2 S^2 \longrightarrow \mathcal{A},$$

where here $\Omega^2 S^2$ is the space of all smooth maps $S^2 \to S^2$ which preserve the basepoint $(1, 0, 0) \in S^2$. Identify the unit sphere $S^2$ with the unit sphere in the Lie algebra $\mathfrak{su}2$. Given a map

$$\alpha : S^2 \longrightarrow S^2 \subset \mathfrak{su}2$$

define the pair $I(\alpha) = (A, \phi)$ by the formula

$$A = \beta(|x|) \left[ \alpha\left(\frac{x}{|x|}\right), d\alpha\left(\frac{x}{|x|}\right) \right]$$

$$\phi = -\beta(|x|)\alpha\left(\frac{x}{|x|}\right).$$

In this formula $\beta : \mathbb{R} \to [0, 1]$ is a smooth cut - off function which is identically 0 if $t \leq 1/2$ and identically 1 if $t \geq 3/4$, and $[,]$ is the Lie bracket in $\mathfrak{su}2$. (See [T1] for details). The following was proved in [T1].

**Theorem 2.1.** *The map $I : \Omega^2 S^2 \longrightarrow \mathcal{A}$ is a homotopy equivalence.*

We remark that the homotopy inverse to the map $I$ is given by sending a pair $(A, \phi)$ to the restriction of the Higgs field $\phi$ to the "sphere at $\infty$" which is to say the map $\bar{\phi} : S^2 \longrightarrow S^2$ given by

$$\bar{\phi}(x) = lim_{t \to \infty} \phi(tx). \tag{2.2}$$

Since the bundle over $\mathbb{R}^3$ is trivial the based gauge group of bundle automorphisms in this setting is given by

$$\mathcal{G}_0 = Map_0(\mathbb{R}^3, SU(2))$$

where here $Map_0$ means smooth maps $g : \mathbb{R}^3 \longrightarrow SU(2)$ which satisfy the basepoint condition

$$\lim_{t \to \infty} g(t, 0, 0) = 1 \in SU(2).$$

The gauge group $\mathcal{G}_0$ acts on $\mathcal{A}$ by the formula

$$g(A, \phi) = (g^{-1}Ag + g^{-1}dg, g^{-1}\phi g).$$

This formula makes sense since $g$ and $\phi$ are matrix valued functions and $A$ is a matrix of one - forms. In this formula the expression $dg$ refers to the differential of the function $g : \mathbb{R}^3 \longrightarrow SU(2)$ which is a one form with coefficients in the Lie algebra $\mathfrak{su}2$. (Here we are identifying the Lie algebra $\mathfrak{su}2$ with the tangent space at the identity of the Lie group $SU(2)$.)

As in previously studied examples, this action of the based gauge group $\mathcal{G}_0$ on $\mathcal{A}$ is free and it is not difficult to see that the quotient map to the orbit space

$$\mathcal{A} \longrightarrow \mathcal{B} = \mathcal{A}/\mathcal{G}_0$$

is a principal $\mathcal{G}_0$ bundle. (Again we refer to [T1] for details.) Moreover since $\mathbb{R}^3$ is contractible, the basepoint condition in $\mathcal{G}_0 = Map_0(\mathbb{R}^3, SU(2))$ implies that $\mathcal{G}_0$ is a contractible topological group. Hence this bundle has contractible fibers. This implies the following.

**Proposition 2.3.** *The projection onto the orbit space*

$$\mathcal{A} \longrightarrow \mathcal{B} = \mathcal{A}/\mathcal{G}_0$$

*is a homotopy equivalence.*

Combining 2.1 and 2.3 we have the following theorem of Taubes proved in [T1].

**Theorem 2.4.** *There is a natural homotopy equivalence*

$$\mathcal{B} \xrightarrow{\simeq} \Omega^2 S^2.$$

In view of this result the path components of $\mathcal{B}$ (and of $\mathcal{A}$) are labelled by the integers $\mathbb{Z}$ corresponding to the degree of the map in $\Omega^2 S^2$. We write $\mathcal{B}_k$ for the component corresponding to $\Omega^2_k S^2$. In the literature the integer $k$ is referred to as the *charge* of the pair $(A, \phi)$.

The Yang - Mills - Higgs functional is invariant under the gauge group action and so defines a function

$$\mathcal{U} : \mathcal{B}_k \longrightarrow \mathbb{R}$$

for each $k \in \mathbb{Z}$. By the definition of $\mathcal{U}$ (actually its local relationship with the Yang - Mills functional on the associated time invariant connections on $\mathbb{R}^4$) it is not difficult to see that $(A, \phi) \in \mathcal{A}$ represents a minimum of $\mathcal{U}$ if and only if the associated four - dimensional connection

$$\alpha = A + \phi dt$$

is self - dual, or anti - self dual:

$$*F_\alpha = \pm F_\alpha.$$

This duality equation, written directly in terms of the pair $(A, \phi)$ is given by the *Bogomolnyi equation*

$$*F_A = \pm D_A \phi. \tag{2.5}$$

Notice that in this equation, since the underlying manifold is $\mathbb{R}^3$, the Hodge star of the curvature two - form $F_A$ is a one form, as is the covariant derivative of the Higgs field $\phi$.

Thus the space of minima $\mathcal{M}_k \subset \mathcal{B}_k$ of the Yang - Mills - Higgs functional $\mathcal{U}$, referred to as *monopoles*, is the space of gauge equivalence classes of pairs $(A, \phi) \in \mathcal{A}$ satisfying the Bogomolnyi equation. From the topological point of view, the basic question inspired by Morse theory is therefore the following.

**Question.** What is the homotopy type of the pair $(\mathcal{B}_k, \mathcal{M}_k)$?

A major step toward the understanding of the spaces of monopoles was taken by Donaldson [D] based on previous work of Hitchin and Nahm [Hi1, Hi2, N]. He proved the following.

**Theorem 2.6.** *The set of monopoles of charge $k$, $\mathcal{M}_k$, is in bijective correspondence with the set of basepoint preserving holomorphic maps of the Riemann sphere $S^2$ to itself of degree $k$.*

We will denote the space of holomorphic maps in this theorem as $Rat_k$, as such holomorphic maps are given by rational functions

$$Rat_k = \left\{ \begin{matrix} p/q : p \text{ and } q \text{ are monic, degree } k \\ \text{polynomials in } \mathbb{C} \text{ with no roots in common.} \end{matrix} \right\}$$

The basepoint condition in this description of the space of holomorphic maps is that they send $\infty \in \mathbb{C} \cup \infty = S^2$ to 1. The fact that Donaldson's correspondence between $\mathcal{M}_k$ and $Rat_k$ is actually a homeomorphism was verified in [BM2].

Thus the above question can be rephrased as follows.

**Question.** What is the homotopy type of the pair $(\Omega^2_k S^2, Rat_k)$?

The first major result concerning the homotopy type of this pair was by Segal [S1] who proved the following.

**Theorem 2.7.** *The pair $(\Omega^2_k S^2, Rat_k)$ is $(k-1)$ - connected.*

Thus the rational function spaces $Rat_k$ (or equivalently the moduli spaces $\mathcal{M}_k$) admit the same homotopy type as the ambient spaces $\Omega^2_k S^2$ (or equivalently $\mathcal{B}_k$) through a "stable range" of dimensions $\le k-1$.

Now the rational function spaces have even more structure. For example we have the following.

**Lemma 2.8.** *There exist pairings*

$$\mu : Rat_k \times Rat_r \longrightarrow Rat_{k+r}$$

*so that the following diagram homotopy commutes:*

$$\begin{array}{ccc} Rat_k \times Rat_r & \xrightarrow{\mu} & Rat_{k+r} \\ \Big\downarrow \cap & & \Big\downarrow \cap \\ \Omega^2_k S^2 \times \Omega^2_r S^2 & \xrightarrow{\sigma} & \Omega^2_{k+r} S^2. \end{array}$$

*where $\sigma$ is the loop sum operation.*

*Proof.* Let $\alpha \in Rat_k$ and $\beta \in Rat_r$. Since based rational functions are determined by their roots and poles, $\alpha$ and $\beta$ can be viewed as being given by configurations of points in the complex plane $\mathbb{C}$ each labelled according to whether it is a root or a pole and by a positive integer determining the multiplicity. Let $\alpha'$ be the configuration of roots and poles in the upper half plane $\mathbb{C}_+ = \{z : Im(z) > 0\}$ given by the image of the roots and poles of $\alpha$ under a fixed diffeomorphism of $\mathbb{C}$ with $\mathbb{C}_+$. Similarly let $\beta''$ be the configuration of roots and poles in the lower half plane $\mathbb{C}_- = \{z : Im(z) < 0\}$ given by the image of the roots and poles of $\beta$ under a fixed diffeomorphism of $\mathbb{C}$ with $\mathbb{C}_-$. Then $\alpha' \cup \beta''$ is a configuration of roots and poles that determines a rational function of degree $k+r$. This procedure defines the map

$$\mu : Rat_k \times Rat_r \longrightarrow Rat_{k+r}$$

which is clearly homotopy compatible with the loop sum operation in the loop space.

