ON MODULI SPACES AND CW STRUCTURES ARISING FROM MORSE THEORY ON HILBERT MANIFOLDS

LIZHEN QIN

Abstract. This paper proves some results on negative gradient dynamics of Morse functions on Hilbert manifolds. It contains the compactness of flow lines, manifold structures of certain compactified moduli spaces, orientation formulas, and CW structures of the underlying manifolds.

1. Introduction

Invented in the 1920s (see [24] and [25]), Morse theory has been a crucial tool in the study of smooth manifolds. In the past two decades, largely due to the influence of A. Floer, there has been a resurgence in activity in Morse theory in its geometrical and dynamical aspects, especially in infinite dimensional situations. An explosion of new ideas produced many "oral theorems" which were apparently widely acknowledged, highly anticipated or even frequently used. Unfortunately, the literature has not kept pace with the oral tradition. Some previously asserted results are still stated without proof and, having asked various experts in the field, the author could not ascertain what is sufficiently proved or what is even regarded as true. The purpose of this paper is to give a self-contained and detailed treatment proving some of these claims.

In the simplest instance, suppose one is given a Morse function on a finite dimensional closed smooth manifold. By choosing a Riemannian metric, one obtains a negative gradient flow. This determines a stratification in which two points lie in the same stratum if they lie on the same unstable manifold. Now each such unstable manifold (or descending manifold) is homeomorphic to an open cell, and it is desirable to know whether this open cell can be compactified in such a way that it becomes the image of a closed cell arising from a CW structure

Key words and phrases. Morse theory, negative gradient dynamics, Hilbert manifold, Condition (C), finite index, Moduli space, compactness, orientation, CW structure, Morse Homology.

on the manifold. This is one of the problems we will be addressing. Another related problem is to consider moduli spaces of flow lines between any pair of critical points. Using piecewise flow lines, one obtains a compactification of these moduli spaces. The question in this case to decide when one obtains a manifold with corner structure from this compactification.

In addition to the finite dimensional case, our results will generalize in two ways. Firstly, all of our results have an infinite dimensional version in which the underlying manifold is a complete Hilbert manifold and the Morse function satisfies Condition (C) and has finite index at each critical point. This situation will be called the *CF case* (see Definition 2.6). Secondly, we will also strengthen some results in the finite dimensional case. For example, we will obtain a certain result about simple homotopy type in Theorem 3.8.

The main results of this paper (see Section 3) consist of nine theorems and one example (Example 3.1). All theorems are considered in CF case. The results on compactness (Theorems 3.1 and 3.2) require no more assumptions. Theorem 3.2 is even true in a more general setting. Other theorems need two additional assumptions, transversality (see Definition 2.7) and the local triviality of the metric (see Definition 2.16). When the compactification of descending manifolds is considered (see Theorems 3.4, 3.7, 3.8 and 3.9 and (2) of Theorem 3.6), the Morse function is furthermore assumed to satisfy a lower bound condition.

The following is a brief description of our main results.

Theorems 3.1 and 3.2 are two results on the compactness. Roughly speaking, compactness means the space of unbroken flow lines can be compactified by adding broken flow lines. When the underlying manifold M is finite dimensional, similar results are well-known, for example, [38, thm. 2.3, p. 798], [8, prop. 3] and [35, prop. 2.35]. For the infinite dimensional Floer case, there are results in [14], [15] and [34]. The referees for this paper also referred the author to [1] and [2] which prove results similar to Theorem 3.1 (see Remark 3.1). Even in the finite dimensional case, some assumptions on M (e.g. compactness, both complete metrics and Condition (c)) are needed in order to prove such results (compare [22, rem., p. 13]).

Some spaces arise naturally from the study of negative gradient dynamics. Let $\mathcal{D}(p)$ and $\mathcal{A}(p)$ be the descending and ascending manifolds of a critical point p respectively. Assuming transversality of the dynamics (see Definition 2.7), let $\mathcal{W}(p,q)$ be the intersection manifold of $\mathcal{D}(p)$ and $\mathcal{A}(q)$, and $\mathcal{M}(p,q)$ be the orbit space of $\mathcal{W}(p,q)$ with respect to the action of the flow (see Definitions 2.4 and 2.8). It's well-known that these manifolds can be compactified in a standard way (see (3.2)).

Theorems 3.3, 3.4 and 3.5 consider the manifold structures of their compactified spaces. Denote the compactified spaces by $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$. A central problem is to equip them with smooth structures in such a way that they are manifolds with corners that are compatible with the given stratifications. The smooth structure of $\overline{\mathcal{M}(p,q)}$ is useful for some geometric constructions in Morse Theory. For example, the papers [11], [12] and [16] use the moduli spaces to recover the topology of the underlying manifold. The smooth structure of $\overline{\mathcal{D}(p)}$ is useful for Witten Deformations, for example, see [20] and [8]. The papers [9] and [21] use the "smooth structure" of $e(\overline{\mathcal{D}(p)})$, where e is the evaluation map defined in (3) of Theorem 3.4. The smooth structure of $\overline{\mathcal{W}(p,q)}$ is useful for computing the cup product of $H^*(M;R)$ via Morse Theory (see [3, sec. 2.4] and [40]). To the best of my knowledge, when M is finite dimensional, and the metric is locally trivial (see Definition 2.16), the cases of $\overline{\mathcal{M}(p,q)}$ and $\overline{\mathcal{D}(p)}$ are solved by [20] and [8]. (Actually, these two papers consider closed 1-forms which are more general than Morse functions.) The paper [8] gives a quick and nice proof. However, this problem still remains open in the general case, in particular, when the metric is nontrivial near the critical points. This problem is closely related to the associative gluing of broken flow lines which is also a well-known open problem. In addition, few papers in the literature study $\overline{\mathcal{W}(p,q)}$.

In this paper, we extend the proof in [8] to the infinite dimensional CF case. This also includes the case of $\overline{\mathcal{W}(p,q)}$. Our proofs of Theorems 3.3 and 3.4 largely follow [8]. Subsection 5.2 presents a detailed remark on the literature, in particular, the relations between this paper and [8].

Example 3.1 is another contribution of this paper to the above problem. It shows that even if the answer to the above problem is positive for a general metric, there are still some remarkable differences from the locally trivial metric case even if the underlying manifold is compact.

Theorem 3.6 is a result on orientations. Since the descending manifolds $\mathcal{D}(p)$ are finite dimensional, we can assign orientations to them arbitrarily. This determines naturally the orientations of $\mathcal{M}(p,q)$, $\mathcal{W}(p,q)$ and the compactified manifolds $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$. The 1-strata (see Definition 2.15) of these compactified manifolds have two types of orientations, boundary orientations and product orientations (see Subsection 6.1 for details). Theorem 3.6 shows the relation between these two. Some results on the finite dimensional case can be found, for example, in [3] and [20] (see Remarks 3.4 and 6.1). These orientation formulas have some applications. As pointed out in [3, prop.

2.8], the formula for $\overline{\mathcal{M}(p,q)}$ ((1) of Theorem 3.6) gives an immediate proof of $\partial^2 = 0$ for the Thom-Smale complex in Morse homology. The formula for $\overline{\mathcal{D}(p)}$ ((2) of Theorem 3.6) tells us how to apply Stokes' theorem correctly when a differential form is integrated on $\overline{\mathcal{D}(p)}$ (compare [21, prop. 6]). In this paper, it together with Theorem 3.7 also gives a straightforward proof of Theorem 3.9. As mentioned above, the papers [3] and [40] compute the cup product of $H^*(M;R)$ via Morse Theory. Both [3, (2.2)] and [40, lem. 2 and 3] neglect signs. If we do care about the signs in their formulas, the formula for $\overline{\mathcal{W}(p,q)}$ ((3) of Theorem 3.6) can tell us the answer (see Remark 6.2).

The proof of Theorem 3.6 is based on subtle computations. If the underlying manifold M is finite dimensional, then it is locally orientable, and the proof follows easily from the geometric constructions in [8] although the details are possibly lengthy. However, the issue is more complicated in the infinite dimensional case since there is no way to give M a local orientation.

Finally, we consider the problem of constructing a CW structure from the descending manifolds of the Morse function. Suppose a Morse function f on M is lower bounded. It's well known that the descending manifolds $\mathcal{D}(p)$ for the critical points p are disjoint. Let $K^a = \bigsqcup_{f(p)\leq a} \mathcal{D}(p)$. A natural question is whether or not K^a is a CW complex with open cells $\mathcal{D}(p)$. This has been considered by Thom ([39]), Bott ([5, p. 104]) and Smale ([37, p. 197]). If the answer is positive, then Morse theory will give a compact manifold a bona fide CW decomposition which is stronger than the homotopical CW approximation in [22, thm. 3.5]. In order to prove this, we have to construct a characteristic map $e : D \longrightarrow M$ such that e maps the interior D° homeomorphically onto $\mathcal{D}(p)$ for each p, where D is a closed disk. This has been solved by [19, thm. 1] and [21, rem. 3] when M is finite dimensional and the metric is locally trivial. In this paper, these results will be further improved as follows.

Actually, the papers [19] and [21] show that there exists such a characteristic map. Theorem 3.4 shows that, even in the infinite dimensional CF case, $\mathcal{D}(p)$ can be compactified to be $\overline{\mathcal{D}(p)}$ and there is the map $e : \overline{\mathcal{D}(p)} \longrightarrow M$ which is explicitly constructed. If $\overline{\mathcal{D}(p)}$ is homeomorphic to a closed disk (this is Theorem 3.7), then K^a is a CW complex, and what's more, the characteristic maps $e : \overline{\mathcal{D}(p)} \longrightarrow M$ are explicit. In order to get an elementary proof of Theorem 3.7, I asked Prof. John Milnor for help. (Actually, there is a quick but nonelementary proof based on the Poincaré Conjecture in all dimensions, see Remark 7.1.) I had not known the existence of characteristic maps had been proved by [19] and [21] at that time. Prof. Milnor helped me greatly. First, he referred me to [19]. Second, he suggested that we may add a vector field to $-\nabla f$ on $\mathcal{D}(p)$ to control the limit behavior of $-\nabla f$. Motivated by his suggestion and [19], I found the desired proof. In particular, the key Lemma 7.8 fulfills his suggestion.

In addition, Theorem 3.7 and Lemma 7.8 help us prove more results. Let $M^a = f^{-1}((-\infty, a])$, the paper [18, cor., p. 543] (see also [19, sec. 4.5]) shows that K^a is a strong deformation retract of M^a when f is lower bounded and proper and a is regular. Theorem 3.8 shows that, in this case, M^a even has a CW decomposition such that K^a expands to M^a by elementary expansions. The last theorem, Theorem 3.9, computes the boundary operator of the CW chain complex associated with K^a. This relates Morse homology to a cellular chain complex (see Remark 3.5). The proofs of Theorems 3.8 and 3.9 reflect the advantage of Theorem 3.7 and Lemma 7.8.

The outline of this paper is as follows. Section 2 gives some definitions, notation and elementary results mostly used in this paper. Section 3 formulates our main results. The subsequent sections are the proofs of the main results.

2. Preliminaries

In this paper, we assume M is a Hilbert manifold with a *complete* Riemannian metric. The completeness of the metric is necessary for Theorem 2.2 (compare [22, rem., p. 13]). Let f be a Morse function on M. Denote the index of a critical point p by $\mathrm{ind}(p)$. Denote $f^{-1}([a, b])$ by $M^{a,b}$. Denote $f^{-1}((-\infty, a])$ by M^a.

We need the well-known Condition (C) or Palais-Smale Condition (see [28]).

Condition (C): *If S is a subset of M on which f is bounded but on which $\|\nabla f\|$ is not bounded away from 0, then there is a critical point of f in the closure of S.*

Assuming this condition, its easy to prove the following results. Good references are [28, thm. 1 and 2], [26] and [29, sec. 9.1].

Theorem 2.1. *If (M, f) satisfies Condition (C), then for all a, b such that $-\infty < a < b < +\infty$, $M^{a,b}$ contains only finite many critical points.*

We cite [26, thm. (3), p. 333] as follows.

Theorem 2.2. *If (M, f) satisfies Condition (C), $x \in M$, and $\phi_t(x)$ is the maximal flow of $-\nabla f$ with initial value x, then $\phi_t(x)$ satisfies one of the following two conditions:*

(1) $f(\phi_t(x))$ has no lower (upper) bound; or

(2) $f(\phi_t(x))$ has a lower (upper) bound, $\phi_t(x)$ can be defined as a function of t on $[0, +\infty)$ ($(-\infty, 0]$), $\lim_{t\to+\infty} \phi_t(x)$ ($\lim_{t\to-\infty} \phi_t(x)$) exists and is a critical point of f.

By Theorem 2.2, we get an immediate corollary.

Corollary 2.3. *Suppose (M, f) satisfies Condition (C) and $-\infty < a < b < +\infty$. Then all flow lines in $M^{a,b}$ are from $f^{-1}(b)$ or a critical point in $M^{a,b}$ to $f^{-1}(a)$ or a critical point in $M^{a,b}$.*

Definition 2.4. *Let $\phi_t(x)$ be the flow generated by $-\nabla f$ with initial value x. Suppose p is a critical point. Define the descending manifold of p to be $\mathcal{D}(p) = \{x \in M \mid \lim_{t\to-\infty} \phi_t(x) = p\}$. Define the ascending manifold of p to be $\mathcal{A}(p) = \{x \in M \mid \lim_{t\to+\infty} \phi_t(x) = p\}$.*

Both $\mathcal{D}(p)$ and $\mathcal{A}(p)$ are embedded submanifolds diffeomorphic to (maybe infinite dimensional) open disks. By Theorem 2.1 and Corollary 2.3, we get the following.

Corollary 2.5. *Suppose (M, f) satisfies Condition (C) and $-\infty < a < b < +\infty$. Suppose $\{p_1, \cdots, p_n\}$ consists of all critical points in $M^{a,b}$. Denote $\mathcal{A}(p_i) \cap f^{-1}(b)$ by S_i^+, and $\mathcal{D}(p_i) \cap f^{-1}(a)$ by S_i^-. Then the flow map can be defined and gives a diffeomorphism:*

$$\psi : f^{-1}(b) - \bigcup_{i=1}^{n} S_i^+ \longrightarrow f^{-1}(a) - \bigcup_{i=1}^{n} S_i^-.$$

In particular, if there is no critical point in $M^{a,b}$, we have the following diffeomorphism:

$$\psi : f^{-1}(b) \longrightarrow f^{-1}(a).$$

Here, if $x \in f^{-1}(b)$, $\phi_t(x) = y \in f^{-1}(a)$ for some t, the flow map is defined by $\psi(x) = y$.

Remark 2.1. *Although we use the notation $S_i^\pm$ in Corollary 2.5, $S_i^\pm$ are not necessarily homeomorphic to spheres.*

Definition 2.6. *If (M, f) satisfies Condition (C) and ind$(p) < +\infty$ for all critical points p, then we call (M, f) a CF pair.*

Definition 2.7. *If the descending manifold $\mathcal{D}(p)$ and the ascending manifold $\mathcal{A}(q)$ are transversal for all critical points p and q, then we say $-\nabla f$ satisfies transversality.*

Remark 2.2. *Some papers in the literature call Definition 2.7 Morse-Smale Condition.*

If $-\nabla f$ satisfies transversality, then $\mathcal{D}(p) \cap \mathcal{A}(q)$ is an embedded submanifold which consists of points on flow lines from p to q. Since a flow line has an R-action, we may take the quotient of $\mathcal{D}(p) \cap \mathcal{A}(q)$ by this R-action, i.e. consider its orbit space acted upon by the flow. This leads to the following definition. (See also [8, observation 4], [11, p. 3], [35, defn. 2.32] and [6, p. 158].)

Definition 2.8. *Suppose $-\nabla f$ satisfies transversality. Define $\mathcal{W}(p,q) = \mathcal{D}(p) \cap \mathcal{A}(q)$. Define the moduli space $\mathcal{M}(p,q)$ to be the orbit space $\mathcal{W}(p,q)/R$.*

Clearly, both $\mathcal{W}(p,q)$ and $\mathcal{M}(p,q)$ are smooth manifolds. Suppose γ_1 and γ_2 are two flow lines such that $\gamma_1(-\infty) = \gamma_2(-\infty) = p$, $\gamma_1(+\infty) = \gamma_2(+\infty) = q$ and $\gamma_1(0) = \gamma_2(t_0)$ for some $t_0 \neq 0$. Then γ_1 and γ_2 are two distinct flow lines which represent the same point of $\mathcal{M}(p,q)$. For convenience and briefness, we identify them as the same flow line. Then $\mathcal{M}(p,q) = \{\gamma \mid \gamma \text{ is a flow line, } \gamma(-\infty) = p \text{ and } \gamma(+\infty) = q.\}$. Suppose $a \in (f(q), f(p))$ is a regular value. For all $\gamma \in \mathcal{M}(p,q)$, it intersects with $f^{-1}(a)$ at a unique point. This gives $\mathcal{M}(p,q)$ a natural identification with $\mathcal{W}(p,q) \cap f^{-1}(a)$ which is a diffeomorphism.

We generalize the concept of flow lines. Suppose γ is a flow line. If it passes through a singularity, it is a constant flow line. Otherwise, it is nonconstant. The following definition is slightly different from the "broken trajectories" in [8, defn. 4].

Definition 2.9. *An ordered sequence of flow lines $\Gamma = (\gamma_1, \cdots, \gamma_n)$, $n \geq 1$, is a generalized flow line if $\gamma_i(+\infty) = \gamma_{i+1}(-\infty)$ and γ_i are constant or nonconstant alternatively according the order of their places in the sequence. γ_i is a component of Γ. Γ is a unbroken generalized flow line if $n = 1$ and a broken generalized flow line if $n > 1$.*

Example 2.1. *Suppose p is a singularity. Assume γ_1, γ_2 and γ_3 are flow lines in which γ_1 and γ_3 are nonconstant and $\gamma_1(+\infty) = \gamma_3(-\infty) = p$, $\gamma_2(t) \equiv p$. Then (γ_1), (γ_1, γ_2), (γ_2, γ_3) and $(\gamma_1, \gamma_2, \gamma_3)$ are generalized flow lines, (γ_1) is unbroken, and others are broken. Furthermore, (γ_1, γ_3) is not a generalized flow line.*

For convenience, we may identify a flow line γ with the generalized flow line (γ). Definition 2.9 is a generalization of flow lines.

Definition 2.10. *Suppose x and y are two points in M. A generalized flow line $(\gamma_1, \cdots, \gamma_n)$ connects x and y if there exist $t_1, t_2 \in (-\infty, +\infty)$*

such that $\gamma_1(t_1) = x$ and $\gamma_n(t_2) = y$. A point z is a point on $(\gamma_1, \cdots, \gamma_n)$ if there exists γ_i and $t \in (-\infty, +\infty)$ such that $\gamma_i(t) = z$.

Example 2.2. *Suppose p and q are two critical points. Let γ_1, γ_2 and γ_3 be flow lines such that $\gamma_1(t) \equiv p$, $\gamma_3(t) \equiv q$, $\gamma_2(-\infty) = p$ and $\gamma_2(+\infty) = q$. Then $(\gamma_1, \gamma_2, \gamma_3)$ is a generalized flow line connecting p and q, while γ_2 is not.*

We need to consider the relations between two critical points.

Definition 2.11. *Suppose p and q are two critical points. We define the relation $p \succeq q$ if there is a flow line from p to q. We define the relation $p \succ q$ if $p \succeq q$ and $p \neq q$.*

Definition 2.12. *An ordered set $I = \{r_0, r_1, \cdots, r_{k+1}\}$ is a critical sequence if r_i $(i = 0, \cdots, k+1)$ are critical points and $r_0 \succ r_1 \succ \cdots \succ r_{k+1}$. We call r_0 the head of I, and r_{k+1} the tail of I. The length of I is $|I| = k$.*

Suppose $I = \{r_0, r_1, \cdots, r_{k+1}\}$ is a critical sequence. We denote the following product manifolds by $\mathcal{M}_I$ and $\mathcal{D}_I$.

$$\mathcal{M}_I = \prod_{i=0}^{k} \mathcal{M}(r_i, r_{i+1}), \qquad \mathcal{D}_I = \prod_{i=0}^{k} \mathcal{M}(r_i, r_{i+1}) \times \mathcal{D}(r_{k+1}). \tag{2.1}$$

We shall consider the manifold structures of compactifications of the spaces $\mathcal{M}(p, q)$, $\mathcal{D}(p)$ and $\mathcal{W}(p, q)$. They usually have corners. For the definition of manifold with corners, we follow [13, p. 2] and [17, sec. 1.1].

Definition 2.13. *A smooth manifold with corners is a space defined in the same way as a smooth manifold except that its atlases are open subsets of $[0, +\infty)^n$.*

If L is a smooth manifold with corners, $x \in L$, a neighborhood of x is differomorphic to $[0, \epsilon)^k \times (0, \epsilon)^{n-k}$, then define $c(x) = k$. Clearly, $c(x)$ does not depend on the choice of atlas. We call a union of some components of $\{x \in L \mid c(x) = 1\}$ a face.

Definition 2.14. *A smooth manifold with faces is a smooth manifold with corners such that each x belongs to the closures of $c(x)$ different connected faces.*

Now we introduce another definition.

Definition 2.15. *Suppose L is a smooth manifold. $\{x \in L \mid c(x) = k\}$ is the k-stratum of L. Denote it by $\partial^k L$.*

Clearly, faces and the k-strata are manifolds in the usual sense. They are also submanifolds of L of codimension 1 and k respectively.

Suppose p is critical point. By the Morse Lemma, there exist $\epsilon > 0$ and a diffeomorphism

$$h : B(\epsilon) \longrightarrow U \tag{2.2}$$

such that

$$f \circ h(v_1, v_2) = f(p) - \frac{1}{2}\langle v_1, v_1\rangle + \frac{1}{2}\langle v_2, v_2\rangle. \tag{2.3}$$

Here $B(\epsilon) = \{(v_1, v_2) \in T_pM \mid v_1 \in V_-, v_2 \in V_+, \|v_1\|^2 < 2\epsilon$ and $\|v_2\|^2 < 2\epsilon\}$, $V_- \times \{0\}$ is the negative spectrum space of $\nabla^2 f$ and $\{0\} \times V_+$ is the positive spectrum space of $\nabla^2 f$, U is a neighborhood of p and $h(0,0) = p$.

Definition 2.16. *If the map h in (2.2) also preserves the metric, then we say that the metric of M is locally trivial at p. If it is locally trivial at each critical point, then we say that the metric on M is locally trivial.*

If the metric is locally trivial at p, then we have

$$-\nabla f|_U = dh \cdot (v_1, -v_2). \tag{2.4}$$

When the metric is locally trivial, Figure 1 shows the standard model of the neighborhood U, where U is identified with $B(\epsilon)$. Here, a and b are regular values such that $b < f(p) < a$, and $f^{-1}(a)$ and $f^{-1}(b)$ are two level surfaces. The arrows indicate the directions of the flow. The points (v_1, v_2), (v_3, v_4) and (v_5, v_6) are on the same flow line, whereas (v_7, v_8), (v_9, v_{10}), (v_{11}, v_{12}) and $(0, 0)$ are on the same broken generalized flow line. Figure 1 will provide geometric intuition for the arguments in this paper.

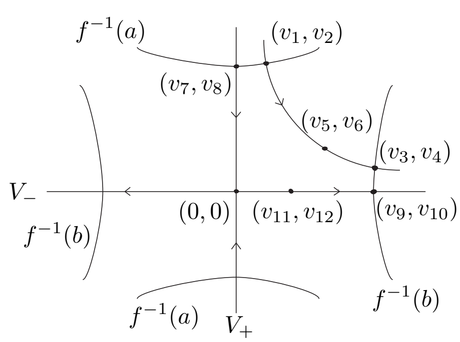

FIGURE 1. Standard Model

3. Main Results

All results are in CF case (see Definition 2.6) in this paper. Theorem 3.2 and Proposition 4.4 hold even in a more general setting. The assumption of transversality (see Definition 2.7), local triviality of the metric (see Definition 2.16) or the lower bound of the Morse function is needed for some results.

3.1. **Compactness.**

Theorem 3.1 shows that the closure of the space of unbroken flow lines is compact and is contained in the (maybe broken) generalized flow lines (see Definition 2.9). Theorem 3.2 considers the set consisting of points on the generalized flow lines with a fixed head and tail. They are essential for proving the compactness of $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$ later.

Theorem 3.1 (Compactness of Flows)**.** *Suppose (M,f) is a CF pair. Suppose p and q are two distinct critical points and $\{\gamma_n\}_{n=1}^{\infty}$ are flow lines such that $\gamma_n(-\infty)=p$ and $\gamma_n(+\infty)=q$. Then there exist finite many distinct critical points r_i $(i=0,\cdots,l+1)$ and flow lines $\hat{\gamma}_i$ $(i=0,\cdots,l)$ such that $\hat{\gamma}_i(-\infty)=r_i$, $\hat{\gamma}_i(+\infty)=r_{i+1}$, $r_0=p$ and $r_{l+1}=q$. There exist a subsequence $\{\gamma_{n_k}\}\subseteq\{\gamma_n\}$ and time $s_{n_k}^0<\cdots<s_{n_k}^l$ such that $\lim\limits_{k\to\infty}\gamma_{n_k}(s_{n_k}^i)=\hat{\gamma}_i(0)$.*

Remark 3.1. *As pointed out by the referees, the papers* [1] *and* [2] *prove results similar to Theorem 3.1. The proof of* [1] *relies on the study of differential operators on vector fields, which is a very different approach from that of this paper. However, the proof of* [2, prop. 2.4, 1.17 and 2.2] *is essentially the same as that of Theorem 3.1. Thus, theoretically, it's unnecessary to include the proof here. Nevertheless, for the sake of completeness, we still keep it.*

Theorem 3.2 (Compactness of Points)**.** *Suppose, for any real numbers $a<b$, $M^{a,b}$ only contains finite many critical points. Suppose, for any two critical points p and q, the conclusion of Theorem 3.1 holds. Let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of generalized flow lines connecting p and q. Then we have the following results.*

(1). Suppose x_n is on Γ_n. Then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\lim\limits_{k\to\infty}x_{n_k}$ exists and is on a generalized flow line connecting p and q.

(2). Suppose x_n^i are on Γ_n and $\lim\limits_{n\to\infty}x_n^i$ exist $(i=1,\cdots,k)$. Then these limit points are on a same generalized flow line connecting p and q.

In particular, if (M,f) is a CF pair, then the above (1) and (2) hold.

Remark 3.2. *Essentially, the compactness of points follows from the compactness of flows. For a more precise description, see Proposition 4.4.*

3.2. **Manifold Structures.** We consider the manifold structures of the compactified spaces of $\mathcal{M}(p,q)$, $\mathcal{D}(p)$ and $\mathcal{W}(p,q)$ (see Definitions 2.4 and 2.8).

First, we introduce some notation. Suppose $I_1 = (p, r_1, \cdots, r_s)$ and $I_2 = (r_{s+1}, \cdots, r_k,\ q)$ are critical sequences (see Definition 2.12) and $r_s \succeq r_{s+1}$. Let $(I, s) = (p, r_1, \cdots, q)$. It is not necessarily a critical sequence since r_s may equal r_{s+1}. Denote the following product manifold by $\mathcal{W}_{I,s}$.

$$\mathcal{W}_{I,s} = \mathcal{M}_{I_1} \times \mathcal{W}(r_s, r_{s+1}) \times \mathcal{M}_{I_2}. \tag{3.1}$$

The compactifications are standard. Define (see (2.1))

$$\overline{\mathcal{M}(p,q)} = \bigsqcup_I \mathcal{M}_I, \quad \overline{\mathcal{D}(p)} = \bigsqcup_I \mathcal{D}_I, \quad \overline{\mathcal{W}(p,q)} = \bigsqcup_{(I,s)} \mathcal{W}_{I,s}. \tag{3.2}$$

Here the first disjoint union is over all critical sequences with head p and tail q; the second one is over all critical sequences with head p; and the third one is over all $(I,s) = (p, r_1, \cdots, r_k, q)$ such that $p \succ r_1 \succ \cdots \succ r_s \succeq r_{s+1} \succ \cdots \succ r_k \succ q$ for all k. Clearly, $\mathcal{M}(p,q) \subseteq \overline{\mathcal{M}(p,q)}$, $\mathcal{D}(p) \subseteq \overline{\mathcal{D}(p)}$ and $\mathcal{W}(p,q) \subseteq \overline{\mathcal{W}(p,q)}$. Since $\mathcal{M}(r_i, r_{i+1})$, $\mathcal{D}(r_k)$ and $\mathcal{W}(r_s, r_{s+1})$ are smooth manifolds, so are $\mathcal{M}_I$, $\mathcal{D}_I$ and $\mathcal{W}_{I,s}$. An example of $\overline{\mathcal{D}(p)}$ is illustrated by Figure 3 in Subsection 5.1.

Suppose $\alpha \in \mathcal{M}_I \subseteq \overline{\mathcal{M}(p,q)}$. Then $\alpha = (\gamma_0, \cdots, \gamma_k)$, where $\gamma_i \in \mathcal{M}(r_i, r_{i+1})$, $r_0 = p$ and $r_{k+1} = q$. By Condition (C), there are only finitely many critical values in $[f(q), f(p)]$. Suppose the critical values of f divide $[f(q), f(p)]$ into $l+1$ intervals $[c_{i+1}, c_i]$ $(i = 0, \cdots, l)$, where $c_0 = f(p)$ and $c_{l+1} = f(q)$. For all $a_i \in (c_{i+1}, c_i)$, they are regular. The union of the components of α intersects with $f^{-1}(a_i)$ at exactly one point $x_i(\alpha)$. There is an evaluation map $E : \overline{\mathcal{M}(p,q)} \longrightarrow \prod_{i=0}^{l} f^{-1}(a_i)$ such that

$$E(\alpha) = (x_0(\alpha), \cdots, x_l(\alpha)). \tag{3.3}$$

If $\alpha_1 \in \prod_{i=0}^{j-1} \mathcal{M}(r_i, r_{i+1}) \subseteq \overline{\mathcal{M}(r_0, r_j)}$ and $\alpha_2 \in \prod_{i=j}^{k} \mathcal{M}(r_i, r_{i+1}) \subseteq \overline{\mathcal{M}(r_j, r_k)}$, then $(\alpha_1, \alpha_2) \in \prod_{i=0}^{k} \mathcal{M}(r_i, r_{i+1}) \subseteq \overline{\mathcal{M}(r_0, r_k)}$. This gives a map $i_{(p,r,q)} : \overline{\mathcal{M}(p,r)} \times \overline{\mathcal{M}(r,q)} \longrightarrow \overline{\mathcal{M}(p,q)}$. We shall prove the following theorem (see Definition 2.12).