The pairing maps in this lemma are often referred to as "gluing maps", in that together with 2.6 it says that there is a natural way of gluing a monopole of charge $k$ to one of charge $r$ to produce a monopole of charge $k+r$. In particular, gluing with a fixed element in $Rat_1$ defines an inclusion map

$$j : Rat_k \hookrightarrow Rat_{k+1}$$

compatible with the homotopy equivalence $j : \Omega^2_k S^2 \xrightarrow{\simeq} \Omega^2_{k+1} S^2$ given by taking the loop sum with the identity map (or any fixed map of degree one.) If

$$Rat_\infty = \lim_{k \to \infty} Rat_k$$

then there is an induced map (well defined up to homotopy) $j : Rat_\infty \to \Omega^2_0 S^2$, or equivalently a map

$$j : \mathbb{Z} \times Rat_\infty \longrightarrow \Omega^2 S^2.$$

We therefore have the following.

**Corollary 2.9.** *The map*

$$j : \mathbb{Z} \times Rat_\infty \longrightarrow \Omega^2 S^2$$

*is a homotopy equivalence.*

As described in [BM], F. Cohen observed that the gluing maps in 2.8 actually define a $\mathcal{C}_2$ - operad structure on the disjoint union $\coprod_k Rat_k$ in the sense of May [May]. Associated to a $\mathcal{C}_2$ - space is a two - fold loop space. This can be thought of as the "group completion" of the space, and can be defined to be the loop space of its classifying space. If we use the Quillen plus construction notation to denote group completion, than 2.9 can be restated as follows.

**Corollary 2.10.** *There is a homotopy equivalence of 2 - fold loop spaces,*

$$\left( \coprod_k Rat_k \right)^+ \simeq \Omega^2 S^2.$$

These results say that the rational function spaces $Rat_k$ define a filtration of the homotopy type of $\Omega^2 S^2$. But also recall that a previous result of May, Milgram, and Segal [May, Mil, S] says that the stable homotopy type of $\Omega^2 S^2$ is filtered by the configuration spaces $C_2(k) = F(\mathbb{R}^2, k)/\Sigma_k$, where

$$\{(x_1, \cdots, x_k) \in (\mathbb{R}^2)^k \,:\, x_i \neq x_j\}.$$

More precisely one has the following.

**Theorem 2.11.** *Let $C_2(\infty) = \lim_k C_2(k)$. Then there is a natural map*

$$\alpha_k : C_2(k) \longrightarrow \Omega^2_k S^2$$

*which induces a homology equivalence*

$$\alpha : \mathbb{Z} \times C_2(\infty) \longrightarrow \Omega^2 S^2.$$

*Phrased in terms of group completions, there is a homotopy equivalence of 2 - fold loop spaces*

$$\left( \coprod_k C_2(k) \right)^+ \simeq \Omega^2 S^2.$$

The homology and other homotopy invariants of these configuration spaces are well understood. Therefore the goal of the work in [CCMM] was to compare the rational function filtration with the configuration space filtration of this loop space. Before we state the results of this work we recall the following well known result.

**Proposition 2.12.** *The configuration space $C_2(k)$ is an Eilenberg - MacLane space of type $K(\beta_k, 1)$, where $\beta_k$ is Artin's braid group on $k$ - strings.*

Thus 2.11 can be rephrased to say that there is a map

$$\alpha : \mathbb{Z} \times K(\beta_\infty, 1) \longrightarrow \Omega^2 S^2$$

which induces an isomorphism in homology, and hence is a stable homotopy equivalence. Combining this with theorem 2.9 we have the following.

**Corollary 2.13.** *The limiting spaces $Rat_\infty$ and $K(\beta_\infty, 1)$ are stably homotopy equivalent.*

The following is the main result of [CCMM].

**Theorem 2.14.** *The spaces $Rat_k$ and $K(\beta_{2k}, 1)$ are stably homotopy equivalent.*

This theorem gives an effective description of the stable homotopy type of the rational function spaces $Rat_k$ and hence the monopole spaces $\mathcal{M}_k$ since quite a bit is known about the stable homotopy type of the braid groups.

We remark that the *unstable* homotopy type of these rational functions has been studied in detail in [CS]. Notice furthermore that theorem 2.11 points to the possibility that many algebraic invariants of the braid groups will lead to interesting invariants of the spaces of monopoles. A particular example of this phenomenon was studied in [CJ, CJ1] where it was seen that a certain class of representations of the braid groups give rise to families of elliptic operators coupled to monopoles. However the full extent of this relationship has yet to be understood.

We end this chapter by outlining the structure of the proof of theorem 2.11. The reader is referred to [CCMM] for details.

The proof revolves around the study of the subspace $Rat_k^0$ of $Rat_k$ consisting of those rational functions with $k$ distinct simple poles; this is the subspace of generic rational functions. If $f = p/q \in Rat_k^0$ then $q$ is a monic, degree $k$ polynomial with $k$ distinct roots, and $p$ is determined by its values at the roots of $q$. Notice that these values all must be nonzero since $p$ and $q$ have no roots in common. This gives a description of $Rat_k^0$ as

$$Rat_k^0 \cong F(\mathbb{C}, k) \times_{\Sigma_k} (\mathbb{C}^*)^k$$

where the coordinates in $F(\mathbb{C}, k)$ correspond to the roots of the denominator and the coordinates of $(\mathbb{C}^*)^k$ are the values of the numerator at these poles. Now notice that there is a fibration

$$(\mathbb{C}^*)^k \longrightarrow Rat_k^0 = F(\mathbb{C}, k) \times_{\Sigma_k} (\mathbb{C}^*)^k \longrightarrow F(\mathbb{C}, k)/\Sigma_k = C_2(k) \simeq K(\beta_k, 1).$$

Since the fiber $(\mathbb{C}^*)^k \simeq (S^1)^k$ is the classifying space of $\mathbb{Z}^k$, this fibration allows us to deduce that $Rat_k^0$ is the classifying space $B\beta_{2,k}$ of the semi - direct product

$$\beta_{2,k} = \beta_k \ltimes (\mathbb{Z})^k$$

where $\beta_k$ acts on $(\mathbb{Z})^k$ through the homomorphism $\beta_k \longrightarrow \Sigma_k$ by permuting factors. This group $\beta_{2,k}$ can be thought of as the group of *framed braids* and it is naturally a subgroup of the braid group $\beta_{2k}$. The inclusion $\beta_{2,k} \hookrightarrow \beta_{2k}$ is given by the "cabling" process which can be described as follows. Start with $k$ pairs of pieces of string and twist the $i^{th}$ pair $n_i$ times, where $n_i$ is the integer in the $i^{th}$ coordinate of an element of $(\mathbb{Z})^k$. Now braid the $k$ pairs according to the braid $b \in \beta_k$. This gives a braid on $2k$ strings and the map which sends $(b; n_1, \cdots n_k)$ to this braid is the inclusion $\beta_{2,k} \hookrightarrow \beta_{2k}$.

We now have the following diagram.

$$\begin{array}{ccc} Rat_k^0 = K(\beta_{2,k}, 1) & \xrightarrow{\phi} & K(\beta_{2k}, 1) \\ \psi \Big\downarrow \cap & & \\ Rat_k. & & \end{array}$$

The following result, which implies theorem 2.11, was proved in [CCMM] using stable splitting theory in loop spaces, as well as homological calculations.

**Proposition 2.15.** *For suitably large $N$ there is a map of suspension spaces*

$$\sigma : \Sigma^N Rat_k \longrightarrow \Sigma^N Rat_k^0$$

*so that the composition*

$$\Sigma^N Rat_k \xrightarrow{\sigma} \Sigma^N Rat_k^0 \xrightarrow{\phi} \Sigma^N K(\beta_{2k}, 1)$$

*is a homotopy equivalence.*

## §3. $S^1$ - invariant Instantons on $S^4$ and Hyperbolic Monopoles

As we saw in section one, the minima of the Yang - Mills functional on a four dimensional manifold satisfy the self duality equations

$$F_A = \pm * F_A.$$

If we consider this equation for connections on the trivial $SU(2)$ - bundle over $\mathbb{R}^4$ and require that these connections have *finite* Yang - Mills energy $\mathcal{YM}(A) = \int_{\mathbb{R}^4} |F_A|^2 < \infty$, then we are essentially considering connections that have a (unique) extension to a self dual connection on a principal $SU(2)$ - bundle over the sphere $S^4$. We will denote by $\mathcal{M}_n = \mathcal{M}_n(S^4)$ the moduli space of gauge equivalence classes of anti - self dual connections on the principal $SU(2)$ - bundle classified by its second Chern class $c_2 = n$. The topology of these moduli spaces is of fundamental importance in Gauge theory. We will discuss what is known about their homotopy type in the next section. In this section we study the relation between these moduli spaces and the moduli spaces of monopoles discussed in the last section. We will study this relationship from both geometric and homotopy theoretic points of view.

As described in the last section, a monopole is given by a time invariant, self dual connection on $\mathbb{R}^4 = \mathbb{R}^3 \times$ "*time*". Now the time invariance of such a connection immediately implies that the Yang - Mills energy, $\mathcal{YM}(A) = \int_{\mathbb{R}^4} |F_A|^2$ is infinite. In particular they do not extend to self dual connections on the one point compactification, $S^4$. However there is another way, due to Atiyah [A2] by which we may view the space of monopoles (or equivalently, rational functions) as a subspace of the the moduli space $\mathcal{M}_n(S^4)$. This is done as follows.

Consider the natural $S^1$ - action on $S^4$ given by the two fold suspension of the action on $S^2$ defined by rotating around the equator. Notice that the fixed point set of this action is a copy of $S^2 \hookrightarrow S^4$. Moreover, if we give $S^4$ the usual Euclidean metric, there is a conformal $S^1$ - equivariant isomorphism of the complement

$$S^4 - S^2 \cong \mathbb{H}^3 \times S^1.$$

where $\mathbb{H}^3$ is hyperbolic 3 - space. Now consider the space of $S^1$ - invariant instantons (self - dual connections) on $S^4 - S^2 \cong \mathbb{H}^3 \times S^1$. Similarly to how time invariant connections were studied in the last section, we see that any $S^1$ -invariant connection on $\mathbb{H}^3 \times S^1$ is given by two pieces of data. The first is a connection $A$ on the trivial $SU(2)$ - bundle over $\mathbb{H}^3$ and the second is a "Higgs field", which is to say a smooth map

$$\phi : \mathbb{H}^3 \longrightarrow \mathfrak{su}(2).$$

The self duality equations can then be translated to the analogue of the three dimensional Bogolmonyi equations on $\mathbb{H}^3$,

$$D_A \phi = * F_A$$

and the solutions are referred to as "Hyperbolic monopoles".

Now associated to the original $SU(2)$ - bundle $E$ over $S^4$ is a vector bundle

$$E \longrightarrow S^4$$

with fiber $\mathbb{C}^2$. The $S^1$ action on $S^4$ lifts (in various ways) to an action on $E$, and if one restricts $E$ to the the fixed point set $S^2 \hookrightarrow S^4$, this action defines a representation of $S^1$ on the fibers over $S^2$. Up to conjugacy this representation is independent of the choice of fiber (over $S^2$) and is of the form

$$\lambda \longrightarrow \begin{pmatrix} \lambda^p & 0 \\ 0 & \lambda^{-p} \end{pmatrix}$$

for some positive integer $p$. This integer is determined by the lifting of the $S^1$ - action. Moreover as described in [A2] it has the following geometric significance.

$$p = \lim_{|x| \to \infty} |\phi(x)|$$

where $\phi : \mathbb{H}^3 \longrightarrow \mathfrak{su}(2)$ is the Higgs field. This is essentially because the Higgs field is the difference between the Lie derivative and the covariant derivative along orbits of the $S^1$ - action. At fixed points the Lie derivative is zero and the covariant derivative is given by the representation along the fibers given above.

If one restricts the original $SU(2)$ - bundle $E \longrightarrow S^4$ to the fixed point set $S^2$ one has a chern class $c_1 = k$ of the monopole. As in the case of Euclidean monopoles this can be viewed as the degree of the asymptotic Higgs field

$$\begin{aligned} \phi_\infty : S^2 &\longrightarrow S^2 \\ x &\longrightarrow \lim_{t \to \infty} \frac{1}{p} \phi(tx). \end{aligned}$$

As is proved in [A2], these integers are related by the formula

$$n = kp \tag{3.1}$$

where $n = c_2$ is the second chern class of the original bundle $E \longrightarrow S^4$.

This asymptotic norm (mass) of the Higgs field is therefore an integer, and according to to a version of Uhlenbeck's removable singularity theorem proved in [SS] a hyperbolic monopole ($S^1$ - invariant self dual connection) on $\mathbb{H} \times S^1 \cong S^4 - S^2$ extends uniquely to a smooth $S^1$ invariant instanton on all of $S^4$ if and only if the its mass is integral. Furthermore, by renormalizing it is easy to see that the topology of the moduli space of (unbased) gauge equivalence classes of such monopoles does not depend on the mass (see [A2]). That is, the topology of the moduli space only depends on the charge $k$, and we denote this moduli space by $Mon_k$. This together with (3.1) and an easy analysis of the induced $S^1$ action on $\mathcal{M}_n$ which is carried out in [CJR] implies that there is a natural embedding

$$\coprod_{k|n} Mon_k \hookrightarrow \mathcal{M}_n \tag{3.2}$$

which describes the inclusion of the fixed points under the $S^1$ action on $\mathcal{M}_n$. Here the disjoint union is taken over all integers $k$ of the dividing $n$, and having the same sign as $n$.

Now according to Atiyah [A2] the moduli spaces of Euclidean monopoles and hyperpolic monopoles of the same charge are homeomorphic and in particular as observed in the last section are describable in terms of rational functions. More explictly, if $\tilde{Mon}_k$ is the space of *based* gauge equivalence classes of hyperbolic monoples, there is a homeomorphism

$$\tilde{Mon}_k \cong Rat_k.$$

This homeomorphism is described explicitly in [A2]. Furthermore there is a natural action of $S^1$ on $\tilde{Mon}_k$ whose orbit space is the space of unbased monopoles,

$$\tilde{Mon}_k/S^1 \cong Mon_k.$$

With respect to the homeomorphism with $Rat_k$ this action is given by scalar multiplication on roots and poles of the rational function. By the results of [CCMM] described in section 2, this gives a complete understanding of the stable homotopy of hyperbolic monopole spaces $Mon_k$, and therefore by (3.2) the stable homotopy type of the fixed points of the natural circle action on the space of instantons $\mathcal{M}_k$. This allowed considerable information about the $S^1$ - equivariant homotopy type of $\mathcal{M}_n$ to be deduced in [CJR]. In particular this allows one to recover quite easily the calculation of Furuta [F] of the Euler characteristic of the instanton space $\mathcal{M}_n$. This is done as follows.

Since the only proper subgroups of $S^1$ are the cyclic groups $\mathbb{Z}_n$, and the orbit spaces $S^1/\mathbb{Z}_n$ are topologically circles and hence have Euler characteristic zero, the Euler characteristic of an $S^1$ - equivariant $CW$ - complex is equal to the Euler characteristic of its space of fixed points. Thus by (3.2) we are reduced to computing the Euler characteristics of the monopole spaces $Mon_k$ for $k|n$. Now as mentioned above, the principal $S^1$ - bundle

$$S^1 \longrightarrow \tilde{Mon}_k \longrightarrow Mon_k$$

is isomorphic to the principal $S^1$ - bundle

$$S^1 \longrightarrow Rat_k \longrightarrow Rat_k/S^1.$$

But by the results of [CCMM] described in section 2, it is easy to see that the inclusion of the fiber $S^1 \hookrightarrow Rat_k$ is a rational homotopy equivalence. This implies that the orbit space $Rat_k/S^1 \cong Mon_k$ has the rational homotopy type of a point, and hence has Euler characteristic one. This together with (3.2) implies the following calculation.

**Theorem 3.3.** *The Euler characteristic of the moduli space of (unbased) gauge equivalence classes of instantons $\mathcal{M}_n$ is equal to the number of integral divisors of $n$.*

This theorem was proved originally by Furuta [F] who did not use knowledge of the homotopy type of the monopole spaces, but instead used a considerably more complicated analysis of a torus action on $\mathcal{M}_n$.

We end this section with one other observation done in [CJR] which gives a homotopy theoretic interpretation of the inclusion of the space of hyperbolic monopoles into the space of instantons given in (3.2). Actually we described the inclusion of the space of *based* gauge equivalence classes of monopoles $\tilde{Mon}_k$ into the space of based gauge equivalence classes of instantons, which we denote similarly by $\tilde{\mathcal{M}}_k$.

As observed above, there is a homeomorphism $\tilde{Mon}_k \cong Rat_k$, which is the space of basepoint preserving holomorphic maps from the Riemann sphere $S^2$ to itself of degree $k$. To emphasize this point we will use the alternate notation $Hol_k(S^2, S^2)$ for this space. Now in [A2] Atiyah proved that there is a similar homeomorphism from the instanton space $\tilde{\mathcal{M}}_k$ to the space of holomorphic maps from $S^2$ to the loop group $\Omega SU(2)$

$$\tilde{\mathcal{M}}_k \cong Hol_k(S^2, \Omega SU(2)).$$

The complex structure of loop groups is described in detail in [PS]. Now the Lie group $SU(2)$ is the unit sphere in the space of quaternions and hence is topologically the three sphere, $S^3$. Now the natural inclusion $S^2 \hookrightarrow \Omega S^3 = \Omega SU(2)$ given by sending a point on the equator of $S^3$ to the loop in $S^3$ which is perpindicular to that point realizes the Freudenthal suspension map in homotopy groups. Moreover, by examining the holomorphic structure of $\Omega SU(2)$ described in [PS] it is shown that this inclusion $S^2 \hookrightarrow \Omega SU(2)$ is holomorphic and hence induces an inclusion of holomorphic mapping spaces

$$Rat_k = Hol_k(S^2, S^2) \hookrightarrow Hol_k(S^2, \Omega SU(2)).$$

Furthermore, the inclusion of holomorphic maps into all continuous maps gives inclusions

$$Hol_k(S^2, S^2) \hookrightarrow \Omega^2_k S^2 \quad \text{and} \quad Hol_k(S^2, \Omega SU(2)) \hookrightarrow \Omega^3_k S^3,$$

where $\Omega^2_k S^2$ and $\Omega^3_k S^3$ denote the space of basepoint preserving degree $k$ self - maps of $S^2$ and $S^3$ respectively. Let

$$E : \Omega^2 S^2 \hookrightarrow \Omega^3 S^3$$

be the suspension map. The following was also verified in [CJR].