Theorem 3.3 (Smooth Structure of $\overline{\mathcal{M}(p,q)}$)**.** *Let (M,f) be a CF pair satisfying transversality and having a locally trivial metric. Then, for* *each pair of critical points (p, q), there is a smooth structure on $\overline{\mathcal{M}(p,q)}$ which satisfies the following properties.*

(1). It is a compact manifold with faces whose k-stratum is exactly $\coprod_{|I|=k} \mathcal{M}_I$, where the disjoint union is over all critical sequences I with head p and tail q.

(2). The smooth structure is compatible with that of $\mathcal{M}_I$ in each stratum.

(3). The evaluation map $E : \overline{\mathcal{M}(p,q)} \longrightarrow \prod_{i=0}^{l} f^{-1}(a_i)$ is a smooth embedding, where E is defined by (3.3).

(4). The smooth structures are compatible with critical pairs, i.e., $i_{(p,r,q)} : \overline{\mathcal{M}(p,r)} \times \overline{\mathcal{M}(r,q)} \longrightarrow \overline{\mathcal{M}(p,q)}$ is a smooth embedding.

We define the evaluation map $e : \overline{\mathcal{D}(p)} \longrightarrow M$ as follows. The restriction of e on $\mathcal{D}_I = \mathcal{M}_I \times \mathcal{D}(r_k)$ is just the coordinate projection $\mathcal{M}_I \times \mathcal{D}(r_k) \longrightarrow \mathcal{D}(r_k)$. This defines the map since $\mathcal{D}(r_k) \subseteq M$.

If $\alpha_1 \in \prod_{i=0}^{j-1} \mathcal{M}(r_i, r_{i+1}) \subseteq \overline{\mathcal{M}(r_0, r_j)}$ and $(\alpha_2, x) \in \prod_{i=j}^{k} \mathcal{M}(r_i, r_{i+1}) \times \mathcal{D}(r_k) \subseteq \overline{\mathcal{D}(r_j)}$, then $(\alpha_1, \alpha_2, x) \in \prod_{i=0}^{k} \mathcal{M}(r_i, r_{i+1}) \times \mathcal{D}(r_k) \subseteq \overline{\mathcal{D}(r_0)}$. This gives a map $i_{(p,r)} : \overline{\mathcal{M}(p,r)} \times \overline{\mathcal{D}(r)} \longrightarrow \overline{\mathcal{D}(p)}$. We shall prove the following theorem.

Theorem 3.4 (Smooth Structure of $\overline{\mathcal{D}(p)}$)**.** *Under the assumptions of Theorem 3.3, suppose f has a lower bound. Then, for each critical point p, there is a smooth structure on $\overline{\mathcal{D}(p)}$ satisfying the following properties.*

(1). It is a compact manifold with faces whose k-stratum is exactly $\coprod_{|I|=k-1} \mathcal{D}_I$ where the disjoint union is over all critical sequences with head p.

(2). The smooth structure is compatible with that of $\mathcal{D}_I$ in each stratum.

(3). The evaluation map $e : \overline{\mathcal{D}(p)} \longrightarrow M$ is smooth, where the restriction of e on $\mathcal{D}_I = \mathcal{M}_I \times \mathcal{D}_{r_k}$ is the coordinate projection onto $\mathcal{D}_{r_k} \subseteq M$.

(4). The smooth structures are compatible with critical pairs, i.e., $i_{(p,r)} : \overline{\mathcal{M}(p,r)} \times \overline{\mathcal{D}(r)} \longrightarrow \overline{\mathcal{D}(p)}$ is a smooth embedding, where the smooth structure of $\overline{\mathcal{M}(p,r)}$ is defined in Theorem 3.3.

Remark 3.3. *It's easy to see that Theorem 3.4 will not be true if we don't assume that f is lower bounded.*

Similarly, we also have the following theorem about $\mathcal{W}(p,q)$. The maps e, $i^1_{(p,r,q)}$ and $i^2_{(p,r,q)}$ are defined in Subsection 5.1.

Theorem 3.5 (Smooth Structure of $\overline{\mathcal{W}(p,q)}$)**.** *Under the assumptions of Theorem 3.3, for each pair of critical points (p,q), there is a smooth structure on $\overline{\mathcal{W}(p,q)}$ satisfying the following properties.*

(1). It is a compact manifold with faces whose k-stratum is exactly $\bigsqcup_{(I,s)} \mathcal{W}_{I,s}$. Here $(I,s) = (p, r_1, \cdots, r_k, q)$ such that $p \succ r_1 \succ \cdots \succ r_s \succeq r_{s+1} \succ \cdots \succ r_k \succ q$. The disjoint union is over all (I,s) which contain $k+2$ components.

(2). The smooth structure is compatible with that of $\mathcal{W}_{I,s}$ in each stratum.

(3). The evaluation map $e : \overline{\mathcal{W}(p,q)} \longrightarrow M$ is smooth.

(4). The smooth structures are compatible with critical pairs, i.e., $i^1_{(p,r,q)} : \overline{\mathcal{W}(p,r)} \times \overline{\mathcal{M}(r,q)} \longrightarrow \overline{\mathcal{W}(p,q)}$ and $i^2_{(p,r,q)} : \overline{\mathcal{M}(p,r)} \times \overline{\mathcal{W}(r,q)} \longrightarrow \overline{\mathcal{W}(p,q)}$ are smooth embeddings.

Here the smooth structure of $\overline{\mathcal{M}(*,*)}$ is defined in Theorem 3.3.

The above theorems are under the assumption that the metric is locally trivial. If this assumption is dropped, can $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$ still be equipped with smooth structures with the desired stratifications? To the best of my knowledge, this question is still open even when M is finite dimensional. However, even if the answer is positive, there is still some difference between the case of a locally trivial metric and that of a general metric.

Consider $E : \overline{\mathcal{M}(p,q)} \longrightarrow \prod_{i=0}^{l} f^{-1}(a_i)$ in (3.3). By (3) of Theorem 3.3, the image of E, $\mathrm{Im}(E) \subseteq \prod_{i=0}^{l} f^{-1}(a_i)$ is a smooth (C^∞) embedded submanifold of $\prod_{i=0}^{l} f^{-1}(a_i)$ when the metric is locally trivial. For a general metric, we have the following counterexample.

Example 3.1 (Not C^1)**.** *Let CP^2 the complex projective plane. Then there exist a metric and a Morse function f on CP^2, where f has three critical points p, q and r such that ind$(p) = 4$, ind$(q) = 0$ and ind$(r) = 2$. $\overline{\mathcal{M}(p,q)} = \mathcal{M}(p,q) \sqcup (\mathcal{M}(p,r) \times \mathcal{M}(r,q))$. And $E(\overline{\mathcal{M}(p,q)})$ is NOT a C^1 embedded submanifold with boundary $E(\mathcal{M}(p,r) \times \mathcal{M}(r,q))$ of $\prod_{i=0}^{1} f^{-1}(a_i)$ (see (3.3)). In other words, it's impossible to give Im(E) a C^1 structure compatible with $\prod_{i=0}^{1} f^{-1}(a_i)$.*

3.3. **Orientation.** We consider the orientation of $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$. Since $\mathrm{ind}(p) < +\infty$, we can assign $\mathcal{D}(p)$ an orientation arbitrarily. The orientations of $\mathcal{D}(p)$ for all p determine the orientations of $\mathcal{M}(p,q)$ and $\mathcal{W}(p,q)$ and then those of their compactified manifolds for all pairs (p,q). Now we consider the 1-stratum (see Definition 2.15) of $\overline{\mathcal{M}(p,q)}$, i.e., $\partial^1\overline{\mathcal{M}(p,q)} = \bigsqcup_r \mathcal{M}(p,r)\times\mathcal{M}(r,q)$. The orientation of $\overline{\mathcal{M}(p,q)}$ gives $\partial^1\overline{\mathcal{M}(p,q)}$ a boundary orientation. On the other hand, the orientations of $\mathcal{M}(p,r)$ and $\mathcal{M}(r,q)$ give $\mathcal{M}(p,r)\times\mathcal{M}(r,q)$ a product orientation. We shall consider the relation between the two orientations of $\partial^1\overline{\mathcal{M}(p,q)}$. Similarly, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$ also have such orientation issues. The definition of the above orientations is given in Subsection 6.1. We have the following orientation formulas.

Theorem 3.6 (Orientation Formulas)**.** *Under the assumption of Theorem 3.3, as oriented manifolds, we have*

(1). $\partial^1\overline{\mathcal{M}(p,q)} = \bigsqcup\limits_{p\succ r\succ q}(-1)^{ind(p)-ind(r)}\mathcal{M}(p,r)\times\mathcal{M}(r,q)$;

(2). $\partial^1\overline{\mathcal{D}(p)} = \bigsqcup\limits_{p\succ r}\mathcal{M}(p,r)\times\mathcal{D}(r)$, *where* f *is lower bounded;*

(3). $\partial^1\overline{\mathcal{W}(p,q)} = \bigsqcup\limits_{p\succeq r\succ q}(-1)^{ind(p)-ind(r)+1}\mathcal{W}(p,r)\times\mathcal{M}(r,q)\sqcup\bigsqcup\limits_{p\succ r\succeq q}\mathcal{M}(p,r)\times\mathcal{W}(r,q)$.

In the above, $\partial^1\square$ *are equipped with boundary orientations,* $\square\times\square$ *are equipped with product orientations.*

Remark 3.4. *The papers* [3, lem. 3.4] *and* [20, sec. 2.14 and 2.15] *announce formulas similar to (1) and (2) of Theorem 3.6 in finite dimensional case (*[3] *even does the Morse-Bott case). Our method to define orientations is different from theirs. Thus our formulas are different from theirs. By our definition of orientations, there is no sign in (2) of Theorem 3.6.*

3.4. **CW Structure.** Finally, we point out that the compatification of $\mathcal{D}(p)$ results in a bona fide smooth CW decomposition of M.

Clearly, $\mathcal{D}(p)$ is diffeomorphic to an open disk of dimension $\mathrm{ind}(p)$, and $\mathcal{D}(p)\cap\mathcal{D}(q)=\emptyset$ when $p\neq q$. Recall the evaluation map $e:\overline{\mathcal{D}(p)}\longrightarrow M$ and that $\overline{\mathcal{D}(p)}=\bigsqcup_I\mathcal{M}_I\times\mathcal{D}(r_k)$ (see Theorem 3.4). The restriction of e to $\mathcal{M}_I\times\mathcal{D}(r_k)$ is just the coordinate projection onto $\mathcal{D}(r_k)$. Thus $e|_{\mathcal{D}(p)}$ is the identity map, and $e(\partial\overline{\mathcal{D}(p)})$ consists of finite number of $\mathcal{D}(q)$ such that $\mathrm{ind}(q)<\mathrm{ind}(p)$. Thus if $\overline{\mathcal{D}(p)}$ is homeomorphic to a closed disk for all p, then, $\forall a\in R$, $K^a=\bigsqcup_{f(p)\leq a}\mathcal{D}(p)$ is a finite CW complex with characteristic maps e. We shall prove the following theorems.

Theorem 3.7 (Topology of $\overline{\mathcal{D}(p)}$)**.** *Under the assumption of Theorem 3.4, there is a homeomorphism* $\Psi:(D^{ind(p)},S^{ind(p)-1})\longrightarrow(\overline{\mathcal{D}(p)},\partial\overline{\mathcal{D}(p)})$, *where $D^{ind(p)}$ is the $ind(p)$ dimensional closed disk and $S^{ind(p)-1} = \partial D^{ind(p)}$.*

For the definition of simple homotopy equivalence and elementary expansion, see [10, p. 14-15]

Theorem 3.8 (CW Structure)**.** *Under the assumption of Theorem 3.4, let a be a regular value of f. Then $K^a = \bigsqcup_{f(p)\leq a} \mathcal{D}(p)$ is a finite CW complex with characteristic maps $e : \overline{\mathcal{D}(p)} \longrightarrow K^a$, where e is defined in (3) of Theorem 3.4. In particular, if f is proper, then the inclusion $K^a \hookrightarrow M^a$ is a simple homotopy equivalence. In fact, in this special case, there is a CW decomposition of M^a such that K^a expands to M^a by elementary expansions.*

The following theorem explicitly computes the boundary operator of the CW chain complex $C_*(K^a)$ associated with the CW structure.

Theorem 3.9 (Boundary Operator)**.** *Let K^a be the CW complex in Theorem 3.8 (we do NOT assume f is proper). Let $C_*(K^a)$ be the associated CW chain complex and $[\overline{\mathcal{D}(p)}]$ be the base element represented by $\overline{\mathcal{D}(p)}$ in $C_*(K^a)$. Then*

$$\partial[\overline{\mathcal{D}(p)}] = \sum_{ind(q)=ind(p)-1} \#\mathcal{M}(p,q)[\overline{\mathcal{D}(q)}],$$

where $\#\mathcal{M}(p,q)$ is the sum of the orientations ± 1 of all points in $\mathcal{M}(p,q)$ defined in Theorem 3.6.

Remark 3.5. *Theorem 3.9 shows that the boundary operator of $C_*(K^a)$ coincides with that of the Thom-Smale complex in Morse homology when M is compact. This shows Morse homology arises from a cellular chain complex. However, unlike the assumption of Theorem 3.9, Morse homology does not require the local triviality of metrics. For Morse homology, see* [23, cor. 7.3], [6] *and* [35]. *For some of its generalizations to Hilbert manifolds, see* [32], [1] *and* [2].

4. Compactness

4.1. **Proof of Theorem 3.1.** In order to prove Theorem 3.1, we need the classical Grobman-Hartman Theorem in Banach spaces.

Suppose U_i ($i = 1, 2$) are two open subsets in two Banach spaces E_i. Let X_i be a smooth vector field on U_i. Let $\phi_t^i(x_i)$ be the associated flow on U_i with initial value x_i. We say ϕ_t^i ($i = 1, 2$) are topologically conjugate if there exists a homeomorphism $h : U_1 \longrightarrow U_2$ such that $h(\phi_t^1(x_1)) = \phi_t^2(h(x_1))$ (see [31, p. 26]). The Grobman-Hartman Theorem states that, if p is a hyperbolic singularity of X on an open subset

U of a Banach space E, then the flow generated by X is locally topologically conjugate to that generated by the linear vector field $\nabla X(p)v$ near 0 on T_pU (see [33, sec. 4], [30, sec. 5] and [31, thm. 4.10, p. 66]. Although the statements in [30] and [31] are only up to topological equivalence, they actually construct the conjugate.)

In our case, $\nabla^2 f(p)$ splits T_pM into two subspace $T_pM = V_- \times V_+$, where $\{0\} \times V_+$ ($V_- \times \{0\}$) is the positive (negative) spectrum space of $\nabla^2 f(p)$. Thus the flow of $-\nabla f$ is topologically conjugate to the flow of $(-\nabla^2 f(p)v_1, -\nabla^2 f(p)v_2)$ on T_pM. Furthermore, $-\nabla^2 f(p)$ is symmetric and negative (positive) definite in $\{0\} \times V_+$ ($V_- \times \{0\}$), thus $-\nabla^2 f(p)v_i$ is transversal to the unit sphere in $V_\pm$. By the method of the proof of [31, prop. 2.15, p. 52], we have the flow of $(-\nabla^2 f(p)v_1, -\nabla^2 f(p)v_2)$ is topologically conjugate to the flow of $(v_1, -v_2)$. Thus we get the flowing lemma (compare (2.4)).

Lemma 4.1. *The flow generated by $-\nabla f$ near a critical point p is locally topologically conjugate to the flow generated by $(v_1, -v_2)$ near 0 on $T_pM = V_- \times V_+$.*

Lemma 4.2. *Suppose $\{\gamma_n(t)\}_{n=1}^{\infty}$ and $\hat{\gamma}_1(t)$ are flow lines such that $\lim\limits_{n\to\infty} \gamma_n(0) = \hat{\gamma}_1(0)$, $\hat{\gamma}_1(+\infty) = p$ with ind$(p) < +\infty$, and, for all n, $\gamma_n(+\infty) \neq p$. Then there exist a subsequence $\{\gamma_{n_k}\} \subseteq \{\gamma_n\}$, time $s_{n_k} > 0$ and a nonconstant flow line $\hat{\gamma}_2(t)$ such that $\hat{\gamma}_2(-\infty) = p$ and $\lim\limits_{k\to\infty} \gamma_{n_k}(s_{n_k}) = \hat{\gamma}_2(0)$.*

Proof. By Lemma 4.1, there exist a neighborhood U_2 of p in M, a neighborhood U_1 of 0 in T_pM and a homeomorphism $h : U_1 \longrightarrow U_2$ such that $h(0) = p$ and h conjugates between the flow generated by $(v_1, -v_2)$ in U_1 and the flow generated by $-\nabla f$ in U_2 (see Figure 2).

Choosing an open subset if necessary, we may assume $U_1 = D_1(\epsilon) \times D_2(\epsilon)$ for some ϵ, where $D_1(\epsilon) = \{v_1 \in V_- \mid \|v_1\| < \epsilon\}$ and $D_2(\epsilon) = \{v_2 \in V_+ \mid \|v_2\| < \epsilon\}$. In U_1, $(V_- \times \{0\}) \cap U_1$ is the unstable submanifold, $(\{0\} \times V_+) \cap U_1$ is the stable submanifold. Thus $h(V_- \times \{0\})$ and $h(\{0\} \times V_+)$ are locally unstable and locally stable submanifolds respectively in U_2.

Since $\hat{\gamma}_1(+\infty) = p$, $\exists t_0$ such that $\forall t \geq t_0$, $\hat{\gamma}_1(t) \in h(\{0\} \times V_+)$. Suppose $h^{-1}(\hat{\gamma}_1(t_0)) = (0, v_{2,0})$. Since $\gamma_n(0) \to \hat{\gamma}_1(t_0)$, we have $\gamma_n(t_0) \in U_2$, $h^{-1}(\gamma_n(t_0)) = (v_{1,n}, v_{2,n})$ and $\|v_{1,n}\| < \frac{\epsilon}{2}$ when n is large enough. Since $\gamma_n(+\infty) \neq p$, we have $v_{1,n} \neq 0$. As a result, in U_1, the flow line passing through $(v_{1,n}, v_{2,n})$ intersects $S_1(\frac{\epsilon}{2}) \times D_2(\epsilon)$ at $\left(\dfrac{\epsilon}{2\|v_{1,n}\|} v_{1,n}, \dfrac{2\|v_{1,n}\|}{\epsilon} v_{2,n}\right)$, where $S_1(\frac{\epsilon}{2}) = \{v_1 \in V_- \mid \|v_1\| = \frac{\epsilon}{2}\}$. When $n \to \infty$, we have

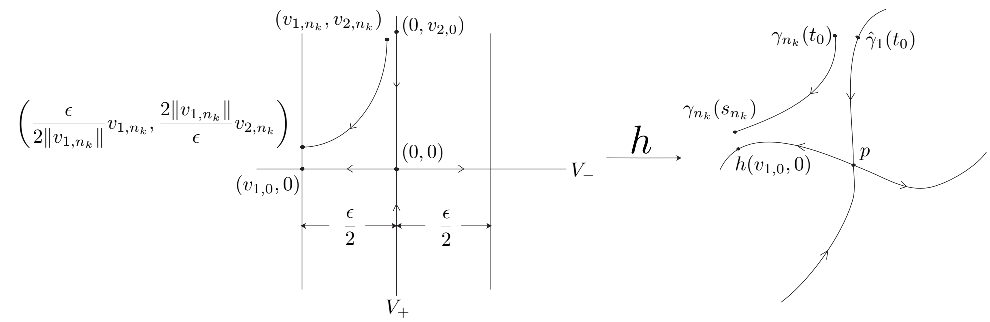

FIGURE 2. Topological Conjugate

$(v_{1,n}, v_{2,n}) \to (0, v_{2,0})$. Thus $\frac{2\|v_{1,n}\|}{\epsilon}v_{2,n} \to 0$. Since it is a $\mathrm{ind}(p)-1$ dimensional sphere and $\mathrm{ind}(p) < +\infty$, we have $S_1\left(\frac{\epsilon}{2}\right)$ is compact. So there exists a subsequence $\left\{\frac{\epsilon v_{1,n_k}}{2\|v_{1,n_k}\|}\right\}$ of $\left\{\frac{\epsilon v_{1,n}}{2\|v_{1,n}\|}\right\}$ such that $\lim_{k\to\infty} \frac{\epsilon}{2\|v_{1,n_k}\|}v_{1,n_k} = v_{1,0}$. Clearly, there exists $s_{n_k} > 0$ such that

$$\gamma_{n_k}(s_{n_k}) = h\left(\frac{\epsilon}{2\|v_{1,n_k}\|}v_{1,n_k}, \frac{2\|v_{1,n_k}\|}{\epsilon}v_{2,n_k}\right).$$

Thus

$$\lim_{k\to\infty} \gamma_{n_k}(s_{n_k}) = h(v_{1,0}, 0).$$

Denote the flow line with initial value $h(v_{1,0}, 0)$ by $\hat{\gamma}_2(t)$. Then $\lim_{k\to\infty} \gamma_{n_k}(s_{n_k}) = \hat{\gamma}_2(0)$. Since $h^{-1}(\hat{\gamma}_2(0)) = (v_{1,0}, 0) \in V_- \times \{0\}$, we know that $h^{-1}(\hat{\gamma}_2(-\infty)) = (0,0)$ or $\hat{\gamma}_2(-\infty) = p$. Since $\hat{\gamma}_2(0) \neq p$, $\hat{\gamma}_2$ is nonconstant. □

Proof of Theorem 3.1. Let a be a regular value such that $a < f(p)$ and there is no critical value in $(a, f(p))$. Let $S_p^- = \mathcal{D}(p) \cap f^{-1}(q)$. Then S_p^- is a sphere with dimension $\mathrm{ind}(p) < +\infty$, and it is compact. Suppose $\gamma_n(s_n^0) \in S_p^-$. Then there exists a subsequence of $\{\gamma_n(s_n^0)\}$, we may still denote it by $\{\gamma_n(s_n^0)\}$, which converges. Suppose $\lim_{n\to\infty} \gamma_n(s_n^0) = x_0$. Then $x_0 \in S_p^-$. Denote the flow line with initial value x_0 by $\hat{\gamma}_0$. Then $\hat{\gamma}_0(-\infty) = p$ because $\hat{\gamma}_0(0) = x_0 \in S_p^- \subseteq \mathcal{D}(p)$. Since $\gamma_n(+\infty) = q$, we have, for all t, $f(\gamma_n(s_n^0 + t)) \geq f(q)$. Thus, for all t,

$$f(\hat{\gamma}_0(t)) = \lim_{n\to\infty} f(\gamma_n(s_n^0 + t)) \geq f(q),$$

i.e., $f(\hat{\gamma}_0(t))$ has a lower bound $f(q)$. By Theorem 2.2, $\lim_{t\to+\infty}\hat{\gamma}_0(t)$ exists and $\hat{\gamma}_0(+\infty) = r_1$ is a critical point in $M^{f(q),f(p)}$. Clearly, $\hat{\gamma}_0$ is nonconstant. Thus $r_1 \neq p$. There are exactly the following two cases.

Case (1): $r_1 = q$. In this case, the proof is finished.

Case (2): $r_1 \neq q$. Since $\gamma_n(+\infty) = q \neq r_1$ and $\mathrm{ind}(r_1) < +\infty$, by Lemma 4.2, there exists a nonconstant flow line $\hat{\gamma}_1$ such that $\hat{\gamma}_1(-\infty) = r_1$. Furthermore, there exists a subsequence of $\{\gamma_n\}$, which we still denote by $\{\gamma_n\}$, and time $s_n^1 > s_n^0$ such that $\lim_{n\to\infty}\gamma_n(s_n^1) = \hat{\gamma}_1(0)$. Similar to the case of $\hat{\gamma}_0$, we have $\lim_{t\to+\infty}\hat{\gamma}_1(t)$ exists and $\hat{\gamma}_1(+\infty) = r_2$ is also a critical point in $M^{f(q),f(p)}$. Since $\hat{\gamma}_1$ is nonconstant, p, r_1 and r_2 are distinct. If $r_2 = q$, the proof is finished. Otherwise, repeat the argument of Case (2).

By Theorem 2.1, there are only finitely many critical points in $M^{f(q),f(p)}$, the process of the above argument terminates in finitely many steps. $\square$

4.2. **Proof of Theorem 3.2.** We first give two results needed for the proof of Theorem 3.2.

Lemma 4.3. *Suppose $\{\gamma_n\}_{n=1}^{\infty}$ and $\hat{\gamma}$ are flow lines such that $\hat{\gamma}(-\infty) = p$, $\hat{\gamma}(+\infty) = q$ and $\lim_{n\to\infty}\gamma_n(s_n) = \hat{\gamma}(0)$. If $\lim_{n\to\infty}(t_n - s_n) = +\infty$ ($\lim_{n\to\infty}(t_n - s_n) = -\infty$), then $\limsup_{n\to\infty} f(\gamma_n(t_n)) \leq f(q)$ ($\liminf_{n\to\infty} f(\gamma_n(t_n)) \geq f(p)$).*

Proof. It suffices to prove the case $\lim_{n\to\infty}(t_n - s_n) = +\infty$.

Since $\hat{\gamma}(+\infty) = q$, then $\forall \epsilon > 0$, $\exists T$, such that $\forall t \geq T$, we have $f(\hat{\gamma}(t)) < f(q) + \epsilon$. By that $\lim_{n\to\infty}(t_n - s_n) = +\infty$, we have $t_n > s_n + T$ and $f(\gamma_n(t_n)) < f(\gamma_n(s_n + T))$ when n is large enough. Since $\lim_{n\to\infty}\gamma_n(s_n) = \hat{\gamma}(0)$, we infer

$$\lim_{n\to\infty} f(\gamma_n(s_n + T)) = f(\hat{\gamma}(T)) < f(q) + \epsilon.$$

Thus

$$\limsup_{n\to\infty} f(\gamma_n(t_n)) \leq \lim_{n\to\infty} f(\gamma_n(s_n + T)) < f(q) + \epsilon.$$

Now let $\epsilon \to 0$. Then we get $\limsup_{n\to\infty} f(\gamma_n(t_n)) \leq f(q)$. $\square$

The following proposition requires neither Condition (C) nor finite indices.

Proposition 4.4. *Suppose p and q are two critical points, $\{\gamma_n\}_{n=1}^{\infty}$ are flow lines such that $\gamma_n(-\infty) = p$ and $\gamma_n(+\infty) = q$, and there exist*

$s_n^0 < \cdots < s_n^l$ such that $\lim_{n\to\infty} \gamma_n(s_n^i) = \hat{\gamma}_i(0)$. Here $\hat{\gamma}_i$ are flow lines such that $\hat{\gamma}_i(-\infty) = r_i$, $\hat{\gamma}_i(+\infty) = r_{i+1}$, and $r_0 = p$, $r_{l+1} = q$. Then we have the following convergence result.

(1). If $\lim_{n\to\infty}(t_n - s_n^i) = \tau$, $|\tau| < +\infty$, then $\lim_{n\to\infty} \gamma_n(t_n) = \hat{\gamma}_i(\tau)$;

(2). If $s_n^i < t_n < s_n^{i+1}$, and $\lim_{n\to\infty}(t_n - s_n^i) = \lim_{n\to\infty}(s_n^{i+1} - t_n) = +\infty$, then $\lim_{n\to\infty} \gamma_n(t_n) = r_{i+1}$, where $s_n^{-1} = -\infty$ and $s_n^{l+1} = +\infty$.

Proof. Case (1) is obvious. We only need to prove Case (2).

We may assume $s_n^i < t_n < s_n^{i+1}$ and $i \geq 0$ because the subcase of $i = -1$ will be converted to the subcase of $i = l$ if f is replaced by $-f$.

We shall prove $\lim_{n\to\infty} \gamma_n(t_n) = r_{i+1}$ by contradiction.

Suppose it doesn't hold, then there exist a subsequence of $\{\gamma_n(t_n)\}$, which we still denote by $\{\gamma_n(t_n)\}$, and a neighborhood U of r_{i+1} such that $\gamma_n(t_n) \notin U$. Choose an open geodesic disk $D(r_{i+1}, \epsilon)$ with center r_{i+1} and radius ϵ such that $\overline{D(r_{i+1}, \epsilon)} \subseteq U$. Since r_{i+1} is a nondegenerate critical point, by the Taylor expansion, we may choose ϵ small enough such that, there exist constants C_1 and C_2, and $0 < C_1 \leq \|\nabla f\| \leq C_2$ in $\overline{D(r_{i+1}, \epsilon)} - D(r_{i+1}, \frac{\epsilon}{2})$ for a fixed ϵ.

Suppose $\gamma(t)$ is a flow line, $\tau_1 < \tau_2$, such that $\gamma(\tau_1) \in D(r_{i+1}, \frac{\epsilon}{2})$ and $\gamma(\tau_2) \notin \overline{D(r_{i+1}, \epsilon)}$. Thus there exist τ_1', τ_2' such that $\tau_1 < \tau_1' < \tau_2' < \tau_2$, $\gamma([\tau_1', \tau_2']) \subseteq \overline{D(r_{i+1}, \epsilon)} - D(r_{i+1}, \frac{\epsilon}{2})$, $\gamma(\tau_1') \in \partial D(r_{i+1}, \frac{\epsilon}{2})$ and $\gamma(\tau_2') \in \partial D(r_{i+1}, \epsilon)$.