**Theorem 3.4.** *The following diagram homotopy commutes.*

$$\begin{tikzcd}
\tilde{Mon}_k \arrow[r, hook] \arrow[d, "\cong"] & \tilde{\mathcal{M}}_k \arrow[d, "\cong"] \\
Hol_k(S^2, S^2) \arrow[r, hook] \arrow[d, "\cap"] & Hol_k(S^2, \Omega SU(2)) \arrow[d, "\cap"] \\
\Omega^2_k S^2 \arrow[r, "E"] & \Omega^3 S^3.
\end{tikzcd}$$

## §4. Instantons on $S^4$

We now return to the problem of studying the geometry of the pairs $(\mathcal{B}_k, \mathcal{M}_k)$ where $\mathcal{B}_k$ is the space of based gauge equivalence classes of connections on the $SU(2)$ bundle $P_k \to S^4$ of Chern class $c_2 = k$, and $\mathcal{M}_k$ is the subspace of self dual connections ("instantons") with respect to the round metric on $S^4$. Recall from section one that there is a homotopy equivalence

$$\mathcal{B}_k \simeq \Omega^3_k S^3.$$

The first solutions of the self duality equations were given by 't Hooft [tH], and were generalized in the work of Atiyah, Drinfeld, Hitchin and Manin, [ADHM], (see also Atiyah's Pisa lecture notes for a more detailed account), to give the following prescription for the space $\mathcal{M}_k$.

We begin with a subspace, $\mathcal{D}_k$, of the space of pairs of quaternion matrices $(B, \Lambda)$, where $B$ is $k \times k$ and $\Lambda$ is $k \times 1$. They are required to satisfy the following constraints:

(1) For all $x \in \mathbb{H}$, $rk(B - xI, \Lambda) = k$, i.e., the $k$ rows are linearly independent in the quaternionic vector space $\mathbb{H}^{k+1}$,
(2) $B$ is symmetric, i.e., $B_{i,j} = B_{j,i}$, $1 \le i, j \le k$,
(3) $BB^* + \Lambda\Lambda^*$ is real where $M^*$ is the conjugate transpose of $M$.

There is an action of the real orthogonal group on $\mathcal{D}_k$ by regarding $O(k)$ as the subgroup of real matrices in $Sp(k)$, and acting via the rule $g(B, \Lambda) = (gBg^{-1}, g\Lambda)$. Then the main result of [ADHM] is an identification $\mathcal{M}_k = O(k)\backslash\mathcal{D}_k$. In terms of this identification 't Hooft's original solutions come from pairs of the form

$$\left(\begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_k \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}\right)$$

as $(\lambda_1, \lambda_2, \dots, \lambda_k)$ wander over $k$-tuples of **distinct** points in $\mathbb{H}$. The action of $O(k)$ serves to unorder the points and this gives us an embedding of the space of unordered configurations $C_4(k) \hookrightarrow \mathcal{M}_k$.

In [AJ] the 't Hooft solutions were applied to show that for large $k$ the map

$$H_*(\mathcal{M}_k; \mathbb{A}) \longrightarrow H_*(\Omega^3_k S^3; \mathbb{A})$$

is **surjective**[1] for $* < k - 2$. This fit with the feeling that there should be some sort of Morse theory associated to the Yang-Mills functional and Atiyah and Jones then made the following conjectures:

(1) The inclusion $\mathcal{M}_k \subset \Omega^3_k S^3$ is a homology isomorphism in dimensions $t \le q(k)$ for some increasing function $q(k)$ with $\lim_{k\to\infty}(q(k)) = \infty$.

[1] This construction was sharpened by Boyer and Mann, [BM]. They showed that the $\mathcal{M}_k$ admit additional structures which allowed them to construct further classes in the image.

(2) The range of the surjection (isomorphism) $q = q(k)$ can be explicitly determined as a function of $k$.
(3) The homology statements can be replaced by homotopy statements in both conjectures 1 and 2.

The last and strongest statement became commonly known as the Atiyah-Jones conjecture. While it is easy to construct maps $j_k : \Omega^3_k SU(2) \xrightarrow{\simeq} \Omega^3_{k+1} SU(2)$, there is no obvious map on the $\mathcal{M}_k$ level that raises the instanton number $k$. But it follows from analytic results of Taubes [T1] that there are compatible inclusions $\iota_k : \mathcal{M}_k \longrightarrow \mathcal{M}_{k+1}$ so that the following diagram homotopy commutes

$$\begin{array}{ccc} \mathcal{M}_k & \xrightarrow{\iota_k} & \mathcal{M}_{k+1} \\ \Big\downarrow{\scriptstyle \vartheta_k} & & \Big\downarrow{\scriptstyle \vartheta_{k+1}} \\ \Omega^3_k SU(2) & \xrightarrow{\simeq} & \Omega^3_{k+1} SU(2). \end{array} \tag{(4.1)}$$

This diagram permits direct limits and hence a stable version of the Atiyah-Jones conjecture. This was verified in [T2]

**Theorem 4.2.** *Let $\mathcal{M}_\infty$ be the homotopy direct limit of the $\mathcal{M}_k$'s under the inclusions $\iota_k$ and let $\vartheta : \mathcal{M}_\infty \longrightarrow \Omega^3_0 SU(2)$ be the direct limit of the inclusions $\mathcal{M}_k \subset \Omega^3_k S^3$. Then $\vartheta$ is a homotopy equivalence.*

More recently the original conjectures have been verified, [BHMM]. In the remainder of this section we discuss the proof and the resulting picture it gives for the geometry of $\mathcal{M}_k$.

A major difficulty with proving the Atiyah-Jones conjectures was the intractibility of the "reality" condition, condition (3), above in the definition of the ADHM-instantons. Attempts were made to live with it, for example in the work of Hartshorne, [Ha], and attempts were made to remove it, [BM3], but each ran into trouble at about $k = 2$ or 3, though Hartshorne was able to characterize $\mathcal{M}_2$.

An alternative approach was initiated by Donaldson [D2, A]. He was able to show that the spaces $\mathcal{M}_k$ could be identified with the moduli spaces of (stable) holomorphic $\mathbb{C}^2$-bundles over $\mathbb{CP}^2$ with a trivialization on a fixed line $L = \mathbb{CP}^1 \subset \mathbb{CP}^2$ and second Chern class $k$. Such bundles had been studied by Barth and classified in terms of "monads", but, as it stood, that description was still too difficult for an effective topological analysis.

A different description of these bundles was then given by Hurtubise, [H]. His idea was to study the local structure of such bundles and then glue together the local data to get the global picture. He proceeded as follows. First, he replaced $\mathbb{CP}^2$ by a $\mathbb{CP}^2$ with a point, $x_\infty$, on $L$ blown up, a space we will denote $\mathcal{E}$. This is a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ and the fibers can be identified with the individual lines in the pencil of lines through the point $x_\infty$.

In particular, lifting the original bundle $\mathbb{C}^2 \hookrightarrow E \longrightarrow \mathbb{CP}^2$ to $\mathcal{E}$ gives a bundle trivialized over $\mathbb{CP}^1 \vee \mathbb{CP}^1$, the first $\mathbb{CP}^1$ being the inverse image of $x_\infty$ and the second being the closure of $L$ in $\mathcal{E}$. The advantage of this replacement is that the fibers decompose $\mathcal{E}$ and we can study the original bundle $E$ by restricting to the fiber $\mathbb{CP}^1$'s.

For $\mathbb{CP}^1$ the classification of holomorphic vector bundles is completely understood. Such bundles have no deformation space, and the classification is given by the Birkhoff-Grothendieck theorem [PS].

**Theorem 4.3.** *Let $\mathbb{C}^t \longrightarrow E \longrightarrow \mathbb{CP}^1$ be a holomorphic vector bundle. Then $E$ is holomorphically isomorphic to a direct sum of line bundles $\xi_1^{\alpha_1} \oplus \cdots \oplus \xi_t^{\alpha_t}$, $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_t$, where $\xi^m$ is the $|m|$-fold tensor product of the Hopf bundle for $m \geq 0$ and of the conjugate Hopf bundle $\xi^{-1}$ for $m < 0$. Also, $\alpha_1, \ldots, \alpha_t$ are uniquely determined by $E$.*

In our case, $\mathbb{C}^2 \longrightarrow E \longrightarrow \mathcal{E}$, the restriction of $E$ to a fiber, $\mathbb{CP}^1_x$, must have the form

$$\xi^\alpha \oplus \xi^{-\alpha}$$

since $c_1 = 0$. Moreover, the function $\mathbb{CP}^1_x \mapsto |\alpha|$ is upper semi-continuous as a function from $\mathbb{CP}^1$ to the non-negative integers. Also, at $x = \infty$, (the point corresponding to the closure of $L$ in $\mathcal{E}$), $\alpha = 0$, so $|\alpha| > 0$ at only a finite number of points, $x_1, \ldots, x_r$ in $\mathbb{C} = \mathbb{CP}^1 - \infty$. The associated fibers $\mathbb{CP}^1_{x_1}, \ldots, \mathbb{CP}^1_{x^r}$ are called the jumping lines of the bundle $E$ and are clearly invariants of the isomorphism type of $E$. Indeed, on the compliment of these lines $\mathcal{E} - \bigsqcup_1^r \mathbb{CP}^1_{x_j}$, E is holomorphically trivial with trivialization uniquely determined by the trivializations on $\mathbb{CP}^1_\infty$ and the other $\mathbb{CP}^1$, $\bar{L}$, [H]. Thus, we can view $E$ as obtained from a trivial bundle $\mathbb{C}^2 \times \mathcal{E}$ by gluing in bundles over $D \times \mathbb{CP}^1$ where $D$ is a small disk neighborhood of the origin in $\mathbb{C}$ replacing the original trivial bundle in small neighborhoods of the jumping lines $\mathbb{CP}^1_{x_j}$.

Let $S_{|\alpha_i|}$ be the space of isomorphism classes of local jumps of type $|\alpha_i|$ on $D \times \mathbb{CP}^1$ where $D$ is a small disk. Then we can view the bundle $E$ as a point in the configuration space

$$C(\mathbb{C}, r) \times_W (S_{|\alpha_1|} \times \cdots \times S_{|\alpha_r|})$$

where $W \subset \mathcal{S}_r$ is a the subgroup of all permutations, permuting those $j$ for which the labels lie in the same $S_{|\alpha_j|}$.

The determination of these $S_{|\alpha|}$ is the next step. We can assume that the bundle, $\mathcal{V}$, on $D \times \mathbb{CP}^1$ is trivial on each $z \times \mathbb{CP}^1$ with $z \neq 0$, has an isolated jump of type $|\alpha|$ when $z = 0$ and has a given framing over $D \times 0$. Such a bundle is actually trivial on $(D - \{0\}) \times \mathbb{CP}^1$, moreover it is holomorphically trivial over $D \times \mathbb{CP}^1 - \infty \equiv D \times \mathbb{C}$ with coordinates $(x, z)$, and over $D \times \mathbb{CP}^1 - \{0\}$ with coordinates $(x, \zeta = z^{-1})$. These last two trivializations allow one to write a clutching function for $\mathcal{V}$ which is given by a matrix defined over $D \times \mathbb{CP}^1 - \{0, \infty\}$, and which is shown in [H] to be of the following form

$$\begin{pmatrix} z^{|\alpha|} & p(x, z, z^{-1}) \\ 0 & z^{-|\alpha|} \end{pmatrix} \tag{(4.4)}$$

where $p(x, z, z^{-1})$ is defined up to multiplication by a non-zero constant,

$$\begin{aligned} p(x, z, z^{-1}) &= p_{-|\alpha|+1}(x) z^{-|\alpha|+1} + p_{-|\alpha|+2}(x) z^{-|\alpha|+2} + \cdots + p_0(x) \\ &\quad + p_1(x) z + \cdots + p_{|\alpha|-1}(x) z^{|\alpha|-1}, \end{aligned}$$

with the $p_j(x)$ analytic in $x$ at $x = 0$. Moreover, $p_j(0) = 0$ for each $j$. The condition that the jump be isolated is described in terms of the projective equivalence class of the Toeplitz matrix associated to $p(x, z, z^{-1})$,

$$((4.5)) \qquad T(p(x,z,z^{-1})) = \begin{pmatrix} p_0(x) & p_1(x) & \dots & p_{|\alpha|-2}(x) & p_{|\alpha|-1}(x) \\ p_{-1}(x) & p_0(x) & \dots & p_{|\alpha|-3}(x) & p_{|\alpha|-2}(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{2-|\alpha|}(x) & p_{3-|\alpha|}(x) & \dots & p_0(x) & p_1(x) \\ p_{1-|\alpha|}(x) & p_{2-|\alpha|}(x) & \dots & p_{-1}(x) & p_0(x) \end{pmatrix}.$$

Precisely, the jump is isolated of type $|\alpha|$ if and only if the determinant

$$((4.6)) \qquad Det\left(T(p(x,z,z^{-1}))\right) = x^{|\alpha|+s}\phi(x)$$

where $\phi(x) \neq 0$ at $x = 0$.

*Remark.*. Since $p(x, z, z^{-1})$ is only defined up to projective equivalence so is the Toeplitz matrix, but the power of $x$, $|\alpha| + s$, appearing in (4.6) is independent of this variation. In fact, it gives the contribution of the local jump to the second Chern class of $E$. Exactly, $k = \sum |\alpha|_j + s_j$, the sum being taken over the jumping lines.

The projective equivalence class of the polynomial $p(x, z, z^{-1})$ is very nearly an invariant of the holomorphic isomorphism type of the local jump, but it does happen that distinct polynomials give isomorphic jumps and further progress on the structure of the spaces $\mathcal{M}_k$ depends on understanding when and how this happens.

This was accomplished in [BHMM] by replacing the analytic functions $p_j(x)$ by formal power series and truncating at the $i^{th}$ power, $i = 1, 2, 3, \dots$. Thus one first solves the equivalence question when $p_j(x) = a_j x$ and the resultinig Toeplitz matrix of constants

$$T(p(x,z,z^{-1})) = \begin{pmatrix} a_0 & a_1 & \dots & a_{|\alpha|-1} \\ a_{-1} & a_0 & \dots & a_{|\alpha|-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1-|\alpha|} & a_{2-|\alpha|} & \dots & a_0 \end{pmatrix}$$

is non-singular. In this case [H] shows that $T(p(x, z, z^{-1}))$ is a complete isomorphism invariant for the jump. In the general situation the constant Toeplitz matrix can be singular, but as we add in higher and higher powers of $x$ the kernel keeps shrinking until, after we have added the terms in $x^i$ it becomes non-singular. If $i > 1$ then we must continue the expansion until about $2i$ before we have entirely determined the local jump, but generally the important topological invariants appear by $i$, and from $i + 1$ to $2i$ the new invariants just product the previous invariants with copies of $\mathbb{C}$.

**Example..** If $|\alpha| = 2$ and $p(x, z, z^{-1}) = x(z^{-1} + x + (x + x^2)z)$ then

$$T(p(x, z, z^{-1})) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x^2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x^3$$

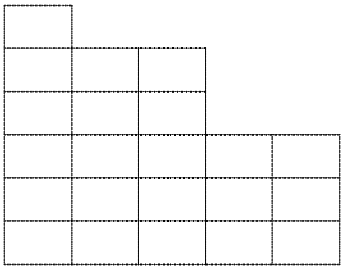
so the first determinant is singular, but the first two together give determinant $x^4 - x^3$ while the final value for the determinant is $-x^3$.

**Definition..** The projective equivalence classes of non-zero $k \times k$-Toeplitz matrices are in 1-1 correpondence with the points of $\mathbb{CP}^{2k-2}$, and the variety $T_{k,s} \subset \mathbb{CP}^{2k-2}$ is the subset consisting of those Toeplitz matrices $t$ with $rk(t) \leq k - s$. Thus, the space of $k \times k$-Toeplitz matrices of rank exactly $k - s$ is

$$\mathcal{T}_{k,s} \;=\; T_{k,s} - T_{k,s+1}.$$

We begin our description of the space of local jumps by restricting consideration to jumps of type $|\alpha|$ with contribution $|\alpha| + s$ to the total $c_2$. In particular, let $q = |\alpha| + s$ and write the space of all local jumps $1 \leq |\alpha| \leq q$, as $Q_q$. Then the ranks of the various truncations of $p(x, z, z^{-1})$ give a further filtration of these jumps. Indeed, they lead to a stratification of these jumps according to a certain graph. For example,

Graph type $J_{(5,5,5,3,3,1)}$, height 6, length 5



((4.7))

Sample Toeplitz expansion: $t_1 x^3 + t_2 x^5 + t_3 x^6 + \cdots$

Then the area under this graph, in this case 22, gives the multiplicity of the graph $G$ which is the contribution, $q$, to $c_2$ and $|\alpha|$ is the length of the base, in this case 5. We thus obtain a decomposition $Q_q = \coprod_G FJ_G$ where each $FJ_G$ is the space of framed jumps of a fixed graph type $G$, and the union is taken over all graphs $G$ of multiplicity $q$. A key result in [BHMM] is now

**Theorem 4.8.** *Let $G$ be a graph with lengths $l = l_0 \geq l_1 \geq \cdots \geq l_{h-1} > l_h = 0$, height $h$, and multiplicity $q = \sum h_i$. The space of framed jumps $FJ_G$ with graph $G$ is a smooth complex manifold of complex dimension $2q + h$.*

**Definition..** The number of ledges in the graph $G$, $L(G)$, is the number of changes in length in $G$. $L(G) \geq 1$ and $L(G) = 1$ if and only if all the lengths $l_j$, $j > 0$ are equal, i.e., if and only if the graph diagram (4.7) is a rectangle.