Consider the distance $d(\gamma(\tau_1'), \gamma(\tau_2'))$ between $\gamma(\tau_1')$ and $\gamma(\tau_2')$. Clearly, $d(\gamma(\tau_1'), \gamma(\tau_2')) \geq \frac{\epsilon}{2}$. Thus

$$\begin{aligned} \frac{\epsilon}{2} \leq d(\gamma(\tau_1'), \gamma(\tau_2')) &\leq \int_{\tau_1'}^{\tau_2'} \left\| \frac{d}{dt}\gamma(t) \right\| dt \\ = \quad \int_{\tau_1'}^{\tau_2'} \|\nabla f(\gamma(t))\| dt &\leq \int_{\tau_1'}^{\tau_2'} C_2 dt = C_2(\tau_2' - \tau_1'). \end{aligned}$$

We have $\tau_2' - \tau_1' \geq \frac{\epsilon}{2C_2}$. Then

$$\int_{\tau_1'}^{\tau_2'} \|\nabla f\|^2 \geq \int_{\tau_1'}^{\tau_2'} C_1^2 \geq \frac{C_1^2 \epsilon}{2C_2}.$$

Thus we get

$$f(\gamma(\tau_1)) - f(\gamma(\tau_2)) = \int_{\tau_1}^{\tau_2} \|\nabla f\|^2 \geq \int_{\tau_1'}^{\tau_2'} \|\nabla f\|^2 \geq \frac{C_1^2 \epsilon}{2C_2} > 0.$$

Denoting $\frac{C_1^2\epsilon}{2C_2}$ by K, we get

$$f(\gamma(\tau_1)) - f(\gamma(\tau_2)) \geq K > 0. \tag{4.1}$$

Since $\hat{\gamma}_i(+\infty) = r_{i+1}$, then there exists t_∞ such that $\hat{\gamma}_i(t_\infty) \in B(r_{i+1}, \frac{\epsilon}{2})$ and $f(\hat{\gamma}_i(t_\infty)) < f(r_{i+1}) + \frac{K}{2}$. Since $\gamma_n(s_n^i) \to \hat{\gamma}_i(0)$, we have $\gamma_n(s_n^i + t_\infty) \in B(r_{i+1}, \frac{\epsilon}{2})$ and $f(\gamma_n(s_n^i + t_\infty)) < f(r_{i+1}) + \frac{K}{2}$ when n is large enough. Also since $(t_n - s_n^i) \to +\infty$, we get $t_n > s_n^i + t_\infty$ when n is large enough. Now we can replace $\gamma(\tau_1)$ and $\gamma(\tau_2)$ in (4.1) by $\gamma_n(s_n^i + t_\infty)$ and $\gamma_n(t_n)$, then $f(\gamma_n(s_n^i + t_\infty)) - f(\gamma_n(t_n)) \geq K$. Furthermore,

$$f(\gamma_n(t_n)) \leq f(\gamma_n(s_n^i + t_\infty)) - K < f(r_{i+1}) - \frac{K}{2}.$$

Thus

$$\limsup_{n\to\infty} f(\gamma_n(t_n)) \leq f(r_{i+1}) - \frac{K}{2} < f(r_{i+1}). \tag{4.2}$$

However, since $\hat{\gamma}_{i+1}(-\infty) = r_{i+1}$, and $(t_n - s_n^{i+1}) \to -\infty$, by Lemma 4.3, we have

$$\liminf_{n\to\infty} f(\gamma_n(t_n)) \geq f(r_{i+1}), \tag{4.3}$$

which is a contradiction. □

Proof of Theorem 3.2. (1). By assumption, there are only finite many critical points in $M^{f(q),f(p)}$. We can find two critical points p' and q', a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that x_{n_k} is on γ_{n_k}, $\gamma_{n_k}(-\infty) = p'$, $\gamma_{n_k}(+\infty) = q'$ and γ_{n_k} is a component of Γ_{n_k}. Clearly, a generalized flow line connecting p' and q' can be extended to one connecting p and q. If there is a cluster point of $\{x_{n_k}\}_{k=1}^\infty$ on a generalized flow line connecting p' and q', this cluster point is also on one connecting p and q. So we may assume that x_n is on γ_n, $\gamma_n(-\infty) = p$ and $\gamma_n(+\infty) = q$.

If $p = q$, this is obviously true. Now we assume $p \neq q$. Suppose $\gamma_n(t_n) = x_n$. Since the conclusion of Theorem 3.1 holds, choosing a subsequence if necessary, we can find $s_n^0 < \cdots < s_n^l$ such that $\lim_{n\to\infty} \gamma_n(s_n^i) = \hat{\gamma}_i(0)$, where $\hat{\gamma}_i(-\infty) = r_i$, $\hat{\gamma}_i(+\infty) = r_{i+1}$, and $r_0 = p$, $r_{l+1} = q$.

Choosing a subsequence again if necessary, we can find a fixed i such that, for all n, we have $t_n \in [s_n^i, s_n^{i+1}]$, where $s_n^{-1} = -\infty$ and $s_n^{l+1} = +\infty$. In addition, we may assume there are exactly the following three cases when $n \to \infty$. By Proposition 4.4, we have:

Case (a): $\lim_{n\to\infty}(t_n - s_n^i) = \tau < +\infty$. Then x_n converges to a point on $\hat{\gamma}_i$;

Case (b): $\lim_{n\to\infty}(s_n^{i+1}-t_n)=\tau<+\infty$. Then x_n converges to a point on $\hat\gamma_{i+1}$;

Case (c): $\lim_{n\to\infty}(t_n-s_n^i)=\lim_{n\to\infty}(s_n^{i+1}-t_n)=+\infty$. Then x_n converges to $r_{i+1}=\hat\gamma_i(+\infty)=\hat\gamma_{i+1}(-\infty)$.

This completes the proof of the first result.

(2). Since the limit of $\{x_n^i\}$ exists, its subsequences share the same limit with it. So we only need to check the limit of a subsequence of $\{x_n^i\}$. Since there are only finitely many critical points in $M^{f(q),f(p)}$, we may argue as in (1): choosing a subsequence if necessary, we may assume $\Gamma_n=(\gamma_{n,1},\cdots,\gamma_{n,m})$, $\gamma_{n,j}(-\infty)=r_j$ and $\gamma_{n,j}(+\infty)=r_{j+1}$ are fixed and independent of n. In addition, $\forall i$, there is a fixed j such that for all n, x_n^i is on $\gamma_{n,j}$. If $r_j=r_{j+1}$, then $\gamma_{n,j}$ converges to the constant flow connecting r_j and r_j. Otherwise, choosing a subsequence again if necessary, $\{\gamma_{n,j}\}_{n=1}^\infty$ converges to a generalized flow line connecting r_j and r_{j+1}. The combination of the limits of $\{\gamma_{n,j}\}_{n=1}^\infty$ for $j=1,\cdots,m$ yields a generalized flow line, Γ, connecting p and q. By an argument similar to that of (1), the limits of all $\{x_n^i\}$ are on Γ. □

5. Manifold Structure

5.1. **Different Viewpoints on Compactified Spaces.** If $\alpha\in\mathcal{M}_I\subseteq\overline{\mathcal{M}(p,q)}$, then $\alpha=(\gamma_0,\cdots,\gamma_k)$, where $\gamma_i\in\mathcal{M}(r_i,r_{i+1})$, $r_0=p$ and $r_{k+1}=q$. Denote the constant flow line passing through r_i by $\beta(r_i)$. We can identify α with the generalized flow line $(\beta(r_0),\gamma_0,\beta(r_1),\cdots,\gamma_k,\beta(r_{k+1}))$ connecting p and q. Thus we get

(5.1)
$$\overline{\mathcal{M}(p,q)}=\{\Gamma\mid\Gamma \text{ is a generalized flow line connecting } p \text{ and } q\}.$$

Suppose $(\alpha,x)\in\mathcal{M}_I\times\mathcal{D}(r_k)\subseteq\overline{\mathcal{D}(p)}$. We can identify α with a generalized flow line connecting p and r_k. Adding the flow line passing through x to the above generalized flow line, we get a generalized flow line connecting p and x. The latter generalized flow line is uniquely determined by (α,x). Thus we get

(5.2)
$$\overline{\mathcal{D}(p)}=\{(\Gamma,x)\mid\Gamma \text{ is a generalized flow line connecting } p \text{ and } x\}.$$

Similarly, we also get

$$\overline{\mathcal{W}(p,q)}=\{(\Gamma,x)\mid\Gamma\in\overline{\mathcal{M}(p,q)},\ x \text{ is on } \Gamma\}. \tag{5.3}$$

From the above viewpoint, $\Gamma\in\mathcal{M}(p,q)$, $(\Gamma,x)\in\mathcal{D}(p)$ (or $\mathcal{W}(p,q)$) if and only if Γ has no intermediate critical points.

By (5.2), we can see that the evaluation map e in (3) of Theorem 3.4 is just defined by $e(\Gamma,x)=x$. If $\Gamma_1\in\overline{\mathcal{M}(p,r)}$ and $\Gamma_2\in\overline{\mathcal{M}(r,q)}$, then the combination of Γ_1 and Γ_2 gives an element in $\overline{\mathcal{M}(p,q)}$. If $\Gamma_1 \in \overline{\mathcal{M}(p,r)}$ and $(\Gamma_2, x) \in \overline{\mathcal{D}(r)}$, then the combination of Γ_1 and Γ_2 is a generalized flow line connecting p and x. These precisely define the maps $i_{(p,r,q)}$ in Theorem 3.3 and $i_{(p,r)}$ in Theorem 3.5.

Similarly, if $(\Gamma_1, x) \in \overline{\mathcal{W}(p,r)}$ and $\Gamma_2 \in \overline{\mathcal{M}(r,q)}$, then the combination of Γ_1 and Γ_2 gives an element in $\overline{\mathcal{M}(p,q)}$ and x is on it. This defines the map $i^1_{(p,r,q)}$ in (4) of Theorem 3.5. $i^2_{(p,r,q)}$ is defined in a similar way. The map e in (3) of Theorem 3.5 is defined as $e(\Gamma, x) = x$. The restriction of e on $\mathcal{W}_{I,s} = \mathcal{M}_{I_1} \times \mathcal{W}(r_s, r_{s+1}) \times \mathcal{M}_{I_2}$ is just the coordinate projection onto $\mathcal{W}(r_s, r_{s+1})$.

Figure 3 shows a standard example on a torus $T^2 = S^1 \times S^1$. Consider S^1 as the unit circle on the complex plane. Define a Morse function on T^2 by $f(z_1, z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$. f has 4 critical points p, r, s and q. Their indices are 2, 1, 1 and 0 respectively. Equip T^2 with the standard metric. The left part of Figure 3 shows the flow on T^2, where the opposite sides of the square are identified with each other. The right part is $\overline{\mathcal{D}(p)}$. $\overline{\mathcal{D}(p)}$ is an octagon. $\mathcal{M}(p,r) \times \mathcal{D}(r)$ (or $\mathcal{M}(p,s) \times \mathcal{D}(s)$) consists of open edges containing r_i (or s_i), where $i = 1, 2$. $\mathcal{M}(p,q) \times \mathcal{D}(q)$ consists of the other 4 open edges. $(\mathcal{M}(p,r) \times \mathcal{M}(r,q) \times \mathcal{D}(q)) \cup (\mathcal{M}(p,s) \times \mathcal{M}(s,q) \times \mathcal{D}(q))$ consists of the 8 vertices. e maps r_i (or s_i) to r (or s).

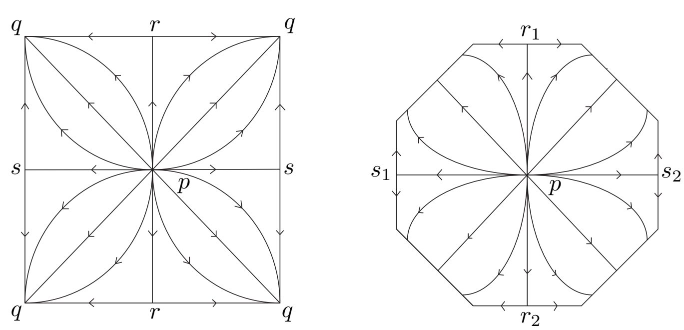

FIGURE 3. Compactification of the Descending Manifolds

5.2. **A Remark on the Literature.** To the best of my knowledge, in the case of a general metric, there is no well developed theory of smooth structures on these compactified spaces. In addition, few papers in the literature study $\overline{\mathcal{W}(p,q)}$.

When M is finite dimensional and the metric is locally trivial, the papers [20, prop. 2.11] and [8, thm. 1] study $\overline{\mathcal{M}(p,q)}$ and $\overline{\mathcal{D}(p)}$. We extend the proof in [8] to the infinite dimensional CF case. Actually [8] proves (1) and (2) of Theorem 3.3 and (1), (2), (3) of Theorem 3.4 except for the face structures. The book [7] contains a proof of the face structures which is different from the proof in this paper. Although (3) and (4) of Theorem 3.3 and (4) of Theorem 3.4 are not pointed out in [8], they are straightforward results from the geometric construction in that proof. Despite the infinite dimensions, our main geometric constructions to prove the smooth structures follow those in [8] except that Corollary 2.5 is elementary in the finite dimensional case. The really big difference between our proof and [8] is to prove the compactness of these manifolds. When M is finite dimensional, both [8, prop. 3] and [35, prop. 2.35] consider generalized flow lines as maps from an interval to M and prove compactness by the Arzela-Ascoli Theorem. However, the Arzela-Ascoli Theorem does not hold when M is infinite dimensional. Our proof is based on Theorems 3.1 and 3.2.

Modifying the geometric construction in [8], we get a proof of Theorem 3.5.

The paper [8] explains its geometric constructions clearly. However, these geometric constructions are important for our proofs of other results (see Lemmas 5.4, 5.6, 6.1, 6.2, 6.3, 7.1, 7.6 and 7.15 and Example 3.1). For the completeness of this paper, we explain the main constructions in [8] again.

5.3. **Preparation Lemmas.** The following two lemmas, Lemmas 5.1 and 5.2 are crucial for our proof. Example 3.1 shows that they necessarily depend on the local triviality of the metric. These two lemmas are announced in [8, observations 8 and 9]. A proof for them in the finite dimensional case is given in [7]. For the importance of them, we present a proof which follows that in [7].

Figure 1 gives an illustration for the following argument. Suppose c is a critical value of f. The critical points with function value c are exactly $p_1, \cdots, p_n$. Just as (2.2), we have diffeomorphisms $h_i : B_i(\epsilon) \longrightarrow U_i$ such that (2.3) and (2.4) hold, where $B_i(\epsilon)$ is the open subset of $T_{p_i}M$ and U_i is the neighborhood of p_i. Choose ϵ small enough such that there is no critical value in $[c-\epsilon, c+\epsilon]$ other than c. Let $M_c^+ = \{x \in M \mid f(x) = c + \frac{1}{2}\epsilon\}$ and $M_c^- = \{x \in M \mid f(x) = c - \frac{1}{2}\epsilon\}$. Let

$$P_c = \{(x^+, x^-) \in M_c^+ \times M_c^- \mid x^+ \text{ and } x^- \text{ are connected by a generalized flow line}\}.$$

Clearly, x^+ and x^- are connected by broken generalized flow lines if and only if $(x^+, x^-) \in \bigsqcup_{p_i} S_{p_i}^+ \times S_{p_i}^-$, where $S_{p_i}^+$ and $S_{p_i}^-$ are $\mathcal{A}(p_i) \cap M_c^+$ and $\mathcal{D}(p_i) \cap M_c^-$ respectively. Suppose the smallest (largest) critical value greater (smaller) than c is c_+ (c_-). Here $c_\pm$ may be $\pm\infty$. Define $M(c) = f^{-1}((c_-, c_+))$. Let

$$Q_c^+ = \{(x^+, z) \in M_c^+ \times M(c) \mid x^+ \text{ and } z \text{ are connected by a generalized flow line}\}.$$

$$Q_c^- = \{(z, x^-) \in M(c) \times M_c^- \mid x^- \text{ and } z \text{ are connected by a generalized flow line}\}.$$

Then $x^\pm$ and z are connected by broken generalized flow lines if and only if $(x^+, z) \in \bigsqcup_{p_i} S_{p_i}^+ \times D_{p_i}$ and $(z, x^-) \in \bigsqcup_{p_i} A_{p_i} \times S_{p_i}^-$ respectively, where $D_{p_i} = \mathcal{D}(p_i) \cap M(c)$ and $A_{p_i} = \mathcal{A}(p_i) \cap M(c)$.

Lemma 5.1. *Suppose the metric is locally trivial. Then P_c is a smoothly embedded submanifold with boundary $\bigsqcup_{p_i} S_{p_i}^+ \times S_{p_i}^-$ of $M_c^+ \times M_c^-$.*

Proof. There is no essential difference in the proof between the case of one critical point and that of several critical points. For convenience, we may assume there is only one critical point p in $M^{c-\epsilon, c+\epsilon}$. We shall prove that P_c is a smooth embedding submanifold with boundary $S_p^+ \times S_p^-$ of $M_c^+ \times M_c^-$.

Firstly, we shall prove $P_c - S_p^+ \times S_p^-$ is a smoothly embedded submanifold of $M_c^+ \times M_c^-$.

Since it is an open subset of M_c^+, $M_c^+ - S_p^+$ is a smooth submanifold of M_c^+. By Corollary 2.5, we can define the flow map $\psi : M_c^+ - S_p^+ \longrightarrow M_c^- - S_p^-$. Define $\varphi : M_c^+ - S_p^+ \longrightarrow M_c^+ \times M_c^-$ by $\varphi(x_+) = (x_+, \psi(x_+))$. Clearly, φ is smooth and $\mathrm{Im}(\varphi) = P_c - S_p^+ \times S_p^-$. Define $\pi_+ : M_c^+ \times M_c^- \longrightarrow M_c^+$ to be the natural projection. We have π_+ is smooth and $\pi_+ \varphi = \mathrm{Id}$, so φ is a homeomorphism to its image. Since $d\pi_+ d\varphi = \mathrm{Id}$, $d\varphi$ is an isomorphism to its image. Thus $P_c - S_p^+ \times S_p^- = \mathrm{Im}(\varphi)$ is a smooth manifold of $M_c^+ \times M_c^-$.

Secondly, we shall prove that there is an open neighborhood W of $S_p^+ \times S_p^-$ in $M_c^+ \times M_c^-$ such that $W \cap P_c$ is a smooth submanifold with boundary $S_p^+ \times S_p^-$ in W.

By local triviality of the metric, there is a diffeomorphism $h : B \longrightarrow U$ given by (2.2) which satisfies (2.3) and (2.4). For convenience, we identify B with U. Then $S_p^+ = \{(0, v_2) \mid \|v_2\|^2 = \epsilon\}$, $S_p^- = \{(v_1, 0) \mid \|v_1\|^2 = \epsilon\}$, $M_c^+ \cap U = \{(v_1, v_2) \mid \|v_1\|^2 < 2\epsilon$, and $\|v_2\|^2 < 2\epsilon, -\|v_1\|^2 + \|v_2\|^2 = \epsilon\}$ and $M_c^- \cap U = \{(v_1, v_2) \mid \|v_1\|^2 < 2\epsilon$, and $\|v_2\|^2 < 2\epsilon, -\|v_1\|^2 + \|v_2\|^2 = -\epsilon\}$. Let $U_+ = \{(v_1, v_2) \mid \|v_1\|^2 < \epsilon$ and $\frac{\epsilon}{2} < \|v_2\|^2 < 2\epsilon.\}$ and $U_- = \{(v_1, v_2) \mid \|v_2\|^2 < \epsilon$ and $\frac{\epsilon}{2} < \|v_1\|^2 < 2\epsilon.\}$. Then $U_+ \times U_-$ is an open neighborhood of $S_p^+ \times S_p^-$ in $M^{c-\epsilon, c+\epsilon} \times M^{c-\epsilon, c+\epsilon}$. For convenience, we identify S_p^- with $\{v_1 \mid (v_1, 0) \in S_p^-\}$ and S_p^+ with $\{v_2 \mid (0, v_2) \in S_p^+\}$.

Consider the map $\varphi : S_p^+ \times S_p^- \times [0,1) \longrightarrow U_+ \times U_-$ satisfying

$$\varphi(v_2, v_1, s) = \Big((sv_1, (1+s^2)^{\frac{1}{2}} v_2), ((1+s^2)^{\frac{1}{2}} v_1, sv_2) \Big).$$

Clearly, φ is smooth, $\mathrm{Im}(\varphi) = P_c \cap (U_+ \times U_-)$ and $\varphi|_{S_p^+ \times S_p^-} = \mathrm{Id}$. On the other hand, consider the map $\alpha : U_+ \times U_- \longrightarrow S_p^+ \times S_p^- \times [0,1)$ satisfying

$$\alpha((z_1, z_2), (z_3, z_4)) = \left(\epsilon^{\frac{1}{2}} \frac{z_2}{\|z_2\|}, \epsilon^{\frac{1}{2}} \frac{z_3}{\|z_3\|}, \epsilon^{-\frac{1}{2}} \|z_1\| \right).$$

Then α is continuous and $\alpha\varphi = \mathrm{Id}$. In addition, α is smooth when $z_1 \neq 0$. Then φ is a homeomorphism to its image, and $d\varphi$ is an isomorphism onto its image when $s \neq 0$.

Now we consider the case of $s = 0$. We shall prove that $d\varphi|_{s=0}$ is an isomorphism onto its image. It suffices to prove that there exists $\lambda > 0$, for all $v \in T(S_p^+ \times S_p^- \times [0,1))$, such that

$$\|d\varphi \cdot v\| \geq \lambda \|v\|. \tag{5.4}$$

Let $\frac{\partial}{\partial s}$ be the positive unit tangent vector of $[0,1)$, e_2 and e_1 are tangent vectors of S_p^+ and S_p^-. Then

$$d\varphi|_{s=0}\left(\frac{\partial}{\partial s}\right) = (v_1, 0, 0, v_2), \quad d\varphi|_{s=0}(e_1) = (0, 0, e_1, 0), \quad d\varphi|_{s=0}(e_2) = (0, e_2, 0, 0).$$

It's easy to see (5.4) holds.

Thus φ is a smooth embedding into $U_+ \times U_-$. Let $W = (U_+ \times U_-) \cap (M_c^+ \times M_c^-)$. Then W is an open neighborhood of $S_p^+ \times S_p^-$ in $M_c^+ \times M_c^-$, $P_c \cap W = \mathrm{Im}(\varphi)$ and $P_c \cap W$ is a smoothly embedded submanifold with boundary $S_p^+ \times S_p^-$. □

Lemma 5.2. *Suppose the metric is locally trivial. Then Q_c^+ (Q_c^-) is a smoothly embedded submanifold with boundary $\bigsqcup_{p_i} S_{p_i}^+ \times D_{p_i}$ ($\bigsqcup_{p_i} A_{p_i} \times S_{p_i}^-$) of $M_c^+ \times M(c)$ ($M(c) \times M_c^-$).*

Proof. We only need to prove the case of Q_c^+.

Let $\tilde{Q}_c^+ = \{(x^+, z) \in Q_c^+ \mid f(z) \in (c_i - \frac{\epsilon}{2}, c_i + \frac{\epsilon}{2})\}$. If we shrink $M(c)$ by an isotopy along flow lines, we get a diffeomorphism from $M(c)$ to $f^{-1}((c_i - \frac{\epsilon}{2}, c_i + \frac{\epsilon}{2}))$. This diffeomorphism preserves flow lines. Thus it induces a diffeomorphism from Q_c^+ to $\tilde{Q}_c^+$. Then we only need to prove that $\tilde{Q}_c^+$ is a submanifold of $M_c^+ \times f^{-1}((c_i - \frac{\epsilon}{2}, c_i + \frac{\epsilon}{2}))$. We can therefore assume $M(c) = f^{-1}((c_i - \frac{\epsilon}{2}, c_i + \frac{\epsilon}{2}))$.

The proof is very similar to that of Lemma 5.1. We assume there is only one critical point in $M(c)$.

Firstly, we prove $Q_c^+ - S_p^+ \times D_p$ is a smooth embedding submanifold of $M_c^+ \times M(c)$. There is a smooth map $\varphi : M(c) - D_p \longrightarrow M_c^+ \times M(c)$ such that $\varphi(z) = (\psi(z), z)$, where ψ is the flow map from $M(c) - D_p$ to M_c^+. Similarly to Lemma 5.1, φ is also a smooth embedding. This gives the proof.

Secondly, we shall find an open neighborhood W of $S_p^+ \times D_p$ such that $Q_c^+ \cap W$ is a smoothly embedded submanifold with boundary $S_p^+ \times D_p$ of $M_c^+ \times M(c)$.

Just as the proof of Lemma 5.1, we use the same notation of h, B, U and U_+, identify U with B, and we define $\widetilde{U}_- = \{(v_1, v_2) \mid \|v_2\|^2 < \epsilon$ and $\|v_1\|^2 < 2\epsilon.\}$. Define $\varphi : S_p^+ \times D_p \times [0, 1) \longrightarrow U_+ \times \widetilde{U}_-$ by

$$\varphi(v_2, v_1, s) = \Big((sv_1, (s^2\|v_1\|^2 + \epsilon)^{\frac{1}{2}} \epsilon^{-\frac{1}{2}} v_2), (v_1, s(s^2\|v_1\|^2 + \epsilon)^{\frac{1}{2}} \epsilon^{-\frac{1}{2}} v_2)\Big).$$

Define $\alpha : U_+ \times \widetilde{U}_- \longrightarrow S_p^+ \times D_p \times [0, 1)$ by

$$\alpha((z_1, z_2), (z_3, z_4)) = \left(\epsilon^{\frac{1}{2}} \frac{z_2}{\|z_2\|}, z_3, \frac{\|z_4\|}{\|z_2\|}\right).$$

Then $\alpha\varphi = \mathrm{Id}$. Similar to the proof of Lemma 5.1, φ is a homeomorphism. And $d\varphi$ is an isomorphism to its image when $s \neq 0$. When $s = 0$,

$$d\varphi|_{s=0}\left(\frac{\partial}{\partial s}\right) = (v_1, 0, 0, v_2), \tag{5.5}$$

$$d\varphi|_{s=0}(e_1) = (0, 0, e_1, 0), \qquad d\varphi|_{s=0}(e_2) = (0, e_2, 0, 0),$$

and $d\varphi$ is also an isomorphism to its image. Thus φ is a smooth embedding. Let $W = (U_+ \times \widetilde{U}_-) \cap (M_c^+ \times M(c))$. This finishes the proof. ☐

We shall cut out a submanifold with corners from a manifold with corners. This requires a result about transversality. (See [27, II. E] for more details about transversality on Hilbert manifolds.) First we recall a classical result about manifold with boundary. Suppose L is a Hilbert manifold with boundary, and N_1 and N_2 are Hilbert manifolds. Assume N_2 is an embedded submanifold of N_1. Suppose $g : L \longrightarrow N_1$ is a smooth manifold transversal to N_2 both in $L^\circ = L - \partial L$ and in ∂L. Then $g^{-1}(N_2)$ is an embedded submanifold with boundary inside L, and $\partial g^{-1}(N_2) = g^{-1}(N_2) \cap \partial L$. Now we extend this result to the product of manifolds with boundary. Suppose L_i ($i = 1, \cdots, n$) are Hilbert manifolds with boundary. Then $\prod_{i=1}^n L_i$ is a Hilbert manifold with corners. Its k-stratum is just $\partial^k \prod_{i=1}^n L_i = \bigsqcup_{|\Lambda|=k} (\prod_{i\in\Lambda} \partial L_i \times \prod_{i\notin\Lambda} L_i^\circ)$, where Λ is a subset of $\{1, \cdots, n\}$. The above extends Definitions 2.13

and 2.15. We have the following result, whose proof is a straightforward extension of that in the case of a manifold with boundary.

Lemma 5.3. *If $g : \prod_{i=1}^n L_i \longrightarrow N_1$ is transversal to N_2 in each stratum of $\prod_{i=1}^n L_i$, then $g^{-1}(N_2)$ is a smoothly embedded submanifold with corners of $\prod_{i=1}^n L_i$ such that $\partial^k g^{-1}(N_2) = g^{-1}(N_2) \cap \partial^k \prod_{i=1}^n L_i$.*

5.4. **Proof of Theorem 3.3.**

Proof. (1) & (2). We only prove the corner structure now. The face structure will follow from (4). Suppose the critical values in $[f(q), f(p)]$ are exactly $c_{l+1} < \cdots < c_1 < c_0$, where $c_0 = f(p)$ and $c_{l+1} = f(q)$. Define

$$P = \prod_{i=1}^{l} P_i, \quad R = S_p^- \times \prod_{i=1}^{l-1} M_i^- \times S_q^+, \quad O = \prod_{i=1}^{l} (M_i^+ \times M_i^-).$$

Here $P_i = P_{c_i}$, $M_i^+ = M_{c_i}^+$, $M_i^- = M_{c_i}^-$, $S_p^- = \mathcal{D}(p) \cap M_1^+$ and $S_q^+ = \mathcal{A}(q) \cap M_l^-$ are defined as before Lemma 5.1. By Lemma 5.1, P is a manifold with corners whose k-stratum is exactly the disjoint union of $\prod_{i=1}^k (S_{r_i}^+ \times S_{r_i}^-) \times \prod_{j \notin \Lambda_I} P_j^\circ$, where $I = \{p, r_1, \cdots, r_k, q\}$ is a critical sequence and $\Lambda_I = \{j \mid c_j = f(r_i), i = 1, \cdots, k\}$. Clearly, P is a submanifold of O, so there is an inclusion $\iota : P \longrightarrow O$. On the other hand, define a smooth embedding $\Delta : R \longrightarrow O$ as follows. Since there is no critical point in $M^{c_{i+1}+\frac{\epsilon}{2}, c_i - \frac{\epsilon}{2}}$, by Corollary 2.5, we have a flow map $\psi_i : M_i^- \longrightarrow M_{i+1}^+$. Define

$$\Delta(y_0^-, y_1^-, \cdots, y_{l-1}^-, y_{l+1}^+) = (\psi_0 y_0^-, y_1^-, \psi_1 y_1^-, \cdots, y_{l-1}^-, \psi_{l-1} y_{l-1}^-, \psi_{l+1}^{-1} y_{l+1}^+).$$

Now we point out that ι is transversal to Δ in each stratum of P. When M is compact, transversality is proved by [8, thm. 1]. (The paper [8] uses different notations from ours. Its $\mathcal{P}$, $\mathcal{S}$ and $\mathcal{O}$ are our P, R and O respectively. Its maps p and s are our ι and Δ respectively.) The proof needs Corollary 2.5 which is trivial in the compact case. Our proof of the transverality duplicates that in [8], so we omit it.

Denote $K = \iota^{-1}(\mathrm{Im}(\Delta))$. By Lemma 5.3, K is a smoothly embedded submanifold of P whose k-stratum is exactly the intersection of K with the k-stratum of P.

Now we identify the strata of K with the disjoint unions of $\mathcal{M}(p, r_1) \times \mathcal{M}(r_1, r_2) \times \cdots \times \mathcal{M}(r_k, q)$ as smooth manifolds.

It's easy to see that

$$(5.6)\quad K = \{(x_1^+, x_1^-, \cdots, x_l^+, x_l^-) \in O \mid x_i^\pm \ (1 \le i \le l) \text{ are on a same generalized flow line connecting } p \text{ and } q\}.$$

Let $I = \{p, r_1, \cdots, r_k, q\}$ be a critical sequence. For all $\Gamma \in \mathcal{M}_I$ (see (5.1)), Γ intersects $M_i^{\pm}$ at exactly one point $x_i^{\pm}(\Gamma)$. Thus there is also an evaluation map $\tilde{E}_I : \mathcal{M}_I \longrightarrow O$ such that $\tilde{E}_I(\Gamma) = (x_1^+(\Gamma), \cdots, x_l^-(\Gamma))$. Clearly, $\tilde{E}_I$ is a smooth embedding, and $\mathrm{Im}(\tilde{E}_I)$ is exactly $K \cap (\prod_{i=1}^k (S_{r_i}^+ \times S_{r_i}^-) \times \prod_{j \notin \Lambda_I} P_j^\circ)$ which is an open subset of the k-stratum of K. This gives an identification preserving smooth structures.