Let $S_{G_1,\dots,G_r}$ denote the set of equivalence classes of framed holomorphic bundles with $r$ jumping lines $(L_1, \cdots, L_r)$ with graphs $\mathcal{G}(r) = (G_1, \dots, G_r)$, respectively. Thus, we have the disjoint union decomposition $\mathcal{M}_k = \bigsqcup S_{\mathcal{G}}$. Furthermore, the map $\Pi$ now restricts to a map $\Pi_{\mathcal{G}(r)} : S_{\mathcal{G}(r)} \longrightarrow \mathbb{DP}^r(\mathbb{C})$ with fibre $FJ_{G_1} \times \cdots \times FJ_{G_r}$. Here $\mathbb{DP}^r(\mathbb{C})$ is the deleted product; that is, the space of $r$ distinct unordered points in $\mathbb{C}$ which is a smooth complex manifold of complex dimension $r$, then

**Theorem 4.9.** *$S_{\mathcal{G}(r)}$ is a smooth complex variety of complex dimension $2k + l + r$.*

It is next proved in [BHMM] that this decomposition of $\mathcal{M}_k$ by the $S_{\mathcal{G}}$ submanifolds is a stratification by complex submanifolds with complex normal bundles.

At this point in the analysis, the codimension of the strata are known from the result above, and the graph types give a natural lexicographical order on the strata which fits well with their closure properties: a stratum is in the closure of a more generic stratum if it is obtained by "moving things to the right". Thus the open stratum corresponds to the graph of type $(1, 1, \dots, 1)$, and the last stratum, in the closure of all the previous ones in $Q_q$ has graph type $(q)$.

As regards the topology of the strata, the result is

**Theorem 4.10.** *Let $F_G$ have graph type*

$$(\underbrace{i_1, \dots, i_1}, \underbrace{i_2, \dots, i_2}, \cdots, \underbrace{i_j \dots, i_j}),$$

*i.e., $G$ has exactly $j$ ledges, then the stratum $F_G$ has the homotopy type of a product of $SO(3)$ and an iterate fibration*

$$\begin{array}{ccccccccc}
E_j = F_G & \longrightarrow & E_{j-1} & \longrightarrow \cdots \longrightarrow & E_3 & \longrightarrow & E_2 \\
\Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow {=} \\
\tilde{\mathcal{T}}_{i_j,0} & & \tilde{\mathcal{T}}_{i_{j-1},i_j} & & \tilde{\mathcal{T}}_{i_2,i_3} & & \mathcal{T}_{i_1,i_2}
\end{array}$$

*where $\tilde{\mathcal{T}}_{l,m}$ denotes the space of actual $l \times l$ Toeplitz matrices of rank $l - m$ and not its projectivization. In particular the generic stratum, $G = (1, 1, \dots, 1)$, has $F_G \simeq SO(3)$, while the minimal stratum, $G = (q)$, has the homotopy type of the product $SO(3) \times \mathcal{T}_{q,0}$.*

At this stage the key step was to observe that the Taubes stabilization procedure becomes simply the addition of a generic jumping line of type 1 at some $x_{r+1}$ very close to $\infty$. This generic jump is given by the clutching matrix

$$\begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix}$$

and its associated $Q_1 \cong SO(3) \times \mathbb{C} \times \mathbb{R}$. Thus, at the level of strata the stabilization becomes

$$C(\mathbb{C}, k) \times_V (Q_{i_1} \times \cdots \times Q_{i_k}) \mapsto C(\mathbb{C}, k+1) \times_{V'} (Q_{i_1} \times \times Q_{i_k} \times Q_1)$$

and the codimensions of these two strata in their respective $\mathcal{M}_s$'s are equal.

But this operation also induces isomorphisms in homology on the separate strata through a range of dimensions which increases with the number of $Q_1$ that appear, i.e., as we iterate the stabilization map. We have, in fact that stabilization occurs through dimension at least $[(N(Q_1) - 1)/2]$ where $N(Q_1)$ is the number of $Q_1$'s appearing.

*Remark.*. The smallest codimension for a stratum at level $k$ which is not in the image of stabilization from $k-1$ is $k$ if $k$ is even and $k+1$ if $k$ is odd. The stratum which achieves it has fiber $F_{(2)} \times \cdots F_{(2)}$ for $k$ even and $F_{(3)} \times F_{(2)} \times \cdots \times F_{(2)}$ if $k$ is odd.

These two facts together can now be applied to the spectral sequences of the stratifications of the $\mathcal{M}_k$'s. For this spetral sequence the $E^1$-term is the direct sum of the homologies of the *Thom complexes of the normal bundles* to the strata. Since the strata have complex, and consequently, oriented, normal bundles each of these Thom complexes has homology equal to that of the stratum just shifted up by the (real) codimension of the stratum in $\mathcal{M}_k$. Then the stabilization for the individual strata together with the increasing codimension of new strata shows that the stabilization map induces a map of spectral sequences which is an isomorphism on $E^1$'s through at least dimension $[(k-1)/2]$. But then standard comparison theorems imply that the map of spaces induces isomorphism in homology through dimension at least $[(k-1)/2] - 1$. From this it is a short step to the complete proof of the Atiyah-Jones conjectures.

## §5. Toeplitz Varieties and the Rational Cohomology of $\mathcal{M}_k$

The building blocks of the strata, the spaces $\mathcal{T}_{m,n}$, have been studied in [Mil2]. We begin by summarizing the main results there. The most complete of these describe the spaces of projective equivalence classes of non-singular Toeplitz matrices.

**Theorem 5.1.** *The space of all projective equivalence classes of non-singular $n \times n$-Toeplitz matrices, $\mathcal{T}_{n,0}$, is homeomorphic to the orbit space under the action of $GL_2(\mathbb{C})$ on the space of pairs $(p_1(z), p_2(z))$ of coprime polynomials with $max(deg(p_1(z)), deg(p_2(z)))$-$= n$ where the action is given as*

$$( \, p_1 \quad p_2 \, ) \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ( \, ap_1 + bp_2 \quad cp_1 + dp_2 \, ).$$

*Remark.*. The set of projective equivalence classes of such coprime pairs,

$$(p_1(z), p_2(z)),$$

with the degree condition can be identified with the space of unbased holomorphic maps $\phi$: $\mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ of degree $n$ which we denote $Rat_n(\mathbb{CP}^1)$, and the quotient above is the same as the quotient of the projective equivalence classes of these pairs by the quotient group $PSL_2(\mathbb{C})$ (acting by composition with linear fractional transformations). When we use this action we see that every orbit contains a based holomorphic map $\phi$ with $\phi(\infty) = 1$, and the quotient is thus the quotient of the based maps, $Rat_n^*(\mathbb{CP}^1)$ by the action of the isotropy group of 1, a Borel subgroup, homotopy equivalent to $S^1$, and $\mathcal{T}_{n,0} \simeq Rat_n^*(\mathbb{CP}^1)/S^1$.

The space $Rat_n^*(\mathbb{CP}^1)$ has already been discussed extensively in this survey. In particular we have described the homology groups $H_*(Rat_n^*(\mathbb{CP}^1); A)$ for all (simple) coefficients $A$. Using this information, it is relatively direct to determine all the differentials in the spectral sequence converging to $H^*(Rat_n^*(\mathbb{CP}^1)/S^1; A)$ for $A = \mathbb{Z}/p$, where $p$ is any prime. Consequently, we have almost complete information about $H^*(\mathcal{T}_{n,0}; A)$.

It is first shown that the action in (5.1) of $S^1$ on $Rat_k^*(\mathbb{CP}^1)$ is free, so there is a homotopy fibration

$$Rat_k^*(\mathbb{CP}^1) \longrightarrow Rat_k^*(\mathbb{CP}^1)/S^1 \longrightarrow \mathbb{CP}^\infty$$

and a spectral sequence with $E_2$-term equal to

$$H^*(\mathbb{CP}^\infty; H^*(Rat_k^*(\mathbb{CP}^1); \mathbb{A})) \cong H^*(\mathbb{CP}^\infty; \mathbb{Z}) \otimes H^*(Rat_k^*(\mathbb{CP}^1); \mathbb{A}).$$

The first differential is $d^2(e^1) = kb$ and the other differentials are described in terms of standard structures such as Dyer-Lashof operations.

Next consider the varieties $\mathcal{T}_{n,k}$ with $k > 0$. It turns out that $T_{n,k}$ is a singular variety with singularity set exactly $T_{n,k+1}$.

The process in [Mil2] for studying the homology of $\mathcal{T}_{n,k}$ is to use Poincaré duality and study instead the pairs $(T_{n,k}, T_{n,k+1})$. A spectral sequence is constructed which converges to $H_*(\mathcal{T}_{n,k}; \mathbb{F})$ for any field and the $E^1$-term and $d^1$-differential are completely determined. The higher differentials are not determined in general but the results are sufficient to determine the rational cohomology groups.

**Theorem 5.2.** *The rational cohomology groups of the projective Toeplitz varieties are given as follows,*

$$H^*(\mathcal{T}_{n,k};\mathbb{Q}) \cong H^*(S^2;\mathbb{Q}), \qquad k \geq 1$$

$$H^*(\mathcal{T}_{n,0};\mathbb{Q}) \cong H^*(pt;\mathbb{Q}).$$

*Moreover, for the space $\tilde{\mathcal{T}}_{n,k}$ consisting of all $n \times n$ complex Toeplitz matrices with $dim(kernel(t_n)) = k$ we have*

$$H^*(\tilde{\mathcal{T}}_{n,0};\mathbb{Q}) = H^*(S^1;\mathbb{Q})$$

$$H^*(\tilde{\mathcal{T}}_{n,k};\mathbb{Q}) = H^*(S^3;\mathbb{Q}) \qquad k \geq 1.$$

**Corollary 5.3.** *The rational cohomology groups of the space $F_G$ associated to the graph $G$ is isomorphic to*

$$H^*(SO(3);\mathbb{Q}) \otimes H^*(\underbrace{S^3 \times \cdots \times S^3}_{L(G)-1\ times};\mathbb{Q})$$

*where $L(G)$ is the number of ledges in $G$.*

In turn, this result leads to a complete determination of the rational cohomology of the strata in the stratification of the $\mathcal{M}_k$ referred to earlier. However, the actual techniques of calculating $H^*(C(\mathbb{C},r) \times_V (F_{G_1} \times \cdots \times F_{G_r});\mathbb{Q})$ are quite computationally involved. The key is to construct a certain filtered Lie algebra and pick out those terms having exact filtration degree $r$. This Lie algebra is the rational cohomology ring of the loop space $\Omega^2(W)$ where $W$ is an explicit wedge of spheres, the numbers and dimensions of which depend on the structures of the distinct $G_t$'s in the set $(G_1,\dots,G_r)$, but not on the number of times each $G_t$ is repeated.

The discussion above allows us to completely describe the $E_1$-term of the spectral sequence converging to $H^*(\mathcal{M}_k;\mathbb{Q})$. For example, the sequence converging to $H^*(\mathcal{M}_3;\mathbb{Q})$ has the form

| | Dim. | $(1)(1)(1)$ | $(1,1)(1)$ | $(2)(1)$ | $(1,1,1)$ | $(2,1)$ | $(3)$ |
|---|---|---|---|---|---|---|---|
| | 1 | $\mathbb{Q}$ | | | | | |
| | 2 | | $\mathbb{Q}$ | | | | |
| | 3 | $\mathbb{Q}$ | $\mathbb{Q}$ | | | | |
| | 4 | $\mathbb{Q}^2$ | | $\mathbb{Q}$ | $\mathbb{Q}$ | | |
| | 5 | $\mathbb{Q}$ | $\mathbb{Q}^2$ | $\mathbb{Q}$ | | | |
| ((5.4)) | 6 | | $\mathbb{Q}^2$ | | | $\mathbb{Q}$ | |
| | 7 | $\mathbb{Q}$ | | $\mathbb{Q}^2$ | $\mathbb{Q}$ | | |
| | 8 | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}^2$ | | | $\mathbb{Q}$ |
| | 9 | | $\mathbb{Q}$ | | | $\mathbb{Q}^2$ | |
| | 10 | | | $\mathbb{Q}$ | | | |
| | 11 | | | $\mathbb{Q}$ | | | $\mathbb{Q}$ |
| | 12 | | | | | $\mathbb{Q}$ | |

Taking the structure of the possible differentials into account we see that there is a rational class in $H^*(\mathcal{M}_3;\mathbb{Q})$ for $*$ equal to either 5 or 7.

The action of $SO(3)$ on $\mathcal{M}_k$ is free for each $k$, and the quotient $\bar{\mathcal{M}}_k$ has, as we have mentioned in §3, Euler characteristic $S(k)$ where $S(k)$ is the number of distinct non-negative divisors of $k$. We will give a proof shortly. But for now, consider the Serre spectral sequence of the fibration

$$SO(3) \longrightarrow \mathcal{M}_k \xrightarrow{\pi} \bar{\mathcal{M}}_k.$$

The $d_4$-differential on $[SO(3)]^* = h_4$, and so for $\lambda \in H^*(\bar{\mathcal{M}}_k;\mathbb{Q})$ we have that

$$d_4([SO(3)]^* \cup \lambda) = h_4 \cup \lambda,$$

and, since $\bar{\mathcal{M}}_k$ is finite dimensional, $h_4^t\lambda = 0$ for some finite $t$. If $h_4^{t-1}\lambda \neq 0$ then $[SO(3)]^* h_4^{t-1}\lambda$ is an infinite cycle, and we see that, when we write $H^*(\bar{\mathcal{M}}_k;\mathbb{Q})$ as a $\mathbb{Q}[h_4]$-module, then the $\mathbb{Q}[h_r]$-indecomposables survive to $H^*(\mathcal{M}_k;\mathbb{Q})$, and for each such indecomposable, there is also a second class as described above.

For example, in the case of $\mathcal{M}_3$ we have $\chi(\bar{\mathcal{M}}_3) = 2$. Also, if there is a 5 dimensional class there must be an 8 dimensional class or a 12 dimensional class. Moreover, there must be a class in dimension 3, 7, or 11 to correspond to the base class in dimension 0. Thus, in this case there are at least 4 generators in $H^*(\mathcal{M}_3;\mathbb{Q})$. However, if there is a 7 class we can draw no further conclusions.

It is clear that there is a considerable amount of information to be obtained once we understand the structure of the differentials in the spectral sequence associated to the stratification. However at this time nothing is known about them, not even partial results.

We now turn to a proof of Furuta's result using the geometry of $\mathcal{M}_k$ described above and in §4.

**Theorem 5.5.** *The Euler characteristic $\chi(\bar{\mathcal{M}}_k)$ is the number of distinct positive divisors of $k$.*

*Proof.*. The action of $SO(3)$ on $\mathcal{M}_k$ preserves the stratification – it is just multiplication on the copy of $SO(3)$ in the stratum. Consequently there is a proper stratification of $\bar{\mathcal{M}}_k$ by strata with complex normal bundles, and each stratum is given as a fibration

$$(F_{G_1} \times \cdots \times F_{G_r})/SO(3) \longrightarrow \bar{\mathcal{S}} \longrightarrow C(\mathbb{C}, k)/V,$$

where the action of $SO(3)$ is via the diagonal action on the $F_{G_t}$'s. If $r \geq 2$ or $r = 1$ but the number of ledges in $G_1$ is greater than one it follows from 5.2 that the Euler characteristic of $(F_{G_1} \times \cdots \times F_{G_r})/SO(3)$ is zero and the Euler characteristic of $\bar{\mathcal{S}}$ is zero as well. But then the associated term in the sectral sequence of the fibration – the rational homology of the Thom space of the normal bundle to the stratum – also has Euler characteristic zero. It remains to consider the case where $r = 1$ and there is exactly 1 ledge. In this case the graph has the form of a rectangle with area $k$, so these are indexed by the positive divisors of $k$. But in this case rationally each fiber has the homology of a point, and since the base is $\mathbb{C}$, $\bar{\mathcal{S}}$ also has the homology of a point so the contribution to the Euler class of the $E_1$-term in the spectral sequence of stratification from this stratum is 1, and the result follows.

## §6. The geometry of higher monopoles and related questions

In two papers, [MM1, MM2], Mann and Milgram extended and applied the techniques of [CCMM] to study further moduli spaces of holomorphic maps $\mathbb{CP}^1 \longrightarrow G/H$ for various symmetric spaces when $G = GL_n(\mathbb{C})$.

Work of [H, HM], suggests that for $G/P$ where $G$ is a complex Lie group and $P$ is a parabolic subgroup, the space of based holomorphic maps $Rat^*_{\bar{k}}(G/P)$ represents $G$-monopoles with specified symmetry breaking at $\infty$.

*Remark.* Actually they construct a one to one correspondence between $Rat^*_{\bar{k}}(G/B)$ and monopoles on $G/B$ with appropriate symmetry breaking at $\infty$. Here $B$ is a Borel subgroup of $G$. They do not show that the correspondence is a homeomorphism, but C. Boyer informs us that techniques of [BM2] should prove that the [HM] correspondence is actually a diffeomorphism.

Additionally, work in linear control theory relates the moduli spaces of linear control systems with $n$ inputs and $m$ outputs to $\coprod_d Rat^*_d(\mathbb{G}_{m,n})$ where $\mathbb{G}_{m,n}$ is the Grassmannian of complex $m$-planes in $\mathbb{C}^{n+m}$.

It is instructive to review this connection. A linear control system is an equivalence class of linear differential systems

$$\begin{aligned} \dot{X} &= AX + BU \\ Y &= CX + LU, \end{aligned} \tag{(6.1)}$$

where $A$ is a $k \times k$ matrix with complex entries, $B$ is $k \times n$, $C$ is $m \times k$ and $L$ is $m \times n$. The column vector $U(t)$ is called the input of the system and $Y(t)$ is the output. Then two systems are equivalent if for all $U$ the resulting $Y$'s are equal. In particular, for $g \in GL_k(\mathbb{C})$, if we make the replacements

$$\begin{aligned} a &\mapsto gAg^{-1} \\ B &\mapsto gB \\ C &\mapsto Cg^{-1} \\ L &\mapsto L, \end{aligned}$$

then the resulting systems are equivalent. Similarly if

$$\begin{aligned} B' &= \begin{pmatrix} B \\ 0 \end{pmatrix} \\ A' &= \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \\ C' &= \begin{pmatrix} C & 0 \end{pmatrix}, \end{aligned}$$

the resulting systems are equivalent.