As a result, identifying $\overline{\mathcal{M}(p,q)}$ with K, we give $\overline{\mathcal{M}(p,q)}$ a smooth structure which is compatible with the smooth structure of $\prod_{i=0}^k \mathcal{M}(r_i, r_{i+1})$ for all critical sequences and its k-stratum is exactly $\bigsqcup_{|I|=k} \mathcal{M}_I$.

Now we prove the compactness of $\overline{\mathcal{M}(p,q)}$.

By (5.6), for all $\{x_n\}_{n=1}^\infty \subseteq K$, $x_n = (x_{n,1}^+, x_{n,1}^-, \cdots, x_{n,l}^+, x_{n,l}^-)$, $x_{n,i}^\pm \in M_i^\pm$ and are on a same generalized flow line connecting p and q. By Theorem 3.2, $\{x_n\}$ has a cluster point $x_0 = (x_1^+, x_1^-, \cdots, x_l^+, x_l^-)$, and $x_i^\pm$ are on a same generalized flow line connecting p and q. Since $M_i^\pm$ is closed, $x_i^\pm \in M_i^\pm$ or $x_0 \in K$. So K and then $\overline{\mathcal{M}(p,q)}$ are compact.

(3). Since a_i, $c_i - \frac{\epsilon}{2}$ and $c_{i+1} + \frac{\epsilon}{2}$ are in (c_{i+1}, c_i) and there is no critical value in (c_{i+1}, c_i), by Corollary 2.5, the flow map gives a smooth map from $f^{-1}(a_i)$ to $M_i^- \times M_{i+1}^+$. This induces a map $\varphi : \prod_{i=0}^l f^{-1}(a_i) \longrightarrow O$. Clearly, $\varphi \circ E : \overline{\mathcal{M}(p,q)} \longrightarrow O$ is exactly the inclusion if we identify $\overline{\mathcal{M}(p,q)}$ with K. So $\varphi \circ E$ and then E are smooth embeddings.

(4). Suppose $f(r) = c_k$. By (3), we have the following commutative diagram. Here $E_{p,q}$, $E_{p,r} : \overline{\mathcal{M}(p,r)} \longrightarrow \prod_{i=0}^{k-1} f^{-1}(a_i)$, and $E_{r,q} : \overline{\mathcal{M}(r,q)} \longrightarrow \prod_{i=k}^l f^{-1}(a_i)$ are evaluation maps.

$$\begin{tikzcd}
\overline{\mathcal{M}(p,r)} \times \overline{\mathcal{M}(r,q)} \arrow[r, "E_{p,r} \times E_{r,q}"] \arrow[d, "i_{(p,r,q)}"'] & \prod_{i=0}^l f^{-1}(a_i) \\
\overline{\mathcal{M}(p,q)} \arrow[ur, "E_{p,q}"'] &
\end{tikzcd}$$

Also by (3), the above three evaluation maps are smooth embeddings. Then so is $i_{(p,r,q)}$. This completes the proof of (4).

Finally, we establish the face structure of $\overline{\mathcal{M}(p,q)}$. Suppose x is in the k-stratum. Then $x \in \mathcal{M}_I$ for some $I = \{p, r_1, \cdots, r_k, q\}$. Thus $x \in \overline{\mathcal{M}(p,r_i)} \times \overline{\mathcal{M}(r_i,q)}$ for $i = 1, \cdots, k$. Clearly, $\mathcal{M}(p,r_i) \times \mathcal{M}(r_i,q)$ are k pairwise disjoint faces. We only need to prove that their closures are $\overline{\mathcal{M}(p,r_i)} \times \overline{\mathcal{M}(r_i,q)}$ respectively. On the one hand, since it is compact, $\overline{\mathcal{M}(p,r_i)} \times \overline{\mathcal{M}(r_i,q)}$ contains the closure of $\mathcal{M}(p,r_i) \times \mathcal{M}(r_i,q)$ in $\overline{\mathcal{M}(p,q)}$. On the other hand, as $\mathcal{M}(p,r_i) \times \mathcal{M}(r_i,q)$ is the 0-stratum (the interior) of $\overline{\mathcal{M}(p,r_i)} \times \overline{\mathcal{M}(r_i,q)}$, we infer that the closure of $\mathcal{M}(p, r_i) \times \mathcal{M}(r_i, q)$ contains $\overline{\mathcal{M}(p, r_i)} \times \overline{\mathcal{M}(r_i, q)}$. Thus the closure is exactly $\overline{\mathcal{M}(p, r_i)} \times \overline{\mathcal{M}(r_i, q)}$. $\square$

5.5. **Proof of Theorem 3.4.** Since f is lower bounded, by Theorem 2.1, there are only finite number of critical values in $(-\infty, f(p)]$. Suppose they are $c_l < \cdots < c_0 = f(p)$. Denote $M(c_i)$ by $M(i)$, P_{c_i} by P_i and $Q^+_{c_i}$ by Q^+_i, where $M(c_i)$, P_{c_i} and $Q^+_{c_i}$ are as defined before Lemma 5.1. Define $U(i) \subseteq \overline{\mathcal{D}(p)}$ as $U(i) = e^{-1}(M(i))$.

Proof. (1), (2) & (3). We shall give each $U(i)$ a smooth structure, and show that $U(i) \cap U(j)$ is open in both $U(i)$ and $U(j)$ and smooth structures are compatible in $U(i) \cap U(j)$.

Firstly, when $i = 0$, $U(0)$ is identified with $\mathcal{D}(p) \cap M(0)$. $\mathcal{D}(p) \cap M(0)$ is a smooth embedded submanifold of M. Thus $U(0)$ has a smooth structure by this identification.

Secondly, when $i > 0$, let $Q(i) = \prod_{j=1}^{i-1} P_j \times Q^+_i$, $O(i) = \prod_{j=1}^{i-1}(M^+_j \times M^-_j) \times M^+_i$ and $R(i) = S^-_p \times \prod_{j=1}^{i-1} M^-_j$. We know that, $\forall x \in Q(i)$, $x = (x^+_1, x^-_1, \cdots, x^-_{i-1}, x^+_i, z_i)$, where $x^\pm_j \in M^\pm_j$ and $z_i \in M(i)$. Define a smooth map $\iota_i : Q(i) \longrightarrow O(i)$ by

$$\iota_i(x^+_1, x^-_1, \cdots, x^-_{i-1}, x^+_i, z_i) = (x^+_1, x^-_1, \cdots, x^-_{i-1}, x^+_i).$$

Define a smooth embedding $\Delta_i : R(i) \longrightarrow O(i)$ by

$$\Delta_i(y^-_0, y^-_1, \cdots, y^-_{i-1}) = (\psi_0 y^-_0, y^-_1, \psi_1 y^-_1, \cdots, y^-_{i-1}, \psi_{i-1} y^-_{i-1}),$$

where ψ_j is the flow map from M^-_j to M^+_{j+1}.

As in the proof of Theorem 3.3, we point out that ι_i is transversal to Δ_i in each stratum of $Q(i)$. The proof is similar to that of Theorem 3.3.

Thus $\tilde{U}(i) = \iota_i^{-1}(\mathrm{Im}(\Delta_i))$ is a smooth embedding submanifold of $Q(i)$ whose k-stratum is exactly the intersection of $\tilde{U}(i)$ with the k-stratum of $Q(i)$.

Now we identify $U(i)$ with $\tilde{U}(i)$. It's easy to see that

$$\begin{aligned} \tilde{U}(i) \quad = \quad & \{(x^+_1, x^-_1, \cdots, x^-_{i-1}, x^+_i, z_i) \in O(i) \times M(i) \mid x^\pm_j \\ & \text{are on a same generalized flow line connecting } p \text{ and } z_i.\}. \end{aligned} \tag{5.7}$$

Let $I = (p, r_1, \cdots, r_k)$ be a critical sequence. For any element $(\Gamma, x) \in \mathcal{D}_I \cap U(i)$ (see (5.2)), Γ intersects $M^\pm_j$ at exactly one point $x^\pm_j(\Gamma)$. Thus there is an evaluation map $\tilde{E}_I : \mathcal{D}_I \cap U(i) \longrightarrow O(i) \times M(i)$ such that $\tilde{E}_I(\Gamma, x) = (x^+_1(\Gamma), x^-_1(\Gamma), \cdots, x^+_i(\Gamma), x)$. Similar to the identification of K with $\overline{\mathcal{M}(p, q)}$ in the proof of Theorem 3.3, this also identifies $U(i)$ with $\tilde{U}(i)$ and preserves the smooth structure of the strata. So we get a desired smooth structure on U_i.

In each $\tilde{U}(i)$, define $\tilde{e}_i : \tilde{U}(i) \longrightarrow M$ by $\tilde{e}_i(x_1^+, \cdots, x_i^+, z_i) = z_i$, then $\tilde{e}_i$ is smooth. When we identify $U(i)$ with $\tilde{U}(i)$, we have $e|_{U(i)} = \tilde{e}_i$. Thus $e|_{U(i)}$ is smooth.

Now we check the compatibility of smooth structures for all $U(i)$ $(0 \leq i \leq l)$. Clearly, if $|i-j| > 1$, then $U(i) \cap U(j) = \emptyset$. We only need to check the compatibility of $U(i)$ and $U(i+1)$.

Denote $M(i)^- = f^{-1}((c_{i+1}, c_i))$. For clarity, when we consider $U(i) \cap U(i+1)$ as a topological subspace of $U(i)$ (or $U(i+1)$), we denote it by $U(i, i+1)$ (or $U(i+1, i)$). Since $U(i, i+1) = e|_{U(i)}^{-1}(M(i)^-)$, it is an open subset of $U(i)$. Furthermore, $U(i+1, i)$ is an open subset of $U(i+1)$. When $i \geq 1$, $U(i, i+1) \subseteq \prod_{j=1}^{i-1}(M_j^+ \times M_j^-) \times M_i^+ \times M(i)^-$ and $U(i+1, i) \subseteq \prod_{j=1}^{i}(M_j^+ \times M_j^-) \times M_{i+1}^+ \times M(i)^-$. Define $\pi : \prod_{j=1}^{i}(M_j^+ \times M_j^-) \times M_{i+1}^+ \times M(i)^- \longrightarrow \prod_{j=1}^{i-1}(M_j^+ \times M_j^-) \times M_i^+ \times M(i)^-$ be the natural projection. Define $\varphi : \prod_{j=1}^{i-1}(M_j^+ \times M_j^-) \times M_i^+ \times M(i)^- \longrightarrow \prod_{j=1}^{i}(M_j^+ \times M_j^-) \times M_{i+1}^+ \times M(i)^-$ such that

$$\varphi(x_1^+, x_1^-, \cdots, x_{i-1}^-, x_i^+, z_i) = (x_1^+, x_1^-, \cdots, x_{i-1}^-, x_i^+, \psi_-(z_i), \psi_+(z_i), z_i),$$

where ψ_- and ψ_+ are flow maps from $M(i)^-$ to M_i^- and M_{i+1}^+ respectively. Then $\pi(U(i+1, i)) = U(i, i+1)$, $\varphi(U(i, i+1)) = U(i+1, i)$, $\pi\varphi|_{U(i,i+1)} = \mathrm{Id}$, and $\varphi\pi|_{U(i+1,i)} = \mathrm{Id}$. Thus π and φ are diffeomorphisms between $U(i, i+1)$ and $U(i+1, i)$, and they are the identity on the set $U(i) \cap U(i+1)$. Thus $U(i)$ and $U(i+1)$ have compatible smooth structures when $i \geq 1$.

Similarly, $U(0,1) \subseteq M(0)^-$ and $U(1,0) \subseteq M_1^+ \times M(0)^-$, and $U(0,1)$ and $U(1,0)$ also have compatible smooth structures.

As a result, we can patch the smooth structures on all $U(i)$ together to give a smooth structure on $\overline{\mathcal{D}(p)}$ satisfying all properties of (1) and (2) but the face structure and compactness. Similar to Theorem 3.3, the face structure will follow from (4). Also e is smooth since $e|_{U(i)}$ is smooth. This proves (3).

Finally, we prove compactness.

Let $K(i) = e^{-1}(L(i))$, where $L(i) = f^{-1}([\frac{c_{i+1}+c_i}{2}, \frac{c_i+c_{i-1}}{2}])$. Then $L(i)$ is closed. Similar to proving the compactness of K in the proof of Theorem 5.2, we get $K(i)$ is compact. Thus $\overline{\mathcal{D}(p)}$ is compact because $\overline{\mathcal{D}(p)} = \bigcup_{i=0}^{l} K(i)$.

This completes the proof of (1), (2) and (3).

(4). First, similar to the proof of (3) of Theorem 3.3, we can prove the following lemma.

Lemma 5.4. *Suppose the critical values in* $(-\infty, f(p)]$ *are* $c_l < \cdots < c_0$. *Let* $U(i) = e^{-1} \circ f^{-1}((c_{i+1}, c_{i-1}))$. *Define* $E(i) : U(i) \longrightarrow \prod_{j=0}^{i-1} f^{-1}(a_j)$

$\times M$ by $E(i)(\Gamma, x) = (x_0(\Gamma), \cdots, x_{i-1}(\Gamma), x)$ for any $(\Gamma, x) \in U(i)$ (see (5.2)), where $x_j(\Gamma)$ is the unique intersection of Γ with $f^{-1}(a_j)$. Then $E(i)$ is a smooth embedding for $i = 0, \cdots, l$. Here $c_{-1} = -\infty$ and $c_{l+1} = +\infty$.

Clearly, $i_{(p,r)}$ is one to one. Suppose $f(r) = c_m$. For clarity, denote the evaluation map from $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{D}(r)}$ to M by e_p and e_r respectively. Let $U_p(k) = e_p^{-1}(M(k))$ and $U_r(k) = e_r^{-1}(M(k))$. Since $\overline{\mathcal{M}(p,r)} \times \overline{\mathcal{D}(r)}$ is compact, we only need to prove that $i_{(p,r)}(k) : \overline{\mathcal{M}(p,r)} \times U_r(k) \longrightarrow \overline{\mathcal{D}(p)}$ is a smooth embedding for $k \geq m$.

By Lemma 5.4, $E_r(k) : U_r(k) \longrightarrow \prod_{j=m}^{k-1} f^{-1}(a_j) \times M$ and $E_p(k) : U_p(k) \longrightarrow \prod_{j=0}^{k-1} f^{-1}(a_j) \times M$ are smooth embeddings. By (3) of Theorem 3.3, we know that $E_{p,r} : \overline{\mathcal{M}(p,r)} \longrightarrow \prod_{j=0}^{m-1} f^{-1}(a_j)$ is also a smooth embedding. Thus $E_{p,r} \times E_r(k) : \overline{\mathcal{M}(p,r)} \times U_r(k) \longrightarrow \prod_{i=0}^{k-1} f^{-1}(a_i) \times M$ is a smooth embedding. In addition, $E_{p,r} \times E_r(k) = E_p(k) \circ i_{(p,r)}(k)$. Thus $i_{(p,r)}(k)$ is a smooth embedding.

Finally, $\overline{\mathcal{D}(p)}$ has the disjoint faces $\mathcal{M}(p,r) \times \mathcal{D}(r)$. Their closures are $\overline{\mathcal{M}(p,r)} \times \overline{\mathcal{D}(r)}$. This gives the face structure of (1). □

5.6. **Proof of Theorem 3.5.** The proof of Theorem 3.5 is a mixture of the proofs of Theorems 3.3 and 3.4. Thus we only need to give the key constructions in the proof. Just as the proofs of the previous two theorems, we still use the notation $M(i)$, P_i and $Q_i^{\pm}$. Suppose the critical values in $[f(q), f(p)]$ are exactly $f(q) = c_{l+1} < \cdots < c_0 = f(p)$. Define $U(i) \subseteq \overline{\mathcal{W}(p,q)}$ as $U(i) = e^{-1}(M(i))$. Use the notation of $S_p^{\pm}$, D_p and A_p as those appearing before Lemma 5.1.

Proof. Similarly to the proof of Theorem 3.4, we shall give each $U(i)$ a smooth structure and then patch them together.

Define $Q(0) = Q_0^- \times \prod_{j=1}^{l} P_j$, $R(0) = D_p \times \prod_{j=0}^{l-1} M_j^- \times S_q^+$ and $O(0) = M(0) \times M_0^- \times \prod_{j=1}^{l} (M_j^+ \times M_j^-)$. Define $\Delta_0 : R(0) \longrightarrow O(0)$ by

$$\Delta_0(z_0, y_0^-, \cdots, y_{l-1}^-, y_{l+1}^+) = (z_0, y_0^-, \psi_0 y_0^-, \cdots, y_{l-1}^-, \psi_{l-1} y_{l-1}^-, \psi_l^{-1} y_{l+1}^+).$$

Define $Q(l+1) = \prod_{j=1}^{l} P_j \times Q_{l+1}^+$, $R(l+1) = S_p^- \times \prod_{j=1}^{l} M_j^- \times A_q$, and $O(l+1) = \prod_{j=1}^{l} (M_j^+ \times M_j^-) \times M_{l+1}^+ \times M(l+1)$. Define $\Delta_{l+1} : R(l+1) \longrightarrow O(l+1)$ by

$$\Delta_{l+1}(y_0^-, \cdots, y_l^-, z_{l+1}) = (\psi_0 y_0^-, y_1^-, \psi_1 y_1^-, \cdots, y_l^-, \psi_l y_l^-, z_{l+1}).$$

When $1 \leq i \leq l$, define $Q(i) = \prod_{j=1}^{i-1} P_j \times Q_i^+ \times Q_i^- \times \prod_{j=i+1}^{l} P_j$, $R(i) = S_p^- \times \prod_{j=1}^{i-1} M_j^- \times M(i) \times \prod_{j=i}^{l-1} M_j^- \times S_q^+$ and $O(i) = \prod_{j=1}^{i-1} (M_j^+ \times$

$M_j^-) \times M_i^+ \times M(i) \times M(i) \times M_i^- \times \prod_{j=i+1}^{l}(M_j^+ \times M_j^-)$. Define $\Delta_i : R(i) \longrightarrow O(i)$ by

$$\begin{aligned} & \Delta_i(y_0^-, y_1^-, \cdots, y_{i-1}^-, z_i, y_i^-, \cdots, y_{l-1}^-, y_{l+1}^+) \\ = \ & (\psi_0 y_0^-, y_1^-, \psi_1 y_1^-, \cdots, y_{i-1}^-, \psi_{i-1} y_{i-1}^-, z_i, z_i, y_i^-, \psi_i y_i^-, \cdots, \psi_{l-1} y_{l-1}^-, \psi_l^{-1} y_{l+1}^+). \end{aligned}$$

In the above, ψ_i are flow maps from M_i^- to M_{i+1}^+.

Define $\iota_i : Q(i) \longrightarrow O(i)$ to be the inclusion for all $i = 0, \cdots, l+1$.

Similar to the proof of Theorems 3.3 and 3.4, ι_i is transversal to Δ_i in each stratum of $Q(i)$. Thus $\tilde{U}(i) = \iota_i^{-1}(\mathrm{Im}(\Delta_i))$ is a smooth manifold with corners. $U(i)$ can be identified with $\tilde{U}(i)$ and the smooth structures are preserved. This gives a smooth structure to each $U(i)$. $e|_{U(i)}$ is smooth, and $U(i)$ and $U(j)$ have compatible smooth structures. Thus e is smooth. The face structures will follow from (4). Let $L(i) = f^{-1}([\frac{c_{i+1}+c_i}{2}, \frac{c_i+c_{i-1}}{2}])$, then $e^{-1}(L(i))$ is compact. Thus $\mathcal{W}(p,q)$ is compact. This finishes the proof of (1), (2) and (3).

Finally, (4) is proved by an argument similar to that in (4) of Theorem 3.4.

This completes the proof. □

5.7. **Proof of Example 3.1.**

Proof. Clearly, there is a Morse function on CP^2 with such three critical points and $f(r) = 0$. By the Morse Lemma, in a neighborhood U of r, there is a local coordinate chart (v_1, v_2, v_3, v_4) such that r has the coordinate $(0,0,0,0)$, $\sum_{i=1}^4 v_i^2 < 4\epsilon^2$ and, in the local chart, we have $f(v) = \frac{1}{2}(-v_1^2 - v_2^2 + v_3^2 + v_4^2)$. We can choose f such that $\epsilon = 1$. We equip CP^2 with a metric such that, in U, it has the form

$$(dx_1)^2 + \frac{1}{2}(dx_2)^2 + \frac{1}{4}(dx_3)^2 + \frac{1}{4}(dx_4)^2. \tag{5.8}$$

Then the flow with initial value (v_1, v_2, v_3, v_4) is $(e^t v_1, e^{2t} v_2, e^{-4t} v_3, e^{-4t} v_4)$.

Consider the map $E : \overline{\mathcal{M}(p,q)} \longrightarrow M_0 \times M_1$, where $M_0 = f^{-1}(\frac{1}{2})$ and $M_1 = f^{-1}(-\frac{1}{2})$. We shall prove $\mathrm{Im}(E)$ is not a C^1 embedded submanifold with boundary $E(\mathcal{M}(p,r) \times \mathcal{M}(r,q))$ of $M_0 \times M_1$.

Clearly, $E(\mathcal{M}(p,r) \times \mathcal{M}(r,q)) = S^+ \times S^-$, where $S^+ = \{(0,0,v_3,v_4) \mid v_3^2 + v_4^2 = 1\}$ and $S^- = \{(v_1, v_2, 0, 0) \mid v_1^2 + v_2^2 = 1\}$. Let $\tilde{S}^+ = \{(v_3, v_4) \mid v_3^2 + v_4^2 = 1\}$ and $\tilde{S}^- = \{(v_1, v_2) \mid v_1^2 + v_2^2 = 1\}$.

The flow map gives a diffeomorphism from an open neighborhood W_0 of S^+ in M_0 onto $U \cap (R^2 \times \tilde{S}^+)$ and a diffeomorphism from an open neighborhood W_1 of S^- in M_1 onto $U \cap (\tilde{S}^- \times R^2)$. Thus there is a diffeomorphism $\psi : W_0 \times W_1 \longrightarrow (U \cap (R^2 \times \tilde{S}^+)) \times (U \cap (\tilde{S}^- \times R^2))$.

Denote $\psi(\mathrm{Im}(E)\cap(W_0\times W_1))$ by P. Then

$$\begin{aligned} P &= \{((v_1,v_2,v_3,v_4),(v_5,v_6,v_7,v_8)) \mid (v_1,v_2,v_3,v_4)\in U\cap(R^2\times\tilde{S}^+) \text{ and} \\ &\quad (v_5,v_6,v_7,v_8)\in U\cap(\tilde{S}^-\times R^2) \text{ are connected by a generalized flow line.}\}. \end{aligned}$$

In order to prove $\mathrm{Im}(E)$ is not a C^1 embedded submanifold of $M_1\times M_2$, we only need to check P is not a C^1 embedded submanifold of $(U\cap(R^2\times\tilde{S}^+))\times(U\cap(\tilde{S}^-\times R^2))$.

Suppose $(v_1,v_2,v_3,v_4)\in U\cap(R^2\times\tilde{S}^+)$ and $(v_1,v_2)\neq(0,0)$, by a direct calculation, (v_1,v_2,v_3,v_4) is connected to

$$(d^{-\frac{1}{2}}v_1, d^{-1}v_2, d^2v_3, d^2v_4) \tag{5.9}$$

by an unbroken flow line, where

$$d=\frac{1}{2}v_1^2+\frac{1}{2}(v_1^4+4v_2^2)^{\frac{1}{2}}. \tag{5.10}$$

We prove our result by contradiction. If P were a C^1 embedded submanifold with boundary $\partial P=S^+\times S^-$, then there is a C^1 collar embedding $\varphi:\tilde{S}^+\times\tilde{S}^-\times[0,\epsilon)\longrightarrow(R^2\times\tilde{S}^+)\times(\tilde{S}^-\times R^2)$ such that

$$\varphi(\cos\theta^+,\sin\theta^+,\cos\theta^-,\sin\theta^-,s)=((v_1,v_2,v_3,v_4),(v_5,v_6,v_7,v_8)),$$

and

$$\varphi(\cos\theta^+,\sin\theta^+,\cos\theta^-,\sin\theta^-,0)=((0,0,\cos\theta^+,\sin\theta^+),(\cos\theta^-,\sin\theta^-,0,0)).$$

When $s\neq0$, $\mathrm{Im}(\varphi)\cap\partial P=\emptyset$, thus $(v_1,v_2)\neq(0,0)$ and (v_5,v_6,v_7,v_8) equals (5.9).

In the following four steps, we will use some estimates. The same notation C or C_i may stand for different constants in different steps.

Firstly, we prove that $\frac{\partial}{\partial s}|_{s=0}v_7=\frac{\partial}{\partial s}|_{s=0}v_8=0$.

Fix θ^+ and θ^-, then v_1 and v_2 are C^1 functions of s, and $v_1=v_2=0$ when $s=0$. So there exist $C_1>0$ and $\delta>0$ such that, for all $s\in[0,\delta)$, we have $|v_1|\leq C_1s$ and $|v_2|\leq C_1s$. Since $(v_3,v_4)\in\tilde{S}^+$, (v_3,v_4) is bounded, by (5.9) and (5.10), there exists $C_2>0$ such that $|v_7|\leq C_2s^2$ and $|v_8|\leq C_2s^2$. This proves our first claim.

Secondly, we claim that $\frac{\partial}{\partial s}|_{s=0}(v_1,v_2)\neq(0,0)$.

If not, then

$$(d\varphi)|_{s=0}\frac{\partial}{\partial s}=\left(0,0,\frac{\partial}{\partial s}|_{s=0}v_3,\frac{\partial}{\partial s}|_{s=0}v_4,\frac{\partial}{\partial s}|_{s=0}v_5,\frac{\partial}{\partial s}|_{s=0}v_6,0,0\right)\in T(S^+\times S^-),$$

So $(d\varphi)|_{s=0}\frac{\partial}{\partial s}$ is not a normal vector of $S^+\times S^-$. This gives a contradiction.

Thirdly, we prove that $\frac{\partial}{\partial s}|_{s=0}v_2=0$.

By the continuity of $\frac{\partial}{\partial s}|_{s=0} v_2$, we only need to prove this is true when $\cos\theta^- \neq 0$. Fix θ^+ and θ^-, then $\lim_{s\to 0} v_5 = \cos\theta^- \neq 0$ and $\lim_{s\to 0} v_6 = \sin\theta^-$. Thus there exist $\delta > 0$, $C_2 > 0$ and $C_3 > 0$, such that, for all $s \in (0, \delta)$, we have $0 < C_2 \leq |v_5|$ and $|v_6| \leq C_3$. By (5.9), $C_2 \leq |d^{-\frac{1}{2}} v_1|$ and $|d^{-1} v_2| \leq C_3$. Then $C_2^2 d \leq |v_1|^2$ and $|v_2| \leq C_3 d$. So $|v_2| \leq C_3 C_2^{-2} |v_1|^2$. In the first step, we showed that there exists $C_1 > 0$, shrinking δ if necessary, we get $|v_1| \leq C_1 s$. Thus $|v_2| \leq C_3 C_2^{-2} C_1^2 s^2$. This gives our third claim.

Finally, we derive the contradiction.

Let $\cos\theta^- = 0$ and $\sin\theta^- = 1$. Fix θ^+. By the second and the third claims, $\frac{\partial}{\partial s}|_{s=0} v_1 \neq 0$. Since $v_1 = 0$ when $s = 0$, then there exist $\delta_1 > 0$ and $C_1 > 0$ such that, for all $s \in [0, \delta_1)$, we have

$$|v_1| \geq C_1 s. \tag{5.11}$$

Since $v_5 = \cos\theta^- = 0$ when $s = 0$, and v_5 is a C^1 function, then there exist $\delta_2 > 0$ and $C_2 > 0$ such that, for all $s \in [0, \delta_2)$, we have

$$|v_5| \leq C_2 s. \tag{5.12}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Combining (5.9), (5.11) and (5.12), we can find $C > 0$ such that, for all $s \in (0, \delta)$, we have $|d^{-\frac{1}{2}} v_1| = |v_5| \leq C|v_1|$ and $v_1 \neq 0$. Thus by (5.10),

$$\frac{2}{v_1^2 + (v_1^4 + 4v_2^2)^{\frac{1}{2}}} = d^{-1} \leq C^2. \tag{5.13}$$

However, when $s \to 0$, we have $v_1 \to 0$, $v_2 \to 0$ and $v_1^2 + (v_1^4 + 4v_2^2)^{\frac{1}{2}} \to 0$, then $d^{-1} \to +\infty$. This gives a contradiction. $\square$

5.8. **Additional Results.** We prove two results which are needed later.

First, we have the following result which follows straightforwardly from the face structure of $\overline{\mathcal{D}(p)}$ (see Definition 2.14).