The Laplace transform of (6.1) is

$$W(s) \;=\; C(I - sA)^{-1}B + L,$$

and this is an invariant of the equivalence class. $W(s)$ is an $m \times n$ matrix of rational functions which satisfies the constraint

$$\lim_{s\to\infty}(W(w)) \;=\; L.$$

Actually, there is no loss in generality if we assume $L = 0$, which we will do from here on. Next, using a combination of rational and integral reduction theory one shows that there are **coprime** polynomial matrices $D$, $N$, where $D$ is $m \times m$ and $N$ is $m \times n$ so that $D^{-1}N = W(s)$. In particular $D$ is invertible. That and the coprime condition show that $(D, N)$ is determined up to left multiplication by $U \in GL_m(\mathbb{C}[s])$, $(D, N) \mapsto (UD, UN)$.

Consequently, since $Det(U) \in \mathbb{C}[s]^* = \mathbb{C}^*$, it follows that $Det(D)$ is a polynomial in $\mathbb{C}[s]$, well defined up to multiplication by a non-zero complex number, and there is a unique monic polynomial associated to the equivalence class of the linear control system. The degree of this polynomial is called the McMillen degree of the system (6.1) in the control theory literature. In fact, it is the degree of the map $\mathbb{CP}^1 \longrightarrow \mathbb{G}_{m,n}$ which corresponds to the equivalence class of (6.1).

Indeed, the map $\mathbb{CP}^1 \longrightarrow \mathbb{G}_{m,n}$ is given by

$$s \mapsto \{(D(s), N(s))\} \subset \mathbb{G}_{m,n},$$

which is an $m$-plane in $\mathbb{C}^{m+n}$ for each $s$ because $D$ and $N$ are coprime. Conversely, given any based ($\infty \mapsto \{(I, 0)\}$), holomorphic map $\mathbb{CP}^1 \longrightarrow \mathbb{G}_{m,n}$, it can be represented by a pair of coprime polynomial matrices $(D, N)$ and the basing condition becomes $\lim_{s\mapsto\infty} D^{-1}(s)N(s) = 0$.

After multiplying $D$ by $U \in GL_n(\mathbb{C}[s])$ we can assume that $D$ has the form

$$D \;=\; \begin{pmatrix} f_{11}(s) & f_{12}(s) & \dots & f_{1,m}(s) \\ 0 & f_{22}(s) & \dots & f_{2,m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_{mm}(s) \end{pmatrix}$$

with each $f_{ii}(s)$ monic and $deg(f_{i,k}(s)) < deg(f_{kk}(s)$ for $i < k$. Now, if we fix the degree, $d$, of $D$, then $\sum deg(f_{ii}(s)) = d$ and if we further fix degrees of the $f_{ii}$ themselves we obtain a finite partition of $Rat_d^*(\mathbb{G}_{m,n})$ into strata.

**Example.** Suppose $m = 2$, $n = 3$, and $d = 2$, then there are three patterns

$$\begin{aligned} R_I \;&=\; \left\{\begin{pmatrix} 1 & a_2s + b_2 \\ 0 & s^2 + bs + c \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ w_1s + w_2 & w_3s + w_4 & w_5s + w_6 \end{pmatrix}\right\} \\ R_{II} \;&=\; \left\{\begin{pmatrix} s + c & d \\ 0 & s + e \end{pmatrix}, \begin{pmatrix} a_2 & a_3 & a_4 \\ w_2 & w_3 & w_4 \end{pmatrix}\right\} \\ R_{III} \;&=\; \left\{\begin{pmatrix} s^2 + bs + c & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a_1s + a_2 & a_3s + a_4 & a_5s + a_6 \\ 0 & 0 & 0 \end{pmatrix}\right\} \end{aligned} \tag{(6.2)}$$

Here $R_I$ has complex dimension 10, $R_{II}$ has complex dimension 9 and $R_{III}$ has complex dimension 8. This gives a stratification of $Rat_2(\mathbb{G}_{2,3})$ consisting of three strata. Of course the constants appearing in (6.2) have some constraints because of the coprime condition. These constraints make $R_I$ a copy of $Rat_2^*(\mathbb{CP}^3)\times\mathbb{C}^2$ and $R_{III}$ a copy of $Rat_2^*(\mathbb{CP}^3)$, while $R_{II}$ is given as a fibration

$$\mathbb{C}\times Rat_1^*(\mathbb{CP}^3)\longrightarrow R_{II}\longrightarrow Rat_1^*(\mathbb{CP}^3).$$

This example illustrates the general situation. The representation $(D, N)$ gives a stratification of $Rat_d^*(\mathbb{G}_{m,n})$ with the strata indexed by ordered partitions of $d$,

$$(d_1, d_2, \dots, d_m), \qquad \sum d_i = d, \qquad d_i \geq 0,\ 1\leq i\leq m,$$

and each stratum is an iterate fibration with $\mathbb{C}^{l+i}\times Rat_{d_i}^*(\mathbb{CP}^n)$ fibers.

A direct counting argument, based on the results of [CCMM] and a theorem of Kirwan [K], (see also [MM3]), then gives the main result of [MM1].

**Theorem 6.3.** *Let $2\leq m<n$, then the forgetful map*

$$e_*\colon H_*(Rat_d^*(\mathbb{G}_{m,n});\mathbb{A})\longrightarrow H_*(\Omega_d^2(\mathbb{G}_{m,n});\mathbb{A})$$

*is injective for all (simple) coefficients $\mathbb{A}$ and all $d>0$. Moreover,*
*(1) The image of $e_*$ is known.*
*(2) $e_*$ is an isomorphism in dimensions $\leq 2d+1$.*

The critical case was $Rat_1^*(\mathbb{G}_{m,n})$, which, similarly to the case in [CCMM], appears to control what happens for higher $d$. But in [MM3] it is proved that $Rat_1^*(\mathbb{G}_{m,n})$ is $\mathbb{C}$ crossed with the space of non-zero vectors in the vector bundle $n\xi^{-1}$ over the complex projective space $\mathbb{CP}^{m-1}$, i.e., the compliment of the 0-section in the Whitney sum of $n$ copies of the adjoint to the Hopf bundle over $\mathbb{CP}^{m-1}$.

In [MM2] the spaces of based holomorphic maps $\mathbb{CP}^1\longrightarrow G/P$ were studied for $G = GL_n(\mathbb{C})$ and $P$ any parabolic subgroup. Once more a stratification was obtained, but this time each stratum is an iterate fibration of strata occuring for $\mathbb{G}_{m,n}$. Using this and various spectral sequence comparison theorems, 6.3 gave a more general form of the stability theorem of [CCMM].

**Theorem 6.4.** *Let $P\subset GL_n(\mathbb{C})$ be any parabolic subgroup. The path components of both $Rat(GL_n(\mathbb{C})/P)$ and $\Omega^2(GL_n(\mathbb{C})/P)$ are indexed by sequences of integers $C = (c_1,\dots,c_n)$ (in the case of $Rat(GL_n(\mathbb{C})/P)$ all $c_i\geq 0$) and the forgetful map*

$$\iota(P;C) : Rat_C(GL_n(\mathbb{C})/P)\longrightarrow\Omega_C^2(GL_n(\mathbb{C})/P)$$

*induces a homotopy equivalence through a range of dimensions that is an increasing function of the minimum of the $c_i$'s.*

Once more the critical case is $Rat^*_{1,1,\ldots,1}(GL_n(\mathbb{C})/P)$, and again in [MM2], these manifolds are explicitly determined. As for the Grassmannians it turns out that

$$H_*(Rat^*_{1,1,\ldots,1}(GL_n(\mathbb{C})/P);\mathbb{Z})$$

is torsion free. But here something new happens. For the first time there is a non-trivial kernel in the homology map

$$H_*(Rat^*_{1,1,\ldots,1}(GL_n(\mathbb{C})/P);\mathbb{Z})\longrightarrow H_*(\Omega^2_{1,1,\ldots,1}(GL_n(\mathbb{C})/P);\mathbb{Z}).$$

What the significance of these kernel classes might be is currently unknown.

*Remark.* Recently, [BHMM2], different techniques have been used to show that for any complex simple Lie group $G$ and parabolic subgroup $P$ we have that the inclusion

$$Rat^*_{\bar{k}}(G/P)\longrightarrow\Omega^2_{\bar{k}}(G/P)$$

becomes more and more highly connected as $min(k_1,\ldots,k_r)\mapsto\infty$.

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Dept. of Mathematics, Stanford University, Stanford, California 94305
*E-mail address*: ralph@gauss.stanford.edu

Dept. of Applied Homotopy Theory, Stanford University, Stanford, Calif. 94305
*Current address*: Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M.
*E-mail address*: milgram@gauss.stanford.edu