Lemma 5.5. *Suppose $I = \{p, r_1, \cdots, r_k\}$ is a critical sequence and $x \in \mathcal{D}_I \subseteq \overline{\mathcal{D}(p)}$. Then there exist an open neighborhood W of x in $\mathcal{D}_I$ and a smooth map $\varphi : W \times [0, \epsilon)^k \longrightarrow \overline{\mathcal{D}(p)}$, where φ is a diffeomorphism onto an open neighborhood of x in $\overline{\mathcal{D}(p)}$ satisfying the following stratum condition. For all $y \in W$, $\rho_I = (\rho_1, \cdots, \rho_k) \in [0, \epsilon)^k$ and $J = \{p, r_{i_1}, \cdots, r_{i_s}\}$, we have $\varphi(x, \rho_I) \in \mathcal{D}_J$ if and only if $\rho_j > 0$ when $r_j \notin J$ and $\rho_j = 0$ when $r_j \in J$.*

Proof. Since $\mathcal{D}_I$ is an open subset of the k-stratum of $\overline{\mathcal{D}(p)}$, there is a smooth map $\varphi : W \times [0, \epsilon)^k \longrightarrow \overline{\mathcal{D}(p)}$ which is a diffeomorphism onto an open neighborhood of x in $\overline{\mathcal{D}(p)}$. As mentioned at the end of the proof of Theorem 3.4, $\mathcal{D}_I$ is contained in $\overline{F_i} = \overline{\mathcal{M}(p, r_i)} \times \overline{\mathcal{D}(r_i)}$, the closure of k disjoint faces $F_i = \mathcal{M}(p, r_i) \times \mathcal{D}(r_i)$ $(i = 1, \cdots, k)$. Furthermore, $W \times [0, \epsilon)^k$ also has k disjoint faces $G_i = W \times (0, \epsilon)^{i-1} \times \{0\} \times (0, \epsilon)^{k-i}$. The closure of G_i is $\overline{G_i} = W \times [0, \epsilon)^{i-1} \times \{0\} \times [0, \epsilon)^{k-i}$. Since it is a diffeomorphism, φ maps a face into a face. Choose W to be connected, then permutating the coordinates of $[0, \epsilon)^k$ if necessary, we have $\varphi(G_i) \subseteq F_i$. Thus $\varphi(\overline{G_i}) \subseteq \overline{F_i}$. Moreover, using the fact that φ is a diffeomorphism again, x is in the i-stratum if and only if $\varphi(x)$ is in the i-stratum. $\square$

Lemma 5.6. *Let $e : \overline{\mathcal{D}(p)} \longrightarrow M$ be the map in (3) of Theorem 3.4, and let $I = \{p, r_1, \cdots, r_k\}$ and $J = \{p, r_1, \cdots, r_{k-1}\}$ be critical sequences. Suppose $(\alpha, r_k) \in \mathcal{M}_I \times \mathcal{D}(r_k) = \mathcal{D}_I$. Let $\mathcal{N} \in T_{(\alpha,r_k)}(\mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})})$ represent an inward normal vector in $N_{(\alpha,r_k)}(\mathcal{D}_I, \mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})})$, and $de(\mathcal{N}) = (\mathcal{N}_1, \mathcal{N}_2) \in V_- \times V_+ = T_{r_k}M$. Then $\mathcal{N}_2 \neq 0$. (Here $N_{(\alpha,r_k)}(\mathcal{D}_I, \mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})}) = \dfrac{T_{(\alpha,r_k)}\mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})}}{T_{(\alpha,r_k)}\mathcal{D}_I}$ is the normal space of $\mathcal{D}_I$ in $\mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})}$, and de is the derivative of e.)*

Proof. Suppose the critical values in $(-\infty, f(p)]$ are exactly $c_0 > c_1 > \cdots > c_l$. Let $c_{-1} = +\infty$ and $c_{l+1} = -\infty$. Suppose $f(r_i) = c_{t_i}$ $(i = 1, \cdots, k)$. Recall the evaluation map e in Theorem 3.4. Let $U(t_k) = e^{-1} \circ f^{-1}((c_{t_k+1}, c_{t_k-1}))$. Then $(\alpha, r_k) \in \mathcal{D}_I \cap U(t_k)$.

Returning to the proof of Theorem 3.4, we have $U(t_k)$ is an embedded submanifold of $\prod_{i=1}^{t_k-1} P_i \times Q^+_{t_k}$. We may assume r_i is the unique critical point with function value c_{t_i}. Otherwise, replace P_{t_i} by its open subset $\{(x, y) \in P_{t_i} \mid \forall r \neq r_{t_i},\ x \notin \mathcal{A}(r) \cap M^+_{t_i}\}$ and replace $Q^+_{t_k}$ by its open subset $\{(x, y) \in Q^+_{t_k} \mid \forall r \neq r_{t_k},\ x \notin \mathcal{A}(r) \cap M^+_{t_i}\}$ in this proof.

Denote $\mathcal{D}_I \cap U(t_k)$ by D_I, and $(\mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})}) \cap U(t_k)$ by D_J. Then $T_{(\alpha,r_k)}\mathcal{D}_I = T_{(\alpha,r_k)}D_I$ and $T_{(\alpha,r_k)}(\mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})}) = T_{(\alpha,r_k)}D_J$. Denote $\prod_{j \neq t_s} P^\circ_j \times \prod_{j<k} \partial P_{t_j} \times Q^+_{t_k}$ by H. Then $\partial H = \prod_{j \neq t_s} P^\circ_j \times \prod_{j<k} \partial P_{t_j} \times \partial Q^+_{t_k}$. Here $P^\circ_j = P_j - \partial P_j$.

Clearly, $D_I = \partial H \cap \iota^{-1}_{t_k}(\mathrm{Im}\Delta_{t_k})$ and $D_J = H \cap \iota^{-1}_{t_k}(\mathrm{Im}\Delta_{t_k})$. We have the following inclusion of pairs

$$(T_{(\alpha,r_k)}D_I, T_{(\alpha,r_k)}D_J) \longrightarrow (T_{(\alpha,r_k)}\partial H, T_{(\alpha,r_k)}H).$$

Since ι_{t_k} is transversal to Δ_{t_k} in ∂H, the above inclusion induce an isomorphism

$$N_{(\alpha,r_k)}(D_I, D_J) \longrightarrow N_{(\alpha,r_k)}(\partial H, H)$$

Thus $\mathcal{N}$ also represents an inward normal vector in $N_{(\alpha,r_k)}(\partial H, H)$.

By the proof of Lemma 5.2 (see (5.5)), another such representative element is

$$\widetilde{\mathcal{N}} = (0, \cdots, 0, (v_1, 0), (0, v_2)) \in T_{(\alpha, r_k)} \left(\prod_{j \neq t_i} P_j^\circ \times \prod_{j=1}^{k-1} \partial P_{t_j} \times Q_{t_k}^+ \right) = T_{(\alpha, r_k)} H,$$

where $((v_1, 0), (0, v_2)) \in TQ_{t_k}^+ \subseteq TM_{t_k}^+ \times TM(t_k)$, and $0 \neq (0, v_2) \in V_- \times V_+ = T_{r_k} M$.

Since both $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are inward normal vectors, we have $\mathcal{N} = a\widetilde{\mathcal{N}} + w$ for some $a > 0$ and $w \in T_{(\alpha, r_k)}(\prod_{j \neq t_i} P_j^\circ \times \prod_{j=1}^{k-1} \partial P_{t_j} \times \partial Q_{t_k}^+) = T_{(\alpha, r_k)} \partial H$. Clearly,

$$w = (w_1, \cdots, w_{t_k - 1}, (0, \tilde{v}_2), (\tilde{v}_1, 0)),$$

where $(0, \tilde{v}_2) \in TS_{t_k}^+$ and $(\tilde{v}_1, 0) \in V_- \times \{0\} = T_{r_k} \mathcal{D}(r_k)$.

Since the evaluation map e on $U(t_k)$ is just the projection $\prod_{j=1}^{t_k - 1} P_j \times Q_{t_k}^+ \longrightarrow Q_{t_k}^+ \subseteq M(t_k)$, we have $de(\mathcal{N}) = (\tilde{v}_1, a v_2)$. Thus $\mathcal{N}_2 = a v_2 \neq 0$. $\square$

6. Orientation

6.1. **Definition of Orientations.** Before defining the orientations of $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$, we give a general way to get an orientation by transversality.

Suppose M_1, M_2 and M_3 are three Hilbert manifolds such that M_2 is embedded in M_3. The normal bundle of M_2 with respect to M_3 is defined as $N(M_2, M_3) = \dfrac{T_{M_2} M_3}{TM_2}$. Here $T_{M_2} M_3$ is the restriction of TM_3 on M_2. If $\varphi : M_1 \longrightarrow M_3$ is transversal to M_2, then $M_0 = \varphi^{-1}(M_2)$ is an embedded submanifold of M_1, and $d\varphi$ induces a bundle map $d\varphi : N(M_0, M_1) \longrightarrow N(M_2, M_3)$, i.e., $d\varphi$ is an isomorphism in each fiber. If M_1 is finite dimensional and oriented and $N(M_2, M_3)$ is a finite dimensional (i.e., the fiber is finite dimensional) and oriented bundle, then we can give an orientation of M_0 as follows. The orientation of $N(M_2, M_3)$ gives an orientation to $N(M_0, M_1)$ via $d\varphi$. Let $\pi : T_{M_0} M_1 \longrightarrow N(M_0, M_1)$ be the natural projection. For all $x \in M_0$, choose $\{e_{k+1}, \cdots, e_n\} \subseteq T_x M_1$ such that $\{\pi(e_{k+1}), \cdots, \pi(e_n)\}$ is a positive base of $N_x(M_0, M_1)$. Choose $\{e_1, \cdots, e_k\} \subseteq T_x M_0$ such that $\{e_1, \cdots, e_k, e_{k+1}, \cdots, e_n\}$ is a positive base of $T_x M_1$, then $\{e_1, \cdots, e_k\}$ gives M_0 an orientation. Clearly, this is well defined and only depends on the orientations of M_1 and $N(M_2, M_3)$.

By the above method, we can derive the orientations of $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$ provided that the orientations of $\mathcal{D}(p)$ and $\mathcal{D}(q)$ have been assigned arbitrarily.

Firstly, we can give $\mathcal{W}(p,q)$ an orientation.

Since $\mathcal{A}(q)$ is transversal to $\mathcal{D}(q)$ at q, then the orientation of $T_q\mathcal{D}(q)$ induces an orientation of $N(\mathcal{A}(q),M)$. Let $i : \mathcal{D}(p) \longrightarrow M$ be the inclusion. i is transversal to $\mathcal{A}(q)$ and $i^{-1}(\mathcal{A}(q)) = \mathcal{W}(p,q)$. The orientations of $\mathcal{D}(p)$ and $N(\mathcal{A}(q),M)$ determine an orientation of $\mathcal{W}(p,q)$.

Secondly, we can give $\mathcal{M}(p,q)$ an orientation.

Choose a regular value $a \in (-\infty, f(p))$. We give $S_p^- = \mathcal{D}(p) \cap f^{-1}(a)$ the **induced orientation** from $\mathcal{D}(p)$ as follows. For all $x \in S_p^-$, $\{e_1, \cdots, e_n\}$ is a positive base of $T_x S_p^-$ if and only if $\{-\nabla f, e_1, \cdots, e_n\}$ is a positive base of $T_x\mathcal{D}(p)$. Suppose $a \in (f(q), f(p))$. Denote $\mathcal{A}(q) \cap f^{-1}(a)$ by S_q^+. Then both S_p^- and S_q^+ are embedded submanifolds of $f^{-1}(a)$ which are transversal to each other. S_p^- has its induced orientation from $\mathcal{D}(p)$ as above. There is a natural bundle map from $N(S_q^+, f^{-1}(a))$ to $N(\mathcal{A}(q), M)$. Thus $N(S_q^+, f^{-1}(a))$ is an oriented bundle. The orientations of S_p^- and $N(S_q^+, f^{-1}(a))$ give $S_p^- \cap S_q^+ = \mathcal{W}(p,q) \cap f^{-1}(a)$ an orientation. The natural identification between $\mathcal{M}(p,q)$ and $\mathcal{W}(p,q) \cap f^{-1}(a)$ (see the comment below Definition 2.8) moves the orientation of $\mathcal{W}(p,q) \cap f^{-1}(a)$ to an orientation of $\mathcal{M}(p,q)$. Clearly, this orientation only depends on those of $\mathcal{D}(p)$ and $\mathcal{D}(q)$.

Thirdly, since $\mathcal{M}(p,q)$, $\mathcal{D}(p)$ and $\mathcal{W}(p,q)$ are the interiors of $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$ respectively, the orientation of each interior determines a unique orientation of each compactified space.

Assign orientations to descending manifolds of all critical points arbitrarily. We can consider the orientations of the 1-strata $\partial^1\overline{\mathcal{M}(p,q)}$, $\partial^1\overline{\mathcal{D}(p)}$ and $\partial^1\overline{\mathcal{W}(p,q)}$ of $\overline{\mathcal{M}(p,q)}$, $\overline{\mathcal{D}(p)}$ and $\overline{\mathcal{W}(p,q)}$. As unoriented manifolds, $\partial^1\overline{\mathcal{M}(p,q)} = \bigsqcup_{p\succ r\succ q} \mathcal{M}(p,r) \times \mathcal{M}(r,q)$. There are two orientations of it. First, since $\overline{\mathcal{M}(p,q)}$ has an orientation, $\mathcal{M}(p,q) \sqcup \partial^1\overline{\mathcal{M}(p,q)}$ is an oriented manifold with boundary $\partial^1\overline{\mathcal{M}(p,q)}$. For all $x \in \partial^1\overline{\mathcal{M}(p,q)}$, let $\mathcal{N}$ be an outward normal vector at x. We define an oriented base $\{e_1, \cdots, e_k\}$ of $T_x\partial^1\overline{\mathcal{M}(p,q)}$ to be positive if and only if $\{\mathcal{N}, e_1, \cdots, e_k\}$ is a positive base of $T_x(\mathcal{M}(p,q) \sqcup \partial^1\overline{\mathcal{M}(p,q)})$. We call this the **boundary orientation** of $\partial^1\overline{\mathcal{M}(p,q)}$. Second, since both $\mathcal{M}(p,r)$ and $\mathcal{M}(r,q)$ have orientations, $\mathcal{M}(p,r) \times \mathcal{M}(r,q)$ has the product orientation of these two orientations. This gives $\partial^1\overline{\mathcal{M}(p,q)}$ the **product orientation**. Similarly, we can also define the boundary orientations and the product orientations for $\partial^1\overline{\mathcal{D}(p)}$ and $\partial^1\overline{\mathcal{W}(p,q)}$.

Theorem 3.6 answers the relations between the boundary orientations and the product orientations of the above 1-strata.

6.2. **Proof of (1) of Theorem 3.6.**

Proof. We only need to prove that, for all r,

$$\partial(\mathcal{M}(p,q)\sqcup\mathcal{M}(p,r)\times\mathcal{M}(r,q))=(-1)^{\mathrm{ind}(p)-\mathrm{ind}(r)}\mathcal{M}(p,r)\times\mathcal{M}(r,q).$$

Denote $\mathcal{M}(p,q)\sqcup\mathcal{M}(p,r)\times\mathcal{M}(r,q)$ by $\widehat{\mathcal{M}(p,q)}$. By local triviality of the metric, we have the diffeomorphism h in (2.2) such that (2.3) and (2.4) hold. In addition, choose ϵ small enough such that $f(r)$ is the only critical value in $[f(r)-\epsilon,f(r)+\epsilon]$. For now on, we identify U with B without any difference. Let $M^+=f^{-1}(f(r)+\frac{1}{2}\epsilon)$ and $M^-=f^{-1}(f(r)-\frac{1}{2}\epsilon)$. Let $S_p^-=\mathcal{D}(p)\cap M^+$, $S_q^+=\mathcal{A}(q)\cap M^-$, $S_r^+=\mathcal{A}(r)\cap M^+$ and $S_r^-=\mathcal{D}(r)\cap M^-$. Then $S_r^+=\{(0,v_2)\in V_-\times V_+\mid \|v_2\|^2=\epsilon\}$ and $S_r^-=\{(v_1,0)\in V_-\times V_+\mid \|v_1\|^2=\epsilon\}$.

Define

$$L=\{(x,y)\in S_p^-\times M^-\mid x \text{ and } y \text{ are connected by a generalized flow line.}\}.$$

We may assume there is only one critical point r in $f^{-1}([f(r)-\epsilon,f(r)+\epsilon])$. Otherwise, define L to be

$$\{(x,y)\in(S_p^- - \bigcup_{r_i\neq r}S_{r_i}^+)\times M^-\mid x \text{ and } y \text{ are connected by a generalized flow line.}\}$$

in this argument. Consider the projection $\pi_+:M^+\times M^-\longrightarrow M^+$, then $L=\pi_+^{-1}(S_p^-)\cap P_c$, where P_c is defined in Lemma 5.1 and $c=f(r)$. By transversality, L is an smoothly embedded submanifold with boundary of $M^+\times M^-$. The interior of L is

$$L^\circ=\{(x,y)\in L\mid x \text{ and } y \text{ are connected by a unbroken flow line.}\},$$

and $\partial L=(S_p^-\cap S_r^+)\times S_r^-$. Clearly, $S_p^-\cap S_r^+$ can be identified with $\mathcal{M}(p,r)$. We consider it as $\mathcal{M}(p,r)$. Then $\partial L=\mathcal{M}(p,r)\times S_r^-$.

Consider the projection $\pi_\pm:M^+\times M^-\longrightarrow M^\pm$. We have $\pi_+(L^\circ)=S_p^- - S_r^+$, $\pi_-(L^\circ)=\mathcal{D}(p)\cap M^-$, and $\pi_\pm$ give diffeomorphisms from L° to its images. Give $S_p^- - S_r^+$ and $\mathcal{D}(p)\cap M^-$ the induced orientations from $\mathcal{D}(p)$ (see Subsection 6.1). Then π_+ and π_- move the above two orientations to L°. These orientations on L° are the same. Thus L° has a preferred orientation.

Clearly, $\pi_-:L\longrightarrow M^-$ is transversal to S_q^+ in L° and ∂L. Just as in (3) of Theorem 3.3, $\pi_-^{-1}(S_q^+)$ can be identified with $\widehat{\mathcal{M}(p,q)}$ because $(x,y)\in\pi_-^{-1}(S_q^+)$ is a pair of points on a generalized flow line $\Gamma\in\widehat{\mathcal{M}(p,q)}$. Likewise $(\pi_-|_{\partial L})^{-1}(S_q^+)$ can be identified with $\partial\widehat{\mathcal{M}(p,q)}$. The boundary of $\pi_-^{-1}(S_q^+)$ is exactly $(\pi_-|_{\partial L})^{-1}(S_q^+)$. We consider the orientation of L first in order to study the one of $\widehat{\mathcal{M}(p,q)}$.

Similarly to $\partial\widehat{\mathcal{M}(p,q)}$, there are two orientations of ∂L. First, the orientation of L gives it a boundary orientation. Second, the orientations of $\mathcal{M}(p,r)$ and S_r^- give $\partial L = \mathcal{M}(p,r)\times S_r^-$ a product orientation, where the orientation of S_r^- is induced from that of $\mathcal{D}(r)$ (see Subsection 6.1). The following key lemma shows the difference between these two orientations of ∂L.

Lemma 6.1. *$\partial L = (-1)^{ind(p)-ind(r)}\mathcal{M}(p,r)\times S_r^-$. Here, ∂L is given the boundary orientation and $\mathcal{M}(p,r)\times S_r^-$ is given the product orientation.*

The proof of Lemma 6.1 is based on a good local collar embedding of ∂L into L and a subtle computation of orientations. The collar embedding is provided by the following two lemmas.

Fix a point$(0,x_2)\in\mathcal{M}(p,r)$. We know $\mathcal{M}(p,r) = S_p^-\cap S_r^+\subseteq\{0\}\times V_+$. Define $\widetilde{\mathcal{M}}(p,r) = \{v_2\in V_+\mid (0,v_2)\in\mathcal{M}(p,r)\}$.

Lemma 6.2. *There exist an open neighborhood Ω of x_2 in V_+, a $\delta>0$, and a map $\tilde\theta : B_1(\delta)\times(\Omega\cap\widetilde{\mathcal{M}}(p,r))\longrightarrow V_-\times V_+$ such that $\tilde\theta(v_1,v_2) = (v_1,\theta(v_1,v_2))$, $\theta(0,v_2)=v_2$ and $\tilde\theta$ is a diffeomorphism from $B_1(\delta)\times(\Omega\cap\widetilde{\mathcal{M}}(p,r))$ to $S_p^-\cap(B_1(\delta)\times\Omega)$. Here $B_1(\delta)=\{v_1\in V_-\mid \|v_1\|^2<\delta\}$.*

Let $\tilde S_r^+ = \{v_2\in V_+\mid (0,v_2)\in S_r^+\}$ and $\tilde S_r^- = \{v_1\in V_-\mid (v_1,0)\in S_r^-\}$. We can identify $\mathcal{M}(p,r)$ with $\widetilde{\mathcal{M}}(p,r)$ and $\tilde S_r^\pm$ with $S_r^\pm$ naturally. Fix a point $(x_1,0)\in S_r^-$.

Lemma 6.3. *There exist $\delta>0$, a neighborhood Ω_2 of x_2 in V_+ and a neighborhood Ω_1 of x_1 in V_+ such that $\varphi : [0,\delta)\times(\Omega_2\cap\widetilde{\mathcal{M}}(p,r))\times(\Omega_1\cap\tilde S_r^-)\longrightarrow V_-\times V_+\times V_-\times V_+$ is a local collar neighborhood embedding of ∂L into L near $((0,x_2),(x_1,0))$. Here*

$$\varphi(s,v_2,v_1) = (sv_1,\theta(sv_1,v_2),\epsilon^{-\frac{1}{2}}\|\theta(sv_1,v_2)\|v_1, s\epsilon^{\frac{1}{2}}\|\theta(sv_1,v_2)\|^{-1}\theta(sv_1,v_2)),$$

and θ is defined in Lemma 6.2.

The proof of these three lemmas will be given later.

Since L and $N(S_q^+,M^-)$ have orientations, $\pi_-^{-1}(S_q^+)$ has an orientation. By the definitions of the orientations of L and $\widehat{\mathcal{M}(p,q)}$, the orientations of $\pi_-^{-1}(S_q^+)$ and $\widehat{\mathcal{M}(p,q)}$ are the same under this identification. The boundary orientation of ∂L and the orientation of $N(S_q^+,M^-)$ also give $(\pi_-|_{\partial L})^{-1}(S_q^+)$ an orientation. This orientation of $(\pi_-|_{\partial L})^{-1}(S_q^+)$ coincides with the boundary orientation induced from $\pi_-^{-1}(S_q^+)$. The reason is as follows. At $((0,x_2),(x_1,0))\in(\pi_-|_{\partial L})^{-1}(S_q^+)$, let $\{e_1,\cdots,e_k\}$ be a base of $T(\pi_-|_{\partial L})^{-1}(S_q^+)$ and $\{e_{k+1},\cdots,e_n\}\subseteq$

$T(\partial L)$ represent a base of $N((\pi_-|_{\partial L})^{-1}(S_q^+), \partial L)$. Let $\mathcal{N}$ be an outward normal vector of $(\pi_-|_{\partial L})^{-1}(S_q^+)$ with respect to $\pi_-^{-1}(S_q^+)$. Then $\{\mathcal{N}, e_1, \cdots, e_k\}$ gives an orientation of $\pi_-^{-1}(S_q^+)$, $\{e_1, \cdots, e_k\}$ gives an orientation of $(\pi_-|_{\partial L})^{-1}(S_q^+)$, $\{\mathcal{N}, e_1, \cdots, e_n\}$ gives an orientation of L, and $\{e_1, \cdots, e_n\}$ gives an orientation of ∂L. When $\{e_{k+1}, \cdots, e_n\}$ is positively oriented, $\{e_1, \cdots, e_k\}$ gives the boundary orientation if and only if $\{e_1, \cdots, e_n\}$ gives the boundary orientation. This is the reason.

Thus $(\pi_-|_{\partial L})^{-1}(S_q^+)$ has the boundary orientation of $\partial\widehat{\mathcal{M}(p,q)}$ under this identification if ∂L is equipped with the boundary orientation.

On the other hand, if we give ∂L the product orientation, i.e., we consider it as $\mathcal{M}(p,r) \times S_r^-$, then $(\pi_-|_{\partial L})^{-1}(S_q^+)$ will have the product orientation of $\mathcal{M}(p,r) \times \mathcal{M}(r,q)$ under this identification.

By Lemma 6.1, we have completed the proof of (1) of Theorem 3.6. $\square$

Proof of Lemma 6.2. Since $\widetilde{\mathcal{M}}(p,r)$ is an embedded submanifold of V_+, there exist a neighborhood Ω of x_2 and a diffeomorphism $\alpha : \Omega \longrightarrow V_+$ such that $V_+ = K_1 \times K_2$, $\alpha(\Omega \cap \widetilde{\mathcal{M}}(p,r)) = K_1 \times \{0\}$ and $\alpha(x_2) = (0,0)$. Here K_1 and K_2 are two Hilbert spaces. Define $\beta : B_1(\delta) \times \Omega \longrightarrow B_1(\delta) \times V_+$ by $\beta(v_1, v_2) = (v_1, \alpha(v_2))$. Then β is also a diffeomorphism. $\beta(B_1(\delta) \times (\Omega \cap \widetilde{\mathcal{M}}(p,r))) = B_1(\delta) \times K_1 \times \{0\}$, $\beta(\{0\} \times \Omega) = \{0\} \times V_+$ and $\beta(0, x_2) = (0,0,0)$.

Since S_p^- is transversal to $\{0\} \times \Omega$, then $\beta(S_p^- \cap (B_1(\delta) \times \Omega))$ is also transversal to $\{0\} \times V_+ = \beta(\{0\} \times \Omega)$. Denote $\beta(S_p^- \cap (B_1(\delta) \times \Omega))$ by S. Then

$$T_{(0,0,0)}S + T_{(0,0,0)}(\{0\} \times V_+) = T_{(0,0,0)}(B_1(\delta) \times V_+) = V_- \times V_+. \tag{6.1}$$

Consider the map $\pi_1 : B_1(\delta) \times V_+ \longrightarrow B_1(\delta) \times \{(0,0)\}$, where $\pi_1(v_1, k_1, k_2) = (v_1, 0, 0)$. By (6.1), we get

$$d\pi_1 : T_{(0,0,0)}S \longrightarrow T_{(0,0,0)}(B_1(\delta) \times \{(0,0)\}) = V_- \times \{(0,0)\}$$

is surjective. In addition, since $\{0\} \times (\Omega \cap \widetilde{\mathcal{M}}(p,r)) \subseteq S_p^- \cap (B_1(\delta) \times \Omega)$, we have

$$\{0\} \times K_1 \times \{0\} = \beta(\{0\} \times (\Omega \cap \widetilde{\mathcal{M}}(p,r))) \subseteq S.$$

Thus

$$\{0\} \times K_1 \times \{0\} = T_{(0,0,0)}(\{0\} \times K_1 \times \{0\}) \subseteq T_{(0,0,0)}S. \tag{6.2}$$

Consider the map $\pi_2 : B_1(\delta) \times V_+ \longrightarrow B_1(\delta) \times K_1 \times \{0\}$, where $\pi_2(v_1, k_1, k_2) = (v_1, k_1, 0)$. By the surjectivity of $d\pi_1$ on S and (6.2), we know that

$$d\pi_2 : T_{(0,0,0)}S \longrightarrow T_{(0,0,0)}(B_1(\delta) \times K_1 \times \{0\}) = V_- \times K_1 \times \{0\}$$

is surjective.

Now we count the dimensions of S and $B_1(\delta) \times K_1 \times \{0\}$.

$$\begin{aligned}\dim(S) &= \dim(S_p^-) = \operatorname{ind}(p) - 1 = \operatorname{ind}(p) - \operatorname{ind}(r) - 1 + \operatorname{ind}(r)\\ &= \dim(\mathcal{M}(p,r)) + \dim(V_-) = \dim(K_1 \times \{0\}) + \dim(B_1(\delta))\\ &= \dim(B_1(\delta) \times K_1 \times \{0\})\end{aligned}$$

By the Inverse Function Theorem, shrinking δ and Ω if necessary, we have that π_2 gives a diffeomorphism from $S = \beta(S_p^- \cap (B_1(\delta) \times \Omega))$ to $B_1(\delta) \times K_1 \times \{0\} = \beta(B_1(\delta) \times (\Omega \cap \widetilde{\mathcal{M}}(p,r)))$. Also, $(\pi_2|_S)^{-1}(v_1, k_1, 0) = (v_1, \hat{\theta}(v_1, k_1, 0))$ for some $\hat{\theta}$. It's easy to see that $S \cap (\{0\} \times V_+) = \{0\} \times K_1 \times \{0\}$. Then $\hat{\theta}(0, k_1, 0) = (k_1, 0)$.

Defining $\tilde{\theta} = \beta^{-1} \circ (\pi_2|_S)^{-1} \circ \beta$ on $B_1(\delta) \times (\Omega \cap \widetilde{\mathcal{M}}(p,r))$, completes the proof. □

Proof of Lemma 6.3. We may assume $\epsilon \leq 1$. Choose δ as in Lemma 6.2. Choose Ω_2 to be Ω in Lemma 6.2. Consider φ as a map defined in $(-\delta, \delta) \times (\Omega_2 \cap \widetilde{\mathcal{M}}(p,r)) \times \tilde{S}_r^-$. By Lemma 6.2, we have $\operatorname{Im}(\varphi) \subseteq L^\circ$ when $s > 0$, $\operatorname{Im}(\varphi) \cap L = \emptyset$ when $s < 0$ and $\varphi(0, v_2, v_1) = ((0, v_2), (v_1, 0)) \in \partial L$.

Now we compute $d\varphi$. First, we introduce some notation. Let $\frac{\partial}{\partial s}$ be the positive unit tangent vector of $(-\delta, \delta)$. Let $\frac{\partial}{\partial x_1}$ be a base of $T_{x_1}\tilde{S}_r^- \subseteq V_-$, i.e.

$$\frac{\partial}{\partial x_1} = \{e_1, \cdots, e_{\operatorname{ind}(r)-1}.\}$$

Let $\frac{\partial}{\partial x_2}$ be a base of $T_{x_2}\widetilde{\mathcal{M}}(p,r)) \subseteq V_+$. The notation $(d\varphi)\frac{\partial}{\partial x_1}$ means

$$(d\varphi)\frac{\partial}{\partial x_1} = \{(d\varphi)e_1, \cdots, (d\varphi)e_{\operatorname{ind}(r)-1}\}.$$

In the following calculation, omit $d\varphi\frac{\partial}{\partial x_2}$ if $\dim(\widetilde{\mathcal{M}}(p,r)) = 0$ and omit $d\varphi\frac{\partial}{\partial x_1}$ if $\dim(\tilde{S}_r^-) = 0$. At (s, x_1, x_2)

$$\begin{aligned}(6.3)(d\varphi)\frac{\partial}{\partial s} &= (x_1, (d\theta)x_1, \epsilon^{-\frac{1}{2}}\|\theta\|^{-1}\langle\theta, (d\theta)x_1\rangle x_1, \epsilon^{\frac{1}{2}}\|\theta\|^{-1}h + s*)\\ (d\varphi)\frac{\partial}{\partial x_2} &= \left(0, (d\theta)\frac{\partial}{\partial x_2}, \epsilon^{-\frac{1}{2}}\|\theta\|^{-1}\left\langle\theta, (d\theta)\frac{\partial}{\partial x_2}\right\rangle x_1, s*\right),\\ (d\varphi)\frac{\partial}{\partial x_1} &= \left(s\frac{\partial}{\partial x_1}, s(d\theta)\frac{\partial}{\partial x_1}, \epsilon^{-\frac{1}{2}}\|\theta\|\frac{\partial}{\partial x_1} + s*, s^2*\right).\end{aligned}$$

Here $*$ stands for some smooth functions which are not important. Since $\theta(0, v_2) \equiv v_2$, we have $(d\theta)(0, x_2)\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_2}$. And since $\frac{\partial}{\partial x_2}$ is contained in $T_{x_2}\tilde{S}_r^+$ and is orthogonal to x_2, we have $\langle\theta(0, x_2), (d\theta)(0, x_2)\frac{\partial}{\partial x_2}\rangle =$

0. In addition, $\|x_2\| = \epsilon^{\frac{1}{2}}$. Thus

$$(d\varphi)(0,x_2,x_1)\frac{\partial}{\partial s} = (x_1, d\theta(0,x_2)x_1, \epsilon^{-1}\langle x_2, d\theta(0,x_2)x_1\rangle x_1, x_2),$$

$$(d\varphi)(0,x_2,x_1)\frac{\partial}{\partial x_2} = \left(0, \frac{\partial}{\partial x_2}, 0, 0\right), \ (d\varphi)(0,x_2,x_1)\frac{\partial}{\partial x_1} = \left(0, 0, \frac{\partial}{\partial x_1}, 0\right). \tag{6.4}$$

Clearly, $d\varphi(0,x_2,x_1)\{\frac{\partial}{\partial s}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\}$ is linear independent. Since $\dim(L) = \dim([0,\,\delta) \times (\Omega_2 \cap \widetilde{\mathcal{M}}(p,r)) \times \tilde{S}_r^-)$, by the Inverse Function Theorem, we have that this lemma is true. $\square$

Proof of Lemma 6.1. Let $((0,x_2),(x_1,0))$ be an arbitrary point in ∂L. We only need to prove the orientation difference is $(-1)^{\mathrm{ind}(p)-\mathrm{ind}(r)}$ at this point.

Suppose $(0, \frac{\partial}{\partial x_2})$ and $(\frac{\partial}{\partial x_1}, 0)$ are a positive basis of $T_{(0,x_2)}\mathcal{M}(p,r)$ and $T_{(x_1,0)}S_r^-$ respectively. We use the locally collar embedding φ in Lemma 6.3. Fix x_2 and x_1, change s. By (6.4), $d\varphi(0,x_2,x_1)\frac{\partial}{\partial x_2} = (0, \frac{\partial}{\partial x_2}, 0, 0)$ and $d\varphi(0,x_2,x_1)\frac{\partial}{\partial x_1} = (0,0,\frac{\partial}{\partial x_1},0)$. So $\{d\varphi(0,x_2,x_1)\frac{\partial}{\partial x_2}, d\varphi(0,x_2,x_1)\frac{\partial}{\partial x_1}\}$ is a positive basis of $\mathcal{M}(p,r) \times S_r^-$. When $\dim(\mathcal{M}(p,r)) = 0$ or $\dim(S_r^-) = 0$, the orientation of $T_{(0,x_2)}\mathcal{M}(p,r)$ or $T_{(x_1,0)}S_r^-$ is a sign ± 1, and $d\varphi(s,x_2,x_1)\frac{\partial}{\partial x_2}$ or $d\varphi(s,x_2,x_1)\frac{\partial}{\partial x_1}$ is replaced by this sign.

Now, $-(d\varphi)(0,x_2,x_1)\frac{\partial}{\partial s}$ is an outward normal vector of ∂L. Thus, when $s=0$, $\{d\varphi\frac{\partial}{\partial x_2}, d\varphi\frac{\partial}{\partial x_1}\}$ is a positive base of ∂L if and only if $\{-d\varphi\frac{\partial}{\partial s}, d\varphi\frac{\partial}{\partial x_2}, d\varphi\frac{\partial}{\partial x_1}\}$ is a positive base of L. This is also equivalent to the statement that, when $s \neq 0$, $\{-d\varphi\frac{\partial}{\partial s}, d\varphi\frac{\partial}{\partial x_2}, d\varphi\frac{\partial}{\partial x_1}\}$ is a positive base of L.

When $s \neq 0$, $\varphi(s,x_2,x_1) \in L^\circ$, and $\pi_+ : L^\circ \longrightarrow S_p^-$ preserves orientation. Thus, by (6.3), the above consideration is equivalent to the statement that,

$$\begin{aligned} & \left\{-d\pi_+ \cdot d\varphi\frac{\partial}{\partial s}, d\pi_+ \cdot d\varphi\frac{\partial}{\partial x_2}, d\pi_+ \cdot d\varphi\frac{\partial}{\partial x_1}\right\} \\ = \ & \left\{-(x_1, d\theta\cdot x_1), \left(0, d\theta\frac{\partial}{\partial x_2}\right), \left(s\frac{\partial}{\partial x_1}, s\cdot d\theta\frac{\partial}{\partial x_1}\right)\right\} \end{aligned}$$

is a positive base of S_p^-. We change this base to another base

$$\left\{-(x_1, d\theta\cdot x_1), \left(0, d\theta\frac{\partial}{\partial x_2}\right), \left(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1}\right)\right\}. \tag{6.5}$$

The new base (6.5) has the same orientation as the old one. Its advantage is that, when $s=0$, (6.5) is still a base of S_p^-. The reason is as follows. When $s \neq 0$, (6.5) is in TS_p^-. Thus, by continuity, it is still in TS_p^- when $s = 0$. In addition, when $s = 0$, $(0, d\theta\frac{\partial}{\partial x_2}) = (0, \frac{\partial}{\partial x_2})$. As a base of $T_{x_1}V_-$ and $T_{x_2}\widetilde{\mathcal{M}}(p, r)$ respectively, both $\{-x_1, \frac{\partial}{\partial x_1}\}$ and $\frac{\partial}{\partial x_2}$ are linearly independent. So (6.5) remains linearly independent when $s = 0$.

When s varies in $[0, \delta)$, the orientation difference between $\{-(x_1, d\theta \cdot x_1), (0, d\theta\frac{\partial}{\partial x_2}), (\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1})\}$ and S_p^- is fixed. So we only need to check the difference when $s = 0$. As a base of $T_{x_2}\mathcal{M}(p, r)$, $(0, \frac{\partial}{\partial x_2})$ contains $\mathrm{ind}(p) - \mathrm{ind}(r) - 1$ vectors. Denote the orientation of a base $\{*\}$ by $\mathrm{Or}\{*\}$. Then, when $s = 0$,

$$\begin{aligned}
& \mathrm{Or}\left\{-(x_1, d\theta \cdot x_1), \left(0, d\theta\frac{\partial}{\partial x_2}\right), \left(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1}\right)\right\} \\
= \; & \mathrm{Or}\left\{-(x_1, d\theta \cdot x_1), \left(0, \frac{\partial}{\partial x_2}\right), \left(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1}\right)\right\} \\
= \; & (-1)^{\mathrm{ind}(p)-\mathrm{ind}(r)}\mathrm{Or}\left\{\left(0, \frac{\partial}{\partial x_2}\right), (x_1, d\theta \cdot x_1), \left(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1}\right)\right\}.
\end{aligned}$$

Since $(x_1, 0) = -\nabla f(x_1, 0)$, $(\frac{\partial}{\partial x_1}, 0)$ is a positive base of $T_{(x_1,0)}S_r^+$, then $\{(x_1, 0), (\frac{\partial}{\partial x_1}, 0)\}$ is a positive base of $T_{(x_1,0)}(V_- \times \{0\}) = V_- \times \{0\} = T_r\mathcal{D}(r)$. Thus $\{(x_1, d\theta \cdot x_1), (\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1})\}$ represents a positive base of the normal space $N_{(0,x_2)}(\mathcal{M}(p, r), S_p^-)$. Since $(0, \frac{\partial}{\partial x_2})$ is a positive base of $T_{(0,x_2)}\mathcal{M}(p, r)$, we infer that $\{(0, \frac{\partial}{\partial x_2}), (x_1, d\theta \cdot x_1), (\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1})\}$ is a positive base of $T_{(0,x_2)}S_p^-$.

As a result, $(-1)^{\mathrm{ind}(p)-\mathrm{ind}(r)}\mathrm{Or}\{d\varphi(0, x_2, x_1)\frac{\partial}{\partial x_2}, d\varphi(0, x_2, x_1)\frac{\partial}{\partial x_1}\}$ represents the orientation of ∂L. This completes the proof. $\square$

6.3. **Proof of (2) of Theorem 3.6.** The proof of (2) is similar to that of (1). In particular, they share many details. We shall only give the outline and the key calculation of this proof.

Proof. We only need to prove that $\partial(\mathcal{D}(p) \sqcup \mathcal{M}(p, r) \times \mathcal{D}(r)) = \mathcal{M}(p, r) \times \mathcal{D}(r)$ as oriented manifolds. Actually, we only need to argue this in an open subset containing $\mathcal{M}(p, r) \times \mathcal{D}(r)$ of $\mathcal{D}(p) \sqcup \mathcal{M}(p, r) \times \mathcal{D}(r)$. Recall the evaluation map $e : \mathcal{D}(p) \sqcup \mathcal{M}(p, r) \times \mathcal{D}(r) \longrightarrow M$ in (3) of Theorem 3.4. We have $e^{-1} \circ f^{-1}((-\infty, f(r) + \epsilon))$ is such an open subset. Moreover, we can simplify this problem again. Let $M(r) = f^{-1}((f(r) - \epsilon, f(r) + \epsilon))$. Consider the open subset $e^{-1}(M(r))$. For all $x \in e^{-1} \circ f^{-1}((-\infty, f(r) + \epsilon)) \cap \mathcal{M}(p, r) \times \mathcal{D}(r)$, there exist $y \in e^{-1}(M(r)) \cap \mathcal{M}(p, r) \times \mathcal{D}(r)$ and a flow map ψ in $\overline{\mathcal{D}(p)}$, such that $\psi(y) = x$ (see Lemma 7.1). From y to x, $d\psi$ preserves the orientations of $\mathcal{D}(p)$

and $\mathcal{M}(p,r)\times\mathcal{D}(r)$ and the outward normal direction. Then $d\psi$ preserves the orientation difference between $\partial(\mathcal{D}(p)\sqcup\mathcal{M}(p,r)\times\mathcal{D}(r))$ and $\mathcal{M}(p,r)\times\mathcal{D}(r)$. Thus we only need to show this is true in $e^{-1}(M(r))$. Now denote $\mathcal{D}(p)\cap M(r)$ by D_p, $\mathcal{D}(r)\cap M(r)$ by D_r and $e^{-1}(M(r))$ by $\widehat{D_p}$. Then $\widehat{D_p}=D_p\sqcup\mathcal{M}(p,r)\times D_r$. We only need to show that $\partial\widehat{D_p}=\mathcal{M}(p,r)\times D_r$ as oriented manifolds.

We use the same notation of $M^{\pm}$, S_p^- and S_r^+ as in the proof of (1). Also identify $S_p^-\cap S_r^+$ with $\mathcal{M}(p,r)$ and define $\widetilde{\mathcal{M}}(p,r)$ as in the proof of (1). Define $\widetilde{D}_r=\{v_2\mid(0,v_2)\in D_r\}$. We also assume that there is only one critical point r in $M(r)$.

Define

$$L=\{(x,y)\in S_p^-\times M(r)\mid x\text{ and }y\text{ are connected by a generalized flow line.}\}.$$

Then $\partial L=\mathcal{M}(p,r)\times D_r$. And

$$L^\circ=\{(x,y)\in L\mid x\text{ and }y\text{ are connected by a unbroken flow line.}\},$$

L is identified with $\widehat{D_p}$ because $(x,y)\in L$ is a pair of points on a generalized flow line connecting p and y. Since $L\subseteq S_p^-\times M(r)$, we may consider the natural projection $\pi:L\longrightarrow M(r)$. Moreover, π identifies L° with D_p, and π coincides with the above identification between L and $\widehat{D_p}$. The orientation of $\widehat{D_p}$ gives L an orientation, and L gives ∂L a boundary orientation. We only need to check the difference between the boundary orientation and the product orientation of ∂L.

Fix $((0,x_2),(x_1,0))\in\mathcal{M}(p,r)\times D_r$. Just as Lemma 6.3, we give a locally collar neighborhood parametrization $\varphi:[0,\delta)\times(\Omega_2\cap\widetilde{\mathcal{M}}(p,r))\times\widetilde{D}_r\longrightarrow V_-\times V_+\times V_-\times V_+$ such that

$$\varphi(s,v_2,v_1)=(sv_1,\theta(sv_1,v_2),v_1,s\theta(sv_1,v_2)),\tag{6.6}$$

where θ is defined in Lemma 6.2. It's necessary to point out that this argument includes the special case of $\mathrm{ind}(r)=0$. In this case, $\widetilde{D}_r=\{0\}$, $\varphi(s,v_2,v_1)=(0,v_2,0,sv_2)$ and $d\varphi\frac{\partial}{\partial x_1}$ is the sign ±1 assigned to D_r.

Suppose $(0,\frac{\partial}{\partial x_2})$ and $(\frac{\partial}{\partial x_1},0)$ are positive basis of $T_{(0,x_2)}\mathcal{M}(p,r)$ and $T_{(x_1,0)}D_r$ respectively. At (s,x_2,x_1), we have

$$d\varphi\frac{\partial}{\partial s}=(x_1,d\theta\cdot x_1,0,\theta+s\cdot d\theta\cdot x_1),\tag{6.7}$$

$$d\varphi\frac{\partial}{\partial x_2}=\left(0,d\theta\frac{\partial}{\partial x_2},0,s\cdot d\theta\frac{\partial}{\partial x_2}\right),\ d\varphi\frac{\partial}{\partial x_1}=\left(s\frac{\partial}{\partial x_1},s\cdot d\theta\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_1},s^2\cdot d\theta\frac{\partial}{\partial x_1}\right).$$

We shall check that, when $s\in[0,\delta)$, $\{-d\varphi\frac{\partial}{\partial s},d\varphi\frac{\partial}{\partial x_2},d\varphi\frac{\partial}{\partial x_1}\}$ coincides with the orientation of L.

When $s \neq 0$, $\varphi(s, x_2, x_1) \in L^\circ$. By the definition of the orientation of L°, $\pi : L^\circ \longrightarrow D_p$ preserves its orientation. Thus, we only need to show that, when $s \neq 0$,

$$\begin{aligned}&\left\{-d\pi \cdot d\varphi\frac{\partial}{\partial s}, d\pi \cdot d\varphi\frac{\partial}{\partial x_2}, d\pi \cdot d\varphi\frac{\partial}{\partial x_1}\right\}\\ = \;&\left\{(0, -\theta - s \cdot d\theta \cdot x_1), \left(0, s \cdot d\theta\frac{\partial}{\partial x_2}\right), \left(\frac{\partial}{\partial x_1}, s^2 \cdot d\theta\frac{\partial}{\partial x_1}\right)\right\}\end{aligned}$$

gives the orientation of D_p at $\pi\varphi(s, x_2, x_1)$. By (6.6), we know that $\pi\varphi(s, x_2, x_1) = (x_1, s\theta(sx_1, x_2))$ is connected with $(sx_1, \theta(sx_1, x_2)) \in S_p^-$ by an unbroken flow line. Consider the flow map ψ in U such that $\psi(v_1, v_2) = (s^{-1}v_1, sv_2)$. Then $\psi(sx_1, \theta(sx_1, x_2)) = (x_1, s\theta(sx_1, x_2))$ and ψ preserves the orientation of D_p. Thus we only need to check that

$$\begin{aligned}&\left\{-d\psi^{-1} \cdot d\pi \cdot d\varphi\frac{\partial}{\partial s}, d\psi^{-1} \cdot d\pi \cdot d\varphi\frac{\partial}{\partial x_2}, d\psi^{-1} \cdot d\pi \cdot d\varphi\frac{\partial}{\partial x_1}\right\}\\ = \;&\left\{(0, -s^{-1}\theta - d\theta \cdot x_1), \left(0, d\theta\frac{\partial}{\partial x_2}\right), \left(s\frac{\partial}{\partial x_1}, s \cdot d\theta\frac{\partial}{\partial x_1}\right)\right\}\end{aligned}$$

gives the orientation of D_p at $(sx_1, \theta(sx_1, x_2))$. Change the above base to the orientation equivalent base $\{(0, -\theta - s \cdot d\theta \cdot x_1), (0, d\theta\frac{\partial}{\partial x_2}), (\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1})\}$. When $s = 0$, it becomes

$$\left\{(0, -x_2), \left(0, \frac{\partial}{\partial x_2}\right), \left(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1}\right)\right\}. \tag{6.8}$$

Since $(\frac{\partial}{\partial x_1}, 0)$ is a positive base of $V_- \times \{0\} = T_r D_r$, $(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1})$ represents a positive base of $N_{(0,x_2)}(\mathcal{M}(p, r), S_p^-)$. At $(0, x_2)$, $(0, -x_2) = -\nabla f$, and $(0, \frac{\partial}{\partial x_2})$ is a positive base of $T_{(0,x_2)}\mathcal{M}(p, r)$. Thus (6.8) gives the orientation of D_r. $\square$

Remark 6.1. *It seems that, in the finite dimensional case, the paper* [20] *gets orientation relations by the same strategy as this paper has. The following key fact is pointed out without explanations in* [20, p. 155]. *"La variété $W^s(c, \varepsilon) \times L_A(c, d)$ est de codimension 0 dans le bord de $\overline{W}^s(d, A + \varepsilon)$ et la normale sortante n_0 à $\overline{W}^s(d, A + \varepsilon)$ en $(c, l) \in W^s(c) \times L_A(c, d)$ s'identifie au vecteur tangent à l orientée par $-\xi$." (Here $\xi = \nabla f$.) This is proved in the paper by moving $-d\varphi\frac{\partial}{\partial s}$ (see (6.7)) to be $(0, -x_2) = -\nabla f$ in (6.8). Thus our work may give the details omitted in* [20].

6.4. **Proof of (3) of Theorem 3.6.** The proof of (3) is a mixture of those of (1) and (2).

Proof. We shall prove that $\partial(\mathcal{W}(p,q)\sqcup\mathcal{M}(p,r)\times\mathcal{W}(r,q)) = \mathcal{M}(p,r)\times\mathcal{W}(r,q)$ and $\partial(\mathcal{W}(p,q)\sqcup\mathcal{W}(p,r)\times\mathcal{M}(r,q)) = (-1)^{\mathrm{ind}(p)-\mathrm{ind}(r)+1}\mathcal{W}(p,r)\times\mathcal{M}(r,q)$. Recall the evaluation map $e : \mathcal{W}(p,q)\sqcup\mathcal{M}(p,r)\times\mathcal{W}(r,q) \longrightarrow M$ (or $\mathcal{W}(p,q)\sqcup\mathcal{W}(p,r)\times\mathcal{M}(r,q) \longrightarrow M$) in (3) of Theorem 3.5. Define $M(r) = f^{-1}((f(r)-\epsilon, f(r)+\epsilon))$, $M(r)^+ = f^{-1}((f(r), f(r)+\epsilon))$ and $M(r)^- = f^{-1}((f(r)-\epsilon, f(r)))$. We have four cases. Just as the proofs of (1) and (2), we will define a manifold L which plays a important role all through this proof, where

$$L = \{(x,y) \mid x \text{ and } y \text{ are connected by a generalized flow line.}\},$$

and (x,y) is contained in some different manifolds in each case. Also, x and y will be connected by a unbroken flow line if and only if $(x,y)\in L^\circ$.

Case (a). The boundary is $\mathcal{M}(p,r)\times\mathcal{W}(r,q)$ and $r\neq q$.

We reduce this problem to considering the case of $e^{-1}(M(r)^-)$. Denote $e^{-1}(M(r)^-)$ by $\widehat{W_{p,q}}$, $\mathcal{W}(r,q)\cap M(r)^-$ by $W_{r,q}$, $\mathcal{D}(p)\cap M(r)^-$ by D_p and $\mathcal{D}(r)\cap M(r)^-$ by D_r. Clearly, as unoriented manifolds, $\partial\widehat{W_{p,q}} = \mathcal{M}(p,r)\times W_{r,q}$.

Define $L\subseteq S_p^-\times M(r)^-$. The natural projection $\pi_2 : L \longrightarrow M(r)^-$ identifies L° with D_p. $\partial L = \mathcal{M}(p,r)\times D_r$. The orientation of D_p gives L an orientation. In the proof of (2), it has been verified that the boundary orientation and the product orientation of ∂L are the same. We identify $\pi_2^{-1}(\mathcal{A}(q))$ with $\widehat{W_{p,q}}$ and identify $(\pi_2|_{\partial L})^{-1}(\mathcal{A}(q))$ with $\partial\widehat{W_{p,q}}$. An argument similar to that in (1) completes the proof.

Case (b). The boundary is $\mathcal{M}(p,q)\times\mathcal{W}(q,q)$.

Replace $M(r)^-$ by $M(q)$ in Case (a). The same argument gives a proof.

Case (c). The boundary is $\mathcal{W}(p,r)\times\mathcal{M}(r,q)$ and $p\neq r$.

Reduce to the case of $e^{-1}(M(r)^+)$. Denote $e^{-1}(M(r)^+)$ by $\widehat{W_{p,q}}$, $\mathcal{W}(p,r)\cap M(r)^+$ by $W_{p,r}$ and $\mathcal{D}(p)\cap M(r)^+$ by D_p.

Define $L\subseteq D_p\times M^-$, where $M^- = f^{-1}(f(r)-\epsilon)$. The projection $\pi_1 : L \longrightarrow D_p$ identifies L° with $D_p - W_{p,r}$, and $\partial L = W_{p,r}\times S_r^-$. Then D_p gives L an orientation. Consider another projection $\pi_2 : L \longrightarrow M^-$. Then $\pi_2^{-1}(S_q^+)$ can be identified with $\widehat{W_{p,q}}$ and $(\pi_2|_{\partial L})^{-1}(S_q^+)$ can be identified with $\partial\widehat{W_{p,q}}$. We reduce the proof to checking the difference of two orientations of ∂L.

Define $\widetilde{W}_{p,r} = \{v_2 \mid (0,v_2)\in W_{p,r}\}$. Similar to Lemma 6.2 and 6.3, there is a neighborhood Ω_2 of x_2 in $\widetilde{W}_{p,r}$ and a parametrization $\tilde{\theta} : B_1(\delta)\times\Omega_2 \longrightarrow D_p$ such that $\tilde{\theta}(v_1,v_2) = (v_1, \theta(v_1,v_2))$ and $\theta(0,v_2) =$

v_2. We also have a local collar embedding $\varphi : [0,\delta) \times \Omega_2 \times \widetilde{S}_r^- \longrightarrow V_- \times V_+ \times V_- \times V_+$ such that

$$\begin{aligned}\varphi(s, v_2, v_1) \;&=\; \Big(sv_1, \theta(sv_1, v_2), (2\epsilon)^{-\frac{1}{2}}(\epsilon + (\epsilon^2 + 4s^2\epsilon\|\theta(sv_1, v_2)\|^2)^{\frac{1}{2}})^{\frac{1}{2}} v_1, \\ &\qquad s(2\epsilon)^{\frac{1}{2}}(\epsilon + (\epsilon^2 + 4s^2\epsilon\|\theta(sv_1, v_2)\|^2)^{\frac{1}{2}})^{-\frac{1}{2}}\theta(sv_1, v_2)\Big).\end{aligned}$$

Just as the proof of (1), we reduce the proof to checking the orientation of $\{-d\pi_1 \cdot d\varphi\frac{\partial}{\partial s}, d\pi_1 \cdot d\varphi\frac{\partial}{\partial x_2}, d\pi_1 \cdot d\varphi\frac{\partial}{\partial x_1}\}$ and then that of $\{-(x_1, d\theta \cdot x_1), (0, \frac{\partial}{\partial x_2}), (\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1})\}$ in $T_{(0,x_2)}D_p$. Here, $\{(0, \frac{\partial}{\partial x_2})\}$ is a positive base of $T_{(0,x_2)}\, W_{p,r}$. It contains $\mathrm{ind}(p) - \mathrm{ind}(r)$ vectors. Thus the orientations are

$$\begin{aligned}&\mathrm{Or}\left\{-(x_1, d\theta \cdot x_1), \left(0, \frac{\partial}{\partial x_2}\right), \left(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1}\right)\right\} \\ =\;& (-1)^{\mathrm{ind}(p)-\mathrm{ind}(r)+1}\mathrm{Or}\left\{\left(0, \frac{\partial}{\partial x_2}\right), (x_1, d\theta \cdot x_1), \left(\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1}\right)\right\}.\end{aligned}$$

Since $\{(0, \frac{\partial}{\partial x_2}), (x_1, d\theta \cdot x_1), (\frac{\partial}{\partial x_1}, d\theta\frac{\partial}{\partial x_1})\}$ is positive, the proof is complete.

Case (d). The boundary is $\mathcal{W}(p,p) \times \mathcal{M}(p,q)$.

Reduce to the case of $e^{-1}(M(p))$. Denote $e^{-1}(M(p))$ by $\widehat{W_{p,q}}$ and $\mathcal{D}(p) \cap M(p)$ by D_p. Then $\partial\widehat{W_{p,q}} = \mathcal{W}(p,p) \times \mathcal{M}(p,q) = \{p\} \times \mathcal{M}(p,q)$.

Define $L \subseteq D_p \times M^-$, where $M^- = f^{-1}(f(p) - \epsilon)$. Then $\pi_1 : L \longrightarrow D_p$ identifies L° with $D_p - \{p\}$, and $\partial L = \mathcal{W}(p,p) \times S_p^-$. Moreover, D_p gives L an orientation. Consider $\pi_2 : L \longrightarrow M^-$. Then $\pi_2^{-1}(S_q^+)$ can be identified with $\widehat{W_{p,q}}$ and $(\pi_2|_{\partial L})^{-1}(S_q^+)$ can be identified with $\partial\widehat{W_{p,q}}$. We reduce the proof to checking the two orientations of ∂L.

Consider the collar embedding $\varphi : [0, \sqrt{2}) \times \widetilde{S}_p^- \longrightarrow V_- \times V_+ \times V_- \times V_+$ such that $\varphi(s, v_1) = (sv_1, 0, v_1, 0)$. Since $\mathcal{W}(p,p)$ has orientation $+1$, we only need to check the orientation difference between $\{-d\varphi\frac{\partial}{\partial s}, d\varphi\frac{\partial}{\partial x_1}\}$ and L. When $s = 1$, $\mathrm{Or}\{-d\pi_1 \cdot d\varphi\frac{\partial}{\partial s}, d\pi_1 \cdot d\varphi\frac{\partial}{\partial x_1}\} = -\mathrm{Or}\{-\nabla f, (\frac{\partial}{\partial x_1}, 0)\}$ is the negative orientation of $T_{(x_1,0)}D_p$. Thus $\partial\widehat{W_{p,q}} = -\mathcal{W}(p,p) \times \mathcal{M}(p,q)$. $\square$

Remark 6.2. *The papers* [3] *and* [40] *compute the cup product of* $H^*(M; R)$ *via Morse Theory. Both* [3, (2.2)] *and* [40, lem. 2 and 3] *neglect signs. Theorem 3.6, (3), can tell us the the signs if we do care about them. The following is an explanation of* [40, lem. 3]. *We shall use notation different from that in* [40]. *Our* $\mathcal{W}(p,q)$ *and* $\#\mathcal{M}(p,q)$ *are* $\mathcal{M}(p,q)$ *and* $n(p,q)$ *in* [40] *respectively. A real coefficients Thom-Smale cochain complex is defined in* [40] *as* $C^* = \bigoplus_n \bigoplus_{ind(p)=n} R[p]$

with coboundary operator

$$\delta q = \sum_{ind(p)=ind(q)+1} \#\mathcal{M}(p,q)p,$$

where $\#\mathcal{M}(p,q)$ *is defined in Theorem 3.9. Let* ω *be a differential form, in* [40], *a cup product action of* ω *on* C^* *is defined as*

$$\pi(\omega)q = \sum_p \left(\int_{\mathcal{W}(p,q)} \omega\right) p.$$

The paper [40, lem. 3] *states that* $\pi(d\omega) = \delta\pi(\omega) \pm \pi(\omega)\delta$. *Actually, (3) of Theorem 3.6 tells us*

$$\pi(d\omega) = \delta\pi(\omega) + (-1)^{|\omega|+1}\pi(\omega)\delta. \tag{6.9}$$

If α *and* β *are two singular cochains, then* $\delta\alpha \cup \beta = \delta(\alpha \cup \beta) + (-1)^{|\alpha|+1}\alpha \cup \delta\beta$. *By comparison with this, (6.9) is reasonable. The proof of (6.9) is as follows.*

$$\begin{aligned}
&\pi(d\omega)q = \sum_p \left(\int_{\mathcal{W}(p,q)} d\omega\right) p \\
= &\sum_p \left(\int_{\overline{\mathcal{W}(p,q)}} e^* d\omega\right) p = \sum_p \left(\int_{\partial^1 \overline{\mathcal{W}(p,q)}} e^*\omega\right) p \\
= &\sum_p \left(\sum_r \int_{\mathcal{M}(p,r)\times\mathcal{W}(r,q)} e^*\omega + \sum_r (-1)^{ind(p)-ind(r)+1} \int_{\mathcal{W}(p,r)\times\mathcal{M}(r,q)} e^*\omega\right) p.
\end{aligned}$$

Here e *is defined in (3) of Theorem 3.5. When* $dim(\mathcal{W}(r,q)) < |\omega|$ *(or* $dim(\mathcal{W}(p,r)) < |\omega|$*),* $e^*\omega = 0$ *on* $\mathcal{M}(p,r)\times\mathcal{W}(r,q)$ *(or* $\mathcal{W}(p,r)\times\mathcal{M}(r,q)$*). Thus*

$$\begin{aligned}
\pi(d\omega)q &= \sum_p \left(\sum_{ind(r)=ind(p)-1} \#\mathcal{M}(p,r) \int_{\mathcal{W}(r,q)} \omega \right. \\
&\qquad \left. + \sum_{ind(r)=ind(q)+1} (-1)^{ind(p)-ind(q)} \#\mathcal{M}(r,q) \int_{\mathcal{W}(p,r)} \omega \right) p \\
&= \delta\pi(\omega)q + (-1)^{ind(p)-ind(q)}\pi(\omega)\delta q.
\end{aligned}$$

This completes the proof since $ind(p) - ind(q) = |\omega| + 1$.

7. CW Structure

7.1. **Proof of Theorem 3.7.** We present an elementary proof here. A non-elementary one is sketched in Remark 7.1.

Recall the evaluation map $e : \overline{\mathcal{D}(p)} \longrightarrow M$ in (3) of Theorem 3.4. We shall "pull back" the vector field $-\nabla f$ on M to $\overline{\mathcal{D}(p)}$ via e. First, we need to explain the definition of the pull back. We know $\overline{\mathcal{D}(p)} = \bigsqcup_I \mathcal{D}_I$, where I are critical sequences with head p. The restriction of e on $\mathcal{D}_I = \mathcal{M}_I \times \mathcal{D}(r_k)$ is the projection $\mathcal{M}_I \times \mathcal{D}(r_k) \longrightarrow \mathcal{D}(r_k)$. For all $(\alpha, x) \in \mathcal{M}_I \times \mathcal{D}(r_k)$, $\{0\} \times T_x\mathcal{D}(r_k) \subseteq T_\alpha \mathcal{M}_I \times T_x\mathcal{D}(r_k) = T_{(\alpha,x)}(\mathcal{M}_I \times \mathcal{D}(r_k))$ and the derivative of e gives an isomorphism $de : \{0\} \times T_x\mathcal{D}(r_k) \longrightarrow T_x\mathcal{D}(r_k)$. Thus there is a unique vector $(0, -\nabla f) \in \{0\} \times T_x\mathcal{D}(r_k)$ such that $de(0, -\nabla f) = -\nabla f$. Then $(0, -\nabla f(x)) \in T_{(\alpha,x)}(\mathcal{M}_I \times \mathcal{D}(r_k))$ is the pull back of $-\nabla f(x)$.

Lemma 7.1. *There is a smooth vector field X on $\overline{\mathcal{D}(p)}$ such that $\forall (\alpha, z) \in \mathcal{M}_I \times \mathcal{D}(r_k)$, $X(\alpha, z) \in \{0\} \times T_z\mathcal{D}(r_k)$ and $de(X) = -\nabla f$.*

Proof. Let X be the pull back of $-\nabla f$ as explained above. We only need to prove that X is smooth.

Suppose the critical values in $(-\infty, f(p)]$ are exactly $f(p) = c_0 > c_1 > \cdots > c_l$. Let $U(i) = e^{-1} \circ f^{-1}((c_{i+1}, c_{i-1}))$, where $c_{-1} = +\infty$ and $c_{l+1} = -\infty$. By Theorem 3.4, each $U(i)$ is open and $\bigcup_i U(i) = \overline{\mathcal{D}(p)}$, and we only need to prove that X is smooth in each $U(i)$. By Lemma 5.4, there is a smooth embedding $E(i) : U(i) \longrightarrow \prod_{j=0}^{i-1} f^{-1}(a_j) \times M(i)$, where $a_j \in (c_{j+1}, c_j)$ is a regular value and $M(i) = f^{-1}((c_{i+1}, c_{i-1}))$. Define a vector field $\widehat{X} = (0, \cdots, 0, -\nabla f) \in \prod_{j=0}^{i-1} Tf^{-1}(a_j) \times TM(i)$ on $\prod_{j=0}^{i-1} f^{-1}(a_j) \times M(i)$. Clearly, $\widehat{X}$ is smooth. For brevity, denote $E(i)$ by E. We shall prove that the restriction of $\widehat{X}$ on $E(U(i))$ is X.

Each $(\alpha, z) \in (\mathcal{M}_I \times \mathcal{D}(r_k)) \cap U(i)$ represents a pair (Γ, z), where Γ is a generalized flow line connecting p and z (see (5.2)). Suppose $\Gamma = (\gamma_0, \cdots, \gamma_n)$, where $\gamma_0 \equiv p$ and $\gamma_n(0) = z$. Suppose the intersection of Γ with $f^{-1}(a_j)$ is z_j. Then $\xi(t) = (z_0, \cdots, z_{i-1}, \gamma_n(t))$ is a curve in $E(U(i)) \subseteq \prod_{j=0}^{i-1} f^{-1}(a_j) \times M(i)$ such that

$$\xi'(0) = (0, \cdots, 0, -\nabla f) = \widehat{X}, \qquad de \cdot \xi'(0) = -\nabla f.$$

Moreover, since $\xi(t) \subseteq E(\{\alpha\} \times \mathcal{D}(r_k))$, we infer $\xi'(0) \in dE(\{0\} \times T_z\mathcal{D}(r_k))$. Identify $U(i)$ with $E(U(i))$, then $\widehat{X} = \xi'(0) = X$ at (α, z). This completes the proof. $\square$

In the following, we use the terminology of [13]. It's easy to see that Definition 7.2 is equivalent to the *secteur tangent* in [13, p. 3].

Definition 7.2. *Suppose L is a manifold with corners. For all $x \in L$,*

$$A_x L = \{v \in T_x L \mid v = \gamma'(0) \text{ for some smooth curve } \gamma : [0, \epsilon) \longrightarrow L.\}$$

is the tangent sector of L at x.

Definition 7.3. *Suppose L is a manifold with corners, $\partial^k L$ is the k-stratum $(k > 0)$ of L, $x \in \partial^k L$ and $v \in T_x L$. v is in the corner if $v \in T_x \partial^k L$. v is outward if $v \notin A_x L$. v is strictly outward if $-v$ is in the interior of $A_x L$.*

Clearly, strictly outward implies outward. We know that $A_x L$ is linear isomorphic to $[0, +\infty)^k \times R^{n-k}$. Under this isomorphism, v is in the corner if and only if $v \in \{0\}^k \times R^{n-k}$; v is strictly outward if and only if $v \in (-\infty, 0)^k \times R^{n-k}$. This does not depend on the isomorphisms. It's easy to see the above vector field X is in the corner. We present the following easy lemma without proof.

Lemma 7.4. *If both v_1 and v_2 are strictly outward, so are $v_1 + v_2$ and lv_1 for $l > 0$. If v_1 is strictly outward and v_2 is in the corner, then $v_1 + v_2$ is strictly outward.*

The proof of the following lemma is in the Appendix.

Lemma 7.5. *Suppose L is a manifold with corners, and $g : L \longrightarrow H$ is a smooth map where H is a Hilbert space. If there exists a smooth map $\tilde{g} : L \longrightarrow S(H)$ such that $g(x) = \|g(x)\|\tilde{g}(x)$, then $\|g(x)\|$ is also smooth, where $S(H)$ is the unit sphere of H.*

Let $\tilde{f} = f \circ e$ defined on $\overline{\mathcal{D}(p)}$ be the pull back of f, then $X \cdot \tilde{f} = -\|(\nabla f)e\|^2 \leq 0$.

Lemma 7.6. *Suppose $x \in \overline{\mathcal{D}(p)}$ be such that $e(x)$ is a critical point. Let U_x be a neighborhood of x. Then there is a smooth vector field Y_x on $\overline{\mathcal{D}(p)}$ such that its support $\mathrm{supp}(Y_x) \subseteq U_x$, $Y_x(x) \neq 0$ and $Y_x \tilde{f} \leq 0$. In addition, for all $y \in \partial\overline{\mathcal{D}(p)}$, $Y_x(y)$ is strictly outward if $Y_x(y) \neq 0$.*

Proof. Suppose $e(x) = r_k$ for some critical point r_k and $x = (\alpha, r_k) \in \mathcal{M}_I \times \mathcal{D}(r_k)$, where $I = \{p, r_1, \cdots, r_k\}$. By Lemma 5.5, there exist a neighborhood W_1 of α in $\mathcal{M}_I$, a neighborhood W_2 of r_k in $\mathcal{D}(r_k)$, an $\epsilon > 0$ and a smooth embedding $\varphi : W_1 \times W_2 \times [0, \epsilon)^k \longrightarrow \overline{\mathcal{D}(p)}$ such that $\mathrm{Im}\varphi \subseteq U_x$, and φ satisfies the stratum condition in Lemma 5.5.

By local triviality of the metric, choose a neighborhood U of r_k as (2.2) such that (2.3) and (2.4) hold. We identify U with B by h in (2.2). We may assume $e(\mathrm{Im}\varphi) \subseteq U$, and W_2 is a neighborhood of 0 in

V_-. Identify $r_k \in \mathcal{D}(r_k)$ with $0 \in V_-$. The key part of the proof is to show φ can be modified so that

$$\tilde{f} \circ \varphi(\tilde{\alpha}, z, \rho_I, \sigma) = f(r_k) - \frac{1}{2}\langle z, z\rangle + \frac{1}{2}\sigma^2, \tag{7.1}$$

where $\tilde{\alpha} \in W_1$, $z \in W_2$, $\rho_I = (\rho_1, \cdots, \rho_{k-1}) \in [0, \epsilon)^{k-1}$ and $\sigma \in [0, \epsilon)$.

Denote $e \circ \varphi(\tilde{\alpha}, z, \rho_I, \sigma) = (e_1(\tilde{\alpha}, z, \rho_I, \sigma), e_2(\tilde{\alpha}, z, \rho_I, \sigma)) \in V_- \times V_+$. Consider the map $\theta : W_1 \times W_2 \times [0, \epsilon)^k \longrightarrow W_1 \times W_2 \times [0, \epsilon)^k$ defined by

$$\theta(\tilde{\alpha}, z, \rho_I, \sigma) = (\tilde{\alpha}, e_1(\tilde{\alpha}, z, \rho_I, \sigma), \rho_I, \|e_2(\tilde{\alpha}, z, \rho_I, \sigma)\|).$$

Firstly, we prove θ is smooth. It suffices to show $\|e_2\|$ is smooth. Since e_2 is smooth, by Lemma 7.5, we only need to find a smooth $\tilde{g}$ such that $e_2 = \|e_2\|\tilde{g}$. By (5.2), an element in $\overline{\mathcal{D}(p)}$ represents a pair (Γ, z), where Γ is a generalized flow line connecting p and $z \in M$. Let $c = f(r_k)$. Define $E : \overline{\mathcal{D}(p)} \cap e^{-1} \circ f^{-1}((c-\epsilon, c+\epsilon)) \longrightarrow f^{-1}(c+\frac{\epsilon}{2}) \times M$ to be the map $E(\Gamma, z) = (s(\Gamma), z)$, where $s(\Gamma)$ is the intersection of Γ with $f^{-1}(c+\frac{\epsilon}{2})$. By Lemma 5.4, E is smooth. Furthermore, $E\varphi(\tilde{\alpha}, z, \rho_I, \sigma) = ((\eta_1, \eta_2), (e_1, e_2)) \in V_- \times V_+ \times V_- \times V_+$. By the stratum condition in Lemma 5.5, $e\varphi(\tilde{\alpha}, z, \rho_I, \sigma) \in \mathcal{D}(r_k)$ or $e_2 = 0$ if and only if $\sigma = 0$. Thus, when $\sigma > 0$, $e_2 \neq 0$ and (e_1, e_2) is connected with (η_1, η_2) by a unbroken flow line. Thus $(e_1, e_2) = (\lambda^{-1}\eta_1, \lambda\eta_2)$ for some $\lambda > 0$ and $e_2/\|e_2\| = \eta_2/\|\eta_2\|$. However, $\eta_2 \neq 0$ even if $\sigma = 0$. Thus $\eta_2/\|\eta_2\|$ is smooth for all $\sigma \in [0, \epsilon)$. Let $\tilde{g}(\tilde{\alpha}, z, \rho_I, \sigma) = \eta_2/\|\eta_2\|$, then $e_2 = \|e_2\|\tilde{g}$ for all $\sigma \in [0, \epsilon)$. Thus $\|e_2\|$ is smooth.

Secondly, we prove that $\frac{\partial}{\partial \sigma}\|e_2\| \neq 0$ at $(\alpha, 0, 0, 0)$. By the stratum condition, $d\varphi\frac{\partial}{\partial \sigma}$ represents an inward normal vector in $N_{(\alpha, r_k)}(\mathcal{M}_I \times \mathcal{D}(r_k), \mathcal{M}_J \times \overline{\mathcal{D}(r_{k-1})})$, where $J = \{p, r_1, \cdots, r_{k-1}\}$. Thus by Lemma 5.6, $0 \neq de_2\frac{\partial}{\partial \sigma} \in V_-$. Denote $de_2\frac{\partial}{\partial \sigma}$ by w. Since $e_2(\alpha, 0, 0, 0) = 0$, we see $e_2(\alpha, 0, 0, \sigma) = \sigma w + O(\sigma^2)$, and

$$\frac{\partial}{\partial \sigma}|_{\sigma=0}\|e_2\| = \lim_{\sigma \to 0+} \frac{\|\sigma w + O(\sigma^2)\|}{\sigma} = \|w\| \neq 0.$$

Thirdly, the Jacobian of θ at $(\alpha, 0, 0, 0)$ is

$$\begin{pmatrix} \frac{\partial}{\partial \tilde{\alpha}} & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial z} & de_1\frac{\partial}{\partial \rho_I} & de_2\frac{\partial}{\partial \sigma} \\ 0 & 0 & \frac{\partial}{\partial \rho_I} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial \sigma}\|e_2\| \end{pmatrix}.$$

Since $\frac{\partial}{\partial \sigma}\|e_2\| \neq 0$, $d\theta$ is nonsingular at $(\alpha, 0, 0, 0)$.

Since $\|e_2\|$ is smooth, $\frac{\partial}{\partial \sigma}|_{\sigma=0}\|e_2\| \neq 0$, and $\|e_2\|$ vanishes if and only if $\sigma = 0$, we can extend $\|e_2\|$ to be defined on $W_1 \times W_2 \times (-\epsilon, \epsilon)^k$ such that

$\|e_2\| < 0$ when $\sigma < 0$. By the Inverse Function Theorem, shrinking W_1, W_2 and ϵ suitably, a smooth θ^{-1} can be defined in $W_1 \times W_2 \times [0,\epsilon)^k$. Modify φ to be $\varphi \circ \theta^{-1}$ to get a smooth embedding $\varphi : W_1 \times W_2 \times [0,\epsilon)^k \longrightarrow \overline{\mathcal{D}(p)}$ such that $e \circ \varphi(\tilde{\alpha}, z, \rho_I, \sigma) = (z, e_2)$ and $\|e_2\| = \sigma$. This gives (7.1).

Consider the vector field $\widetilde{Y} = \sum_{i=1}^{k-1}(\rho_i - \epsilon)\frac{\partial}{\partial \rho_i} + (\sigma - \epsilon)\frac{\partial}{\partial \sigma}$ in $W_1 \times W_2 \times [0,\epsilon)^k$. It's strictly outward at corners, $\widetilde{Y}(\varphi^{-1}(x)) \neq 0$ and $\widetilde{Y}(\tilde{f} \circ \varphi) = (\sigma - \epsilon)\sigma \leq 0$.

By Lemma 7.4, using the partition of the unity, we can move $\widetilde{Y}$ to $\overline{\mathcal{D}(p)}$. This defines the desired smooth vector field Y_x. □

Lemma 7.7. *Suppose $x \in \overline{\mathcal{D}(p)}$ is such that $e(x)$ is a regular point. Let U_x be a neighborhood of x. Then there is a smooth vector field Y_x on $\overline{\mathcal{D}(p)}$ such that its support* $\mathrm{supp}(Y_x) \subseteq U_x$*, $Y_x(x) \neq 0$ and $Y_x \tilde{f} = 0$. In addition, $\forall y \in \partial\overline{\mathcal{D}(p)}$, $Y_x(y)$ is strictly outward if $Y_x(y) \neq 0$.*

Proof. Suppose $x \in \mathcal{M}_I \times \mathcal{D}(r_k)$. By Lemma 5.5, there is a smooth embedding $\varphi : W \times [0,\epsilon)^k \longrightarrow \overline{\mathcal{D}(p)}$ such that $\mathrm{Im}\varphi \subseteq U_x$ where W is a neighborhood of x in $\mathcal{M}_I \times \mathcal{D}(r_k)$. Since $e(x)$ is a regular point, $X\tilde{f}(x) = -\|\nabla f(e(x))\|^2 < 0$. Shrinking W and ϵ suitably, we may assume $X\tilde{f} < 0$ in $\mathrm{Im}\varphi$.

Denote the coordinates of $[0,\epsilon)^k$ by $(\rho_1, \cdots, \rho_k)$. Then $\sum_{i=1}^{k}(\rho_i - \epsilon)\frac{\partial}{\partial \rho_i}$ defines a vector field on $W \times [0,\epsilon)^k$ which is strictly outward at corners. Move this one to $\mathrm{Im}\varphi$ to get a strictly outward vector field Y_1 on $\mathrm{Im}\varphi$. Let $Y_2 = Y_1 - \frac{Y_1\tilde{f}}{X\tilde{f}}X$. Then $Y_2\tilde{f} = 0$. Since Y_1 is strictly outward, and X is in the corner, we get, by Lemma 7.4, Y_2 is strictly outward and $Y_2(x) \neq 0$. Using a partition of the unity, we get Y_x. □

As mentioned in Introduction, the following key lemma fulfills Milnor's suggestion of adding a vector field to X.

Lemma 7.8. *Suppose $K \subseteq \mathcal{D}(p) \subseteq \overline{\mathcal{D}(p)}$, K is closed and p is an interior point of K. Then there is a smooth vector field $\widetilde{X}$ on $\overline{\mathcal{D}(p)}$ such that $\widetilde{X}\tilde{f} \leq X\tilde{f} = (-\nabla f)f$, $\widetilde{X}$ equals X and $-\nabla f$ on K, and $\widetilde{X}$ is strictly outward on $\partial\overline{\mathcal{D}(p)}$.*

Proof. Since K is closed, $\overline{\mathcal{D}(p)} - K$ is open. Since $K \subseteq \mathcal{D}(p)$, then $\overline{\mathcal{D}(p)} - K \supseteq \partial\overline{\mathcal{D}(p)}$. Thus $\forall x \in \partial\overline{\mathcal{D}(p)}$, by Lemmas 7.6 and 7.7, there is a vector field Y_x such that $\mathrm{supp}(Y_x) \subseteq \overline{\mathcal{D}(p)} - K$ and satisfies the conclusions of those lemmas. Define $W_x = \{y | Y_x(y) \neq 0\}$, we have W_x is a neighborhood of x. Since $\partial\overline{\mathcal{D}(p)}$ is compact, it can be covered by finite many W_{x_i} $(i = 1, \cdots, n)$. Let $Y = \sum_{i=1}^{n} Y_{x_i}$. Since $Y_{x_i}\tilde{f} \leq 0$, we get $Y\tilde{f} \leq 0$. Since Y_{x_i} vanishes on K, so does Y. Also since $\{W_{x_i} \mid i = 1, \cdots, n\}$ covers $\partial\overline{\mathcal{D}(p)}$, and Y_{x_i} is strictly outward if it's nonzero, by Lemma 7.4, we have that Y is strictly outward. Recall that X is in the corner on $\partial\overline{\mathcal{D}(p)}$. We complete the proof by defining $\widetilde{X} = X + Y$. □

Lemma 7.9. *Let $\phi_t(x)$ be the flow line of $\widetilde{X}$ with initial value x and $x \neq p$. Then $\phi_t(x)$ reaches $\partial\overline{\mathcal{D}(p)}$ at a unique time $0 \leq \omega(x) < +\infty$. Furthermore, $\omega(x)$ is continuous with respect to x in $\overline{\mathcal{D}(p)} - \{p\}$.*

Proof. Above all, we prove the following claim: If $\phi_t(x)$ cannot reach $\partial\overline{\mathcal{D}(p)}$ when $t \geq 0$, then $\phi_t(x)$ exists for $t \in [0, +\infty)$.

If not, the maximal positive flow of $\phi_t(x)$ can only be defined in $[0, s]$ or $[0, s)$, where $s < +\infty$. If the domain is $[0, s]$, then $\phi_s(x) \in \partial\overline{\mathcal{D}(p)}$. This is a contradiction. If the domain is $[0, s)$, by the compactness of $\overline{\mathcal{D}(p)}$, $\phi_t(x)$ has a cluster point y_0 when $t \to s$. There are two cases. Case (1): $y_0 \in \mathcal{D}(p)$. In this case, there is a neighborhood U_{y_0} of y_0 such that there exists $\delta > 0$ such that, for all $y \in U_{y_0}$, and for all $t \in (-\delta, +\delta)$, $\phi_t(y)$ exists. Thus $\phi_t(x)$ can be defined in $[0, s + \delta)$. This is a contradiction. Case (2): $y_0 \in \partial\overline{\mathcal{D}(p)}$. In this case, a neighborhood U_{y_0} of y_0 is diffeomorphic to an open subset of $[0, +\infty)^k \times R^{n-k}$ for some k and n. The vector field in U_{y_0} can be smoothly extended to an open subset of R^n. Then we may consider y_0 as an interior point. This converts the argument to the first case. We can define $\phi_t(x)$ for $t \in [0, s]$ with $\phi_s(x) = y_0$. This is also a contradiction. This gives the claim.

Secondly, we prove that $\phi_t(x)$ reaches $\partial\overline{\mathcal{D}(p)}$ at some time $0 \leq \omega(x) < +\infty$ by contradiction.

Suppose $\phi_t(x)$ doesn't reach $\partial\overline{\mathcal{D}(p)}$. By the claim, $\phi_t(x)$ exists for $t \in [0, +\infty)$. By the assumption, $m = \inf_M f > -\infty$. For all $y \in \overline{\mathcal{D}(p)}$, $\tilde{f}(y) \leq \tilde{f}(p) = f(p)$. For all $T \geq 0$,

$$\int_0^T \widetilde{X}\tilde{f}(\phi_t(x))dt = \tilde{f}(\phi_T(x)) - \tilde{f}(\phi_0(x)) \geq m - f(p) > -\infty. \tag{7.2}$$

Since $\widetilde{X}\tilde{f} \leq X\tilde{f} \leq 0$, then there exists $\{t_n\} \subseteq [0, +\infty)$, $t_n \to +\infty$ and $\widetilde{X}\tilde{f}(\phi_{t_n}(x)) \to 0$. Since $\overline{\mathcal{D}(p)}$ is compact, we may assume $\phi_{t_n}(x) \to y_0$. Then $0 = \widetilde{X}\tilde{f}(y_0) \leq X\tilde{f}(y_0) \leq 0$. Since $X\tilde{f}(y_0) = -\|\nabla f(e(y_0))\|^2$, we see that $e(y_0)$ is a critical point. Thus $y_0 \in \partial\overline{\mathcal{D}(p)}$. Choose a neighborhood U_{y_0} of y_0 which is diffeomorphic to $[0, \epsilon)^k \times B(0, \epsilon)$, where $B(0, \epsilon) = \{v \in R^{n-k} \mid \|v\| < \epsilon\}$ and y_0 is identified with $0 \in [0, \epsilon)^k \times B(0, \epsilon)$. Identify U_{y_0} with $[0, \epsilon)^k \times B(0, \epsilon)$. We may assume $\widetilde{X}$ can be extended smoothly to $(-\epsilon,\epsilon)^k \times B(0,\epsilon)$. Denote the flow of the extended vector field by φ_t. Then $\varphi_t(y_0) = \varphi_t(0) = t\widetilde{X}(0) + O(t^2)$. Since $\widetilde{X}(0)$ is outward, there exists $\delta_1 > 0$, such that for all $\delta \in (0, \delta_1]$, $\varphi_\delta(0) \in (-\epsilon,\epsilon)^k \times B(0,\epsilon) - [0,\epsilon)^k \times B(0,\epsilon)$. Fixing δ, there exists $\epsilon_1 > 0$, for all $y \in [0,\epsilon_1)^k \times B(0,\epsilon_1)$, $\varphi_t(y)$ exists for $t \in [-\delta,\delta]$ and $\varphi_\delta(y) \in (-\epsilon,\epsilon)^k \times B(0,\epsilon) - [0,\epsilon)^k \times B(0,\epsilon)$. Since $(0,\epsilon)^k \times B(0,\epsilon)$ and $(-\epsilon,\epsilon)^k \times B(0,\epsilon) - [0,\epsilon)^k \times B(0,\epsilon)$ are disconnected, we have $\varphi_{t_0}(y) \in [0,\epsilon)^k \times B(0,\epsilon) - (0,\epsilon)^k \times B(0,\epsilon)$ at some time $t_0 \in [0,\delta)$. Since $\phi_{t_n}(x) \in [0,\epsilon_1)^k \times B(0,\epsilon_1)$ for some t_n, we have $\phi_{t_n+t_0}(x) \in \partial\overline{\mathcal{D}(p)}$ for some $t_0 \in [0,\delta)$. This gives a contradiction.

Finally, we prove that $\omega(x)$ is unique and continuous.

Since $\widetilde{X}$ is outward, $\phi_t(x)$ does not exist after it reaches $\partial\overline{\mathcal{D}(p)}$. Thus $\omega(x)$ is unique. Denote $y_0 = \phi_{\omega(x_0)}(x_0) \in \partial\overline{\mathcal{D}(p)}$, by the argument at the end of the second step, we have, $\forall\delta > 0$, there is a neighborhood U_{y_0} of y_0 such that, for all $y \in U_{y_0}$, $\phi_{t_0}(y) \in \partial\overline{\mathcal{D}(p)}$ for some $t_0 \in [0,\delta)$. Then there exist a neighborhood U_{x_0} of x_0 and $\delta_2 > 0$ such that, for all $x \in U_{x_0}$, $\phi_{\omega(x_0)-\delta_2}(x)$ exists and is in U_{y_0}. Thus $\omega(x) \le \omega(x_0)+\delta$. Since $\omega(x) \ge 0$, and $\omega(x) = 0$ when $x \in \partial\overline{\mathcal{D}(p)}$, we get $\omega(x)$ is continuous at $x_0 \in \partial\overline{\mathcal{D}(p)}$. If $x_0 \in \mathcal{D}(p)$, then for all $\delta > 0$, there exists $\delta_2 \in (0,\delta)$, such that $\phi_{\omega(x_0)-\delta_2}(x_0)$ exists and is in $\mathcal{D}(p)$. Also, there exists U_{x_0} such that, for all $x \in U_{x_0}$, $\phi_{\omega(x_0)-\delta_2}(x)$ exists and is in $\mathcal{D}(p)$. Thus $\omega(x) \ge \omega(x_0) - \delta$. We have now proved $\omega(x)$ is continuous in general. □

Actually, the above lemma only requires $\widetilde{X}$ to be outward. However, the following one requires $\widetilde{X}$ to be strictly outward.

Lemma 7.10. *Let $\phi_t(x)$ be the flow line of $\widetilde{X}$ with initial value x. Then $\phi_t(x)$ exists for $t \in (-\infty,0]$ and $\lim_{t\to-\infty}\phi_t(x) = p$.*

Proof. Firstly, we prove that $\phi_t(x)$ exists for $t \in (-\infty,0]$ by contradiction. If not, the maximal negative flow can only be defined for $[s,0]$ or $(s,0]$, where $s > -\infty$.

Suppose the domain is $[s,0]$. If $\phi_s(x) \in \mathcal{D}(p)$, then $\phi_t(x)$ can be defined in $(s-\delta,0]$ for some $\delta > 0$. This is a contradiction. Suppose $\phi_s(x) = x_0 \in \partial\overline{\mathcal{D}(p)}$. Like the proof of Lemma 7.9, a neighborhood of x_0 is identified with $[0,\epsilon)^k \times B(0,\epsilon)$ and x_0 is identified with 0. Extend the vector field in $[0,\epsilon)^k \times B(0,\epsilon)$ smoothly to be defined in $(-\epsilon,\epsilon)^k \times B(0,\epsilon)$. Denote the flow of the extended vector field by φ_t. Since $\widetilde{X}(0) = \widetilde{X}(x_0)$ is strictly outward, then $-\widetilde{X}(0) \in (0,+\infty)^k \times R^{n-k}$. Since $\varphi_t(x_0) = \varphi_t(0) = t\widetilde{X}(0) + O(t^2)$, there exists $\delta > 0$ such that, for all $t \in [-\delta,0]$, we have $\varphi_t(0) \in [0,\epsilon)^k \times B(0,\epsilon)$. Thus $\phi_t(x)$ exists for $t \in [s-\delta,0]$. This gives a contradiction.

Suppose the domain is $(s, 0]$. Using the same argument as in the proof of Lemma 7.9, we can extend the domain to be $[s, 0]$. This gives a contradiction.

As a result, we proved the first assertion.

Secondly, we prove by contradiction that $\phi_t(x)$ has no cluster point in $\partial\overline{\mathcal{D}(p)}$ when $t \to -\infty$.

Suppose $\phi_t(x)$ has a cluster point $x_0 \in \partial\overline{\mathcal{D}(p)}$. By the continuity of $\omega(x)$ in Lemma 7.9, there exists a neighborhood U_{x_0} of x_0 such that, for all $x \in U_{x_0}$, we have $\omega(x) \in [0, 1)$. Since x_0 is a cluster point, there exist $T < -1$, and $\phi_T(x) \in U_{x_0}$. Thus $\phi_{T+t_0}(x) \in \partial\overline{\mathcal{D}(p)}$ for some $t_0 \in [0, 1)$. Then $\phi_t(x)$ does not exist when $t > T + t_0$. In particular, $\phi_t(x)$ does not exist when $t = 0$. This gives a contradiction.

Thirdly, we prove by contradiction that $\phi_t(x)$ has no cluster point in $\mathcal{D}(p) - \{p\}$ when $t \to -\infty$.

Suppose $x_0 \in \mathcal{D}(p) - \{p\}$ is a cluster point. Clearly, $\widetilde{X}\tilde{f}(x_0) \leq Xf(x_0) = -\|\nabla f(e(x_0))\|^2 = A < 0$. Thus there exists a neighborhood U_{x_0} of x_0, a $\delta > 0$, for all $x \in U_{x_0}$, such that $\phi_t(x)$ exists for $t \in [-\delta, \delta]$ and $\widetilde{X}\tilde{f}(\phi_t(x)) \leq \frac{A}{2}$ in this interval. Since x_0 is a cluster point, there exists $\{t_n\} \subseteq (-\infty, 0]$ such that $t_{n+1} < t_n - \delta$ and $\phi_{t_n}(x) \in U_{x_0}$. Then

$$\int_{-\infty}^{0} \widetilde{X}\tilde{f}(\phi_t(x))dt \leq \sum_{n=1}^{\infty} \int_{t_n-\delta}^{t_n} \widetilde{X}\tilde{f}(\phi_t(x))dt \leq \sum_{n=1}^{\infty} \int_{t_n-\delta}^{t_n} \frac{A}{2} = -\infty.$$

On the other hand, similar to (7.2), we have for all $T < 0$, $\int_T^0 \widetilde{X}\tilde{f}(\phi_t(x))dt \geq \tilde{f}(x) - \tilde{f}(p) > -\infty$. This gives a contradiction.

Finally, since $\overline{\mathcal{D}(p)}$ is compact, $\forall\{t_n\} \subseteq (-\infty, 0]$, there must be a cluster point of $\phi_{t_n}(x)$. Thus $\phi_t(x) \to p$ when $t \to -\infty$. $\square$

Now we are ready to prove Theorem 3.7. The idea of this proof is as follows. Choose a closed neighborhood K of p in $\overline{\mathcal{D}(p)}$ which is diffeomorphic to $D^{\mathrm{ind}(p)}$. The flow line ϕ_t of the above $\widetilde{X}$ expands K homeomorphically onto $\overline{\mathcal{D}(p)}$. We also explain this idea by the previous example on T^2. The flow generated by X on $\overline{\mathcal{D}(p)}$ is as the right part of Figure 3. The flow generated by $\widetilde{X}$ is illustrated by Figure 4.

Proof of Theorem 3.7. Choose a closed neighborhood K of p in $\overline{\mathcal{D}(p)}$ satisfying the following two properties: (1). $K \subseteq \mathcal{D}(p)$. (2). There is a diffeomorphism $\theta : D(\epsilon) \longrightarrow K$ such that $\theta(0) = p$, $\tilde{f} \circ \theta(v) = f(p) - \frac{1}{2}\langle v, v\rangle$ and $((d\theta)^{-1}X)(v) = v$, where $D(\epsilon) = \{v \in R^{\mathrm{ind}(p)} \mid \|v\| \leq \epsilon\}$.

We only need to construct a homeomorphism $\Psi : (D(\epsilon), S(\epsilon)) \longrightarrow (\overline{\mathcal{D}(p)}, \partial\overline{\mathcal{D}(p)})$, where $S(\epsilon) = \partial D(\epsilon)$.

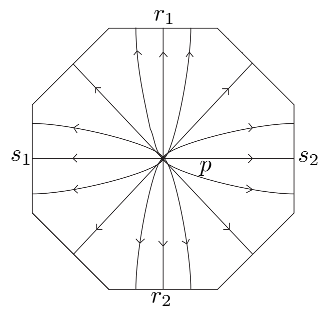

FIGURE 4. Flow Generated by $\widetilde{X}$

By Lemmas 7.8, 7.9 and 7.10, there is a vector field $\widetilde{X}$ on $\overline{\mathcal{D}(p)}$ satisfying the following four properties: (1). We have $\widetilde{X} = X$ in K. (2). We have $\widetilde{X}\tilde{f} < 0$ in $\mathcal{D}(p) - \{p\}$. (3). The flow $\phi_t(x)$ generated by $\widetilde{X}$ reaches the boundary at a unique time $\omega(x) \in [0, +\infty)$ when $x \neq p$, and $\omega(x)$ is continuous in $\overline{\mathcal{D}(p)} - \{p\}$. (4). For all x, $\phi_t(x) \to p$ when $t \to -\infty$.

Denote φ_t the flow generated by the vector field $Z(v) = v$ on $D(\epsilon)$. Then $\theta(\varphi_t(v)) = \phi_t(\theta(v))$.

Define $\beta(s)$ in $[0, \epsilon]$ to be

$$\beta(s) = \begin{cases} 0 & t \in [0, \frac{\epsilon}{2}], \\ \frac{2t-\epsilon}{\epsilon} & t \in [\frac{\epsilon}{2}, \epsilon]. \end{cases}$$

Define $\Psi : D(\epsilon) \longrightarrow \overline{\mathcal{D}(p)}$ to be

$$\Psi(v) = \begin{cases} \theta(v) & \|v\| \in [0, \frac{\epsilon}{2}], \\ \phi[\omega[\theta(\frac{v}{\|v\|}\epsilon)]\beta(\|v\|), \theta(v)] & \|v\| \in [\frac{\epsilon}{2}, \epsilon]. \end{cases}$$

Here we use the notation $\phi(t, x) = \phi_t(x)$.

Firstly, Ψ is continuous, $\Psi(S(\epsilon)) \subseteq \partial\overline{\mathcal{D}(p)}$ and $\Psi^{-1}(\partial\overline{\mathcal{D}(p)}) \subseteq S(\epsilon)$.

Secondly, we prove that Ψ is injective. Consider the orbits of the flows. The orbits in $D(\epsilon)$ are $\{0\}$ and $\{sv \mid \|v\| = \epsilon, s \in (0, 1]\}$. We have $\Psi(0) = p$ and $\Psi(sv) = \phi(l(s, v), \theta(v))$, where $l(s, v) = \omega(\theta(v))\beta(s\epsilon) + \log s$ and $\|v\| = \epsilon$. When $\|v\| \equiv \epsilon$, $\tilde{f}(\theta(v)) \equiv \frac{1}{2}f(p) - \frac{1}{2}\epsilon^2$, by the above property (2) of $\widetilde{X}$, we have Ψ maps distinct orbits to distinct orbits. Since $l(s, v)$ is a strictly increasing function with respect to s, by the above property (2) of $\widetilde{X}$ again, we have Ψ is injective.

Thirdly, we prove that Ψ is surjective. Clearly, $\Psi(0) = p$. For all $x \in \overline{\mathcal{D}(p)}$ and $x \neq p$, since $\phi_t(x) \to p$ when $t \to -\infty$, we have that there exist $t_0 \in R$ and $v_0 \in D(\epsilon)$ such that $\phi_{-t_0}(x) \in K$, $\|v_0\| = \epsilon$, and $v_0 =$

$\theta^{-1}(\phi_{-t_0}(x))$. Then $t_0 \leq \omega(\theta(v_0))$. Since $l(s, v_0) \to -\infty$ when $s \to 0$ and $l(s, v)$ is continuous, the range of $l(s, v_0)$ is $(-\infty, \omega(\theta(v_0))]$. Then there exists s_0 such that $l(s_0, v_0) = t_0$. Thus $\Psi(s_0 v_0) = x$. Therefore, Ψ is surjective.

Finally, Ψ is a map from a compact space to a Hausdorff space, so Ψ is a homeomorphism. □

Remark 7.1. *The following is a quick but non-elementary proof of Theorem 3.7, which is based on the Poincaré Conjecture in all dimensions. Clearly, $\overline{\mathcal{D}(p)}$ is a compact topological manifold with boundary whose interior is an open disk. We can prove that $\partial\overline{\mathcal{D}(p)}$ is a homotopy sphere. By the Poincaré Conjecture, $\partial\overline{\mathcal{D}(p)}$ is a topological sphere. Consider a collar embedding $\partial\overline{\mathcal{D}(p)} \times [0,1] \longrightarrow \overline{\mathcal{D}(p)}$ which identifies $\partial\overline{\mathcal{D}(p)} \times \{0\}$ with $\partial\overline{\mathcal{D}(p)}$. By the Generalized Schoenflies Theorem (see* [4, thm. 5]*), we can prove that $\overline{\mathcal{D}(p)} - \partial\overline{\mathcal{D}(p)} \times [0, \frac{1}{2})$ is a closed disk. This completes the proof.*

7.2. **Proof of Theorem 3.8.** As mentioned in Subsection 3.4, the CW complex structure of K^a immediately results from Theorems 3.4 and 3.7. We only need to prove that, when f is proper, M^a has the desired CW decomposition. By Theorem 3.7, we can always construct a CW decomposition from a good vector field. The key part of this proof is to find a good vector field for M^a (see Lemma 7.14). This is heavily based on Milnor's dealing with gradient-like dynamics in [23].

Definition 7.11. *Suppose f is a Morse function on a Hilbert manifold. A vector field X is a gradient-like vector field of f if $X = \nabla f$ near each critical point of f and $Xf > 0$ at each regular point of f.*

Remark 7.2. *Some papers in the literature include the local triviality of X into the definition of a gradient-like vector field. We follow the style of* [36] *and exclude it.*

Up until now, we haven't assumed that M^a is compact and we have considered only negative gradient dynamics. In this subsection, we take M^a to be compact because we take f to be proper. The results proved before this subsection still hold for negative gradient-like dynamics when the underlying manifold is compact. There are two reasons. Both are sufficient. Firstly, on the compact manifold, Smale points out in [36, remark after thm. B] that all gradient-like dynamics are actually gradient dynamics (this is even true on a Hilbert manifold, see Lemma 7.12), and (M, f) is a CF pair automatically. Secondly, we can formally replace "gradient" by "gradient-like" in the above proofs when M is compact.

The proof of the following lemma is in the Appendix.

Lemma 7.12. *If X is a gradient-like vector field of a Morse function f on a Hilbert manifold M, then there is a metric on M such that this metric equals the original one associated with X near each critical point of f and $\nabla f = X$ for this metric.*

Since a is a regular value of f and f is proper, M^a is a compact manifold with boundary $f^{-1}(a)$. There is a smooth collar embedding $\varphi : [0, \epsilon_0) \times \partial M^a \longrightarrow M^a$ such that $f \circ \varphi(s, x) = a - s$. Clearly, all critical points of f are in $M^a - \mathrm{Im}\varphi$. Double M^a to be a compact manifold $2M^a$ without boundary such that the above φ can be extended in the obvious way to a smooth embedding $\varphi : (-\epsilon_0, \epsilon_0) \times \partial M^a \longrightarrow 2M^a$.

For convenience, we identify $(-\epsilon_0, \epsilon_0) \times \partial M^a$ with $\mathrm{Im}\varphi$ from now on.

There is an evident Z_2-symmetry group acting on $2M^a$. For all $x \in M^a \subseteq 2M^a$, denote $\bar{x} \in 2M^a$ the copy of x. Define $\sigma : 2M^a \longrightarrow 2M^a$ by $\sigma(x) = \bar{x}$ and $\sigma(\bar{x}) = x$. Then

$$Z_2 = \{\mathrm{Id}, \sigma\} \tag{7.3}$$

is the group. By the smooth structure of $2M^a$, Z_2 acts smoothly. The set of fixed points of Z_2 is $\mathrm{Fix}(Z_2; 2M^a) = \partial M^a$.

We omit the proof of the following, which is straightforward.

Lemma 7.13. *There exists a Morse Function F on $2M^a$ satisfying the following properties. (1). It is invariant under the Z_2 action. (2). It equals f on $M^a - \mathrm{Im}\varphi$. (3). We have $F(s, x) = a - \frac{1}{2}s^2 + g(x)$ in $(-\delta, \delta) \times \partial M^a$ for some $\delta \in (0, \epsilon_0)$, and g (and then $F|_{\partial M^a}$) is a Morse function on ∂M^a. (4). The critical points of F are exactly the critical points of f (which are in $M^a - \mathrm{Im}\varphi$) together with their images under the Z_2 action, and the critical points of g. (5) The function values of F on ∂M^a are greater than the function values at critical points off ∂M^a.*

We can define a metric G on $2M^a$ satisfying the following properties. (1). It is invariant under the Z_2 action. (2). It equals the original metric on $M^a - \mathrm{Im}\varphi$. (3). It is a product metric on $(-\delta, \delta) \times \partial M^a$, where $(-\delta, \delta)$ is given the standard metric. (4). It is locally trivial.

Lemma 7.14. *There is a negative gradient-like vector field ξ of F on $2M^a$ satisfying the following properties. (1). The vector field ξ is invariant under the Z_2 action. (2). It equals $-\nabla f$ on $M^a - Im\varphi$. (3). It satisfies local triviality and transversality. (4). For all $x \in \partial M^a$, $\xi(x) \in T_x \partial M^a$, and $\xi|_{\partial M^a}$ is a negative gradient-like vector field of $F|_{\partial M^a}$ on ∂M^a satisfying local triviality and transversality.*

Proof. We shall modify $-\nabla F$ to be ξ. The proof follows closely those of [23, thm. 4.4, lem. 4.6 and thm. 5.2] plus arguing in the Z_2 invariant setting. The book [23] uses gradient-like vector fields, we use negative ones.

Clearly, if ξ is Z_2 invariant, then $\xi(x) \in T_x\partial M^a$ for all $x \in \partial M^a$. Since both F and the metric on $2M^a$ are Z_2 invariant, so is $-\nabla F$. By the constructions of F and the metric, $-\nabla F$ and $-\nabla F|_{\partial M^a}$ satisfy everything but transversality.

Suppose the critical points on ∂M^a have function values $c_1 < \cdots < c_l$. Suppose c_0 is the maximum of function values on critical points off ∂M^a. By (5) of Lemma 7.13, $c_0 < c_1$. By induction on k, we shall modify the vector field ξ on M^{a_k,b_k} for some a_k, $b_k \in (c_{k-1}, c_k)$ such that the vector field on M^{c_k} satisfies the conclusion (in M^{c_k}, we don't consider $\mathcal{D}(p)\cap M^{c_k}$ for $p \notin M^{c_k}$), and the vector field globally satisfies everything but transversality.

Firstly, by (2) and (4) of Lemma 7.13 and the construction of the metric, the vector field on M^{c_0} satisfies the conclusion automatically.

Secondly, supposing we have finished the construction for $M^{c_{k-1}}$, we shall modify ξ for M^{c_k} by the method in [23]. Suppose the critical points with function value c_k are exactly p_i $(i = 1, \cdots, n)$. Denote the descending and ascending manifolds of p in ∂M^a with respect to $\xi|_{\partial M^a}$ by $\widetilde{\mathcal{D}(p)}$ and $\widetilde{\mathcal{A}(p)}$ respectively.

By (3) of Lemma 7.13 and local triviality of ξ, there is a neighborhood U_i of p_i such that U_i has a coordinate chart (s, v_1, v_2), $s^2 < 4\epsilon$, $\|v_1\|^2 < 4\epsilon$, $\|v_2\|^2 < 4\epsilon$, $F(s, v_1, v_2) = c_k - \frac{1}{2}s^2 - \frac{1}{2}\|v_1\|^2 + \frac{1}{2}\|v_2\|^2$, the metric on U_i is standard, and the action of σ is $\sigma(s, v_1, v_2) = (-s, v_1, v_2)$. Here ϵ is uniform for all i. We may assume U_i are disjoint for different i. Then $\mathcal{D}(p_i) \cap U_i = \{(s, v_1, 0)\}$ and $\widetilde{\mathcal{D}(p_i)} \cap U_i = \{(0, v_1, 0)\}$. Denote $S_i^- = \mathcal{D}(p_i) \cap F^{-1}(c_k - \epsilon) = \{(s, v_1, 0) \mid s^2 + \|v_1\|^2 = 2\epsilon\}$ and $\widetilde{S_i^-} = \widetilde{\mathcal{D}(p_i)} \cap F^{-1}(c_k - \epsilon) = \{(0, v_1, 0) \mid \|v_1\|^2 = 2\epsilon\}$. Let $B_2 = \{v_2 \mid \|v_2\|^2 < \epsilon\}$. Then we have a map $\alpha_i : S_i^- \times B_2 \longrightarrow F^{-1}(c_k - \epsilon) \cap U_i$ defined by

$$\alpha_i(s, v_1, 0, v_2) = ((\|v_2\|^2 + 2\epsilon)^{\frac{1}{2}}(2\epsilon)^{-\frac{1}{2}}s, (\|v_2\|^2 + 2\epsilon)^{\frac{1}{2}}(2\epsilon)^{-\frac{1}{2}}v_1, v_2).$$

Clearly, α_i is a diffeomorphism to its image, and $\alpha_i(\widetilde{S_i^-} \times B_2) \subseteq \partial M^a$. There is a $v_{2,i} \in B_2$ such that, for all critical points $q \in M^{c_{k-1}}$, $\alpha_i : S_i^- \times \{v_{2,i}\} \longrightarrow F^{-1}(c_k - \epsilon)$ is transverse to $\mathcal{A}(q) \cap F^{-1}(c_k - \epsilon)$, and $\alpha_i : \widetilde{S_i^-} \times \{v_{2,i}\} \longrightarrow F^{-1}(c_k - \epsilon) \cap \partial M^a$ is transverse to $\widetilde{\mathcal{A}(q)} \cap F^{-1}(c_k - \epsilon)$. Define $\alpha_{t,i} : S_i^- \longrightarrow F^{-1}(c_k - \epsilon)$ by $\alpha_{t,i}(s, v_1, 0) = \alpha_i(s, v_1, 0, tv_{2,i})$ for $t \in [0, 1]$. When t varies in $[0, 1]$, $\alpha_{t,i} : S_i^- \longrightarrow F^{-1}(c_k - \epsilon)$ is an isotopy of embeddings, and its restriction to $\widetilde{S_i^-}$ is also an isotopy of embeddings $\alpha_{t,i} : \widetilde{S_i^-} \longrightarrow F^{-1}(c_k - \epsilon) \cap \partial M^a$. Moreover, $\alpha_{t,i}$ is Z_2 equivariant. Following [23], we can extend $\alpha_{t,i}$ to be a Z_2 equivariant isotopy of $F^{-1}(c_k - \epsilon)$, which we still denote by $\alpha_{t,i}$, such that $\alpha_{0,i}$ is the identity, and $\alpha_{t,i}$ is the identity outside of U_i for all t.

Since U_i are disjoint for all i, composing these isotopies $\alpha_{t,i}$, we get a Z_2 equivariant isotopy α_t of $F^{-1}(c_k - \epsilon)$ such that $\alpha_t(U_i) = U_i$ and $\alpha_t|_{U_i} = \alpha_{t,i}|_{U_i}$. We have α_0 is the identity, and for all critical points $q \in M^{c_{k-1}}$, $\alpha_1 : S_i^- \longrightarrow F^{-1}(c_k - \epsilon)$ is transverse to $\mathcal{A}(q) \cap F^{-1}(c_k - \epsilon)$, and $\alpha_1 : \widetilde{S_i^-} \longrightarrow F^{-1}(c_k - \epsilon) \cap \partial M^a$ is transverse to $\widetilde{\mathcal{A}(q)} \cap F^{-1}(c_k - \epsilon)$.

By this isotopy α_t and its Z_2 equivariance, following [23], we can modify ξ in $M^{c_k - \epsilon, c_k - \frac{1}{2}\epsilon}$ such that the new ξ is still Z_2 invariant, and $\mathcal{D}(p_i)$ $(\widetilde{\mathcal{D}(p_i)})$ is transverse to $\mathcal{A}(q)$ $(\widetilde{\mathcal{A}(q)})$ for all p_i and all critical points $q \in M^{c_{k-1}}$. Since ξ only changed in $M^{c_k - \epsilon, c_k - \frac{1}{2}\epsilon}$, we have ξ and $\xi|_{\partial M^a}$ are still locally trivial, and on $M^{c_{k-1}}$ nothing has changed. Thus we get a desired ξ for M^{c_k}.

The above two steps complete the induction. $\square$

By Lemma 7.14, ξ and $\xi|_{\partial M^a}$ give $2M^a$ and ∂M^a a CW decomposition respectively. We shall consider the relation between these two decompositions. Use the same notations as in the proof of Lemma 7.14, denote the descending and ascending manifolds of $\xi|_{\partial M^a}$ by $\widetilde{\mathcal{D}(p)}$ and $\widetilde{\mathcal{A}(p)}$. It's easy to see that $\mathcal{D}(p) \cap \mathcal{A}(q) = \widetilde{\mathcal{D}(p)} \cap \widetilde{\mathcal{A}(q)}$ when p, $q \in \partial M^a$. Thus the moduli spaces $\mathcal{M}(p, q)$ of ξ and $\xi|_{\partial M^a}$ are the same. Then $\overline{\mathcal{D}(p)} = \bigsqcup_I \mathcal{M}_I \times \mathcal{D}(r_k)$ and $\overline{\widetilde{\mathcal{D}(p)}} = \bigsqcup_{I \subseteq \partial M^a} \mathcal{M}_I \times \widetilde{\mathcal{D}(r_k)}$. Since $\widetilde{\mathcal{D}(r_k)} \subseteq \mathcal{D}(r_k)$, there is a natural embedding $\theta : \overline{\widetilde{\mathcal{D}(p)}} \hookrightarrow \overline{\mathcal{D}(p)}$. In addition, suppose Γ is a generalized flow line connecting p and x, then $\sigma\Gamma$ is a generalized flow line connecting $\sigma p = p$ and σx. Thus there is a Z_2 action on $\overline{\mathcal{D}(p)}$.

Lemma 7.15. *Suppose $p \in \partial M^a$. Then $\theta : \overline{\widetilde{\mathcal{D}(p)}} \hookrightarrow \overline{\mathcal{D}(p)}$ is a smooth embedding. The action of Z_2 on $\overline{\mathcal{D}(p)}$ is smooth and* $\mathrm{Im}\theta = \mathrm{Fix}(Z_2; \overline{\mathcal{D}(p)})$. *In addition, $\widetilde{e} = e\theta$, where $\widetilde{e}$ is the characteristic map $\widetilde{e} : \overline{\widetilde{\mathcal{D}(p)}} \longrightarrow \partial M^a$ and e is the characteristic map $e : \overline{\mathcal{D}(p)} \longrightarrow 2M^a$.*

Proof. Except for smoothness, this lemma is obviously true. We only need to prove smoothness. This is a local property.

Suppose the critical values in $(-\infty, f(p)]$ are $c_l < \cdots < c_0$. Denote $M(i) = F^{-1}((c_{i+1}, c_{i-1}))$, $U(i) = e^{-1}(M(i))$ and $\widetilde{U}(i) = \widetilde{e}^{-1}(M(i))$. Choose $a_i \in (c_{i-1}, c_{i+1})$, by Lemma 5.4, we have the following commutative diagram, and both $E(i)$ and $\widetilde{E}(i)$ are smooth embeddings. Thus

θ is a smooth embedding.

$$\begin{tikzcd} \widetilde{U}(i) \arrow[r, "\widetilde{E}(i)"] \arrow[d, "\theta"'] & \prod_{j=0}^{i-1} F^{-1}(a_j) \times M(i) \\ U(i) \arrow[ur, "E(i)"'] & \end{tikzcd}$$

Since F is Z_2 invariant, there is a smooth Z_2 action on $\prod_{j=0}^{i-1} F^{-1}(a_j)\times M(i)$ and $E(i)$ is Z_2 equivariant. Thus the action of Z_2 on $U(i)$ is smooth. $\square$

Proof of Theorem 3.8. For briefness, we shall not distinguish between a CW complex and its underlying space in this proof.

The function F in Lemma 7.13 and the vector fields ξ and $\xi|_{\partial M^a}$ in Lemma 7.14 give two CW decompositions. They are $2M^a = \bigsqcup_p \mathcal{D}(p)$ with characteristic maps $e : \overline{\mathcal{D}(p)} \longrightarrow 2M^a$ and $\partial M^a = \bigsqcup_{p\in\partial M^a} \widetilde{\mathcal{D}(p)}$ with characteristic maps $\widetilde{e} : \overline{\widetilde{\mathcal{D}(p)}} \longrightarrow \partial M^a$. The decomposition of $2M^a$ is Z_2 invariant, $K^a = \bigsqcup_{p\in M^a-\partial M^a} \mathcal{D}(p)$ is a subcomplex of $2M^a$, and $\bigsqcup_{p\in 2M^a-M^a} \mathcal{D}(p) = \sigma(K^a) \subseteq 2M^a - M^a$. However, there is still no CW structure on M^a. We shall expand K^a to M^a by a sequence of elementary expansions (compare [10, p. 14]), which gives M^a a CW structure.

For clarity, denote the characteristic map for $\overline{\mathcal{D}(p)}$ by e_p. Suppose $p \in \partial M^a$, and denote $e_p^{-1}(M^a)$ by $\frac{1}{2}\overline{\mathcal{D}(p)}$.

Construct a vector field $\widetilde{X}$ on $\overline{\mathcal{D}(p)}$ as Lemma 7.8, i.e., $\widetilde{X}(F\circ e_p) \leq \xi F$, $\widetilde{X}$ equals ξ near p in $\mathcal{D}(p)$, and $\widetilde{X}$ is strictly outward on $\partial\overline{\mathcal{D}(p)}$. By Lemma 7.15, $\sigma\widetilde{X}$ has the same property as $\widetilde{X}$ does. By Lemma 7.4, and replacing $\widetilde{X}$ by $\frac{1}{2}(\widetilde{X}+\sigma\widetilde{X})$ if necessary, we may assume $\widetilde{X}$ is Z_2 invariant. By the Z_2 invariance of F, Lemma 7.15 and the proof of Theorem 3.7, the Z_2 equivariant flow generated by $\widetilde{X}$ gives a homeomorphism

$$\Psi : \left(\frac{1}{2}D^{\mathrm{ind}(p)}, D^{\mathrm{ind}(p)-1}\right) \longrightarrow \left(\frac{1}{2}\overline{\mathcal{D}(p)}, \overline{\widetilde{\mathcal{D}(p)}}\right),$$

where $\frac{1}{2}D^{\mathrm{ind}(p)} = \{(s, v_1) \in [0,+\infty)\times V_- \mid s^2 + \|v_1\|^2 \leq \epsilon\}$, $D^{\mathrm{ind}(p)-1} = \{(0, v_1) \in \{0\}\times V_- \mid \|v_1\|^2 \leq \epsilon\}$, and $V_- \times \{0\}$ is the descending subspace of $T_p\partial M^a$.

Denote the k skeletons of $2M^a$ and ∂M^a by L_k and $\widetilde{L}_k$ respectively. If $\mathrm{ind}(p) = n$, then $e_p(\partial\overline{\widetilde{\mathcal{D}(p)}}) \subseteq \widetilde{L}_{n-2}$, $e_p(\partial(\frac{1}{2}\overline{\mathcal{D}(p)})) \subseteq (L_{n-1}\cap M^a)\cup\widetilde{L}_{n-1}$ and $e_p(\partial(\frac{1}{2}\overline{\mathcal{D}(p)}) - \widetilde{\mathcal{D}(p)}) \subseteq L_{n-1}\cap M^a$.

Expand K^a by attaching cell pairs $e_p : (\frac{1}{2}\overline{\mathcal{D}(p)}, \widetilde{\overline{\mathcal{D}(p)}}) \longrightarrow (M^a, \partial M^a)$ for critical points $p \in \partial M^a$ by induction on $\operatorname{ind}(p)$. Then K^a expands by elementary expansions to a CW complex N such that K^a and ∂M^a are its subcomplexes. Clearly, $N \subseteq M^a$. In addition, if $x \in M^a - K^a$, then $x \in \mathcal{D}(p)$ for some $p \in \partial M^a$ because $\mathcal{D}(q) \subseteq 2M^a - M^a$ when $q \notin M^a$. Since $\frac{1}{2}\overline{\mathcal{D}(p)} = e_p^{-1}(M^a)$, then $x \in e_p(\frac{1}{2}\overline{\mathcal{D}(p)}) \subseteq N$. Thus $N = M^a$ as sets. Finally, N and M^a share the same topology since N is a finite complex. □

7.3. **Proof of Theorem 3.9.**

Proof. In this proof, all critical points have function values less than a.

Suppose $\operatorname{ind}(p) = k$. $[\overline{\mathcal{D}(p)}]$ is a base of $H_k(\overline{\mathcal{D}(p)}, \partial\overline{\mathcal{D}(p)})$. It's well known that $\partial[\overline{\mathcal{D}(p)}]$ is the image of $[\overline{\mathcal{D}(p)}]$ under the following composition of homomorphisms

$$\begin{aligned} H_k(\overline{\mathcal{D}(p)}, \partial\overline{\mathcal{D}(p)}) &\longrightarrow H_{k-1}(\partial\overline{\mathcal{D}(p)}) \longrightarrow \widetilde{H}_{k-1}(\partial\overline{\mathcal{D}(p)}/e^{-1}(K^a_{k-2})) \\ &\longrightarrow \widetilde{H}_{k-1}(K^a_{k-1}/K^a_{k-2}) = H_{k-1}(K^a_{k-1}, K^a_{k-2}), \end{aligned}$$

where $K^a_n = \bigsqcup_{\operatorname{ind}(q)\leq n} \mathcal{D}(q)$, and $\partial\overline{\mathcal{D}(p)} = \bigsqcup_i \partial^i\overline{\mathcal{D}(p)}$ is the full boundary of $\overline{\mathcal{D}(p)}$. The first homomorphism follows from the homology long exact sequence, the second one follows from the quotient map $\partial\overline{\mathcal{D}(p)} \longrightarrow \partial\overline{\mathcal{D}(p)}/e^{-1}(K^a_{k-2})$, and the third one follows from the map $\partial\overline{\mathcal{D}(p)}/e^{-1}(K^a_{k-2}) \longrightarrow K^a_{k-1}/K^a_{k-2}$ induced by e. Denote the first homomorphism by φ_1 and the composition of the first two by φ_2. The composition of all of them is the boundary operator ∂.

We have $e^{-1}(K^a_{k-2}) = \bigsqcup_{\operatorname{ind}(q)<k-1} \mathcal{M}(p,q) \times \mathcal{D}(q) \sqcup \bigsqcup_{|I|>0} \mathcal{D}_I$. Thus there is the following wadge of spheres with dimension $k-1$

$$\partial\overline{\mathcal{D}(p)}/e^{-1}(K^a_{k-2}) = \bigvee_{\operatorname{ind}(q)=k-1} \bigvee_{x\in\mathcal{M}(p,q)} \{x\} \times \overline{\mathcal{D}(q)}/\partial(\{x\} \times \overline{\mathcal{D}(q)}),$$

where the base points of spheres are $\partial(\{x\} \times \overline{\mathcal{D}(q)})/\partial(\{x\} \times \overline{\mathcal{D}(q)})$.

Clearly, $\overline{\mathcal{D}(p)}$ is a topological manifold with boundary $\partial\overline{\mathcal{D}(p)}$, and $[\overline{\mathcal{D}(p)}]$ represents an orientation of $\overline{\mathcal{D}(p)}$. So $\varphi_1([\overline{\mathcal{D}(p)}])$ represents the boundary orientation of $\partial\overline{\mathcal{D}(p)}$ induced from $[\overline{\mathcal{D}(p)}]$. Give $\{x\} \times \overline{\mathcal{D}(q)}$ the orientation $[\overline{\mathcal{D}(q)}]$ of $\overline{\mathcal{D}(q)}$ by the natural identification. Denote by $[\{x\} \times \overline{\mathcal{D}(q)}]$ the element in $\widetilde{H}_{k-1}(\{x\} \times \overline{\mathcal{D}(q)}/\partial(\{x\} \times \overline{\mathcal{D}(q)})) \subseteq \widetilde{H}_{k-1}(\partial\overline{\mathcal{D}(p)}/e^{-1}(K^a_{k-2}))$ which represents this orientation. Then by (2) of Theorem 3.6, we have

$$\varphi_2([\overline{\mathcal{D}(p)}]) = \sum_{\operatorname{ind} q=k-1} \sum_{x\in\mathcal{M}(p,q)} \varepsilon(x)[\{x\} \times \overline{\mathcal{D}(q)}],$$

where $\varepsilon(x)$ is the orientation ± 1 at $x \in \mathcal{M}(p,q)$. Thus

$$\partial[\overline{\mathcal{D}(p)}] = \sum_{\text{ind}(q)=k-1} \sum_{x\in\mathcal{M}(p,q)} \varepsilon(x)[\overline{\mathcal{D}(q)}] = \sum_{\text{ind}(q)=\text{ind}(p)-1} \#\mathcal{M}(p,q)[\overline{\mathcal{D}(q)}].$$

□

Appendix

Proof of Lemma 7.5. Define $\varphi : [0,+\infty) \times S(H) \longrightarrow H \times S(H)$ by $\varphi(\lambda, v) = (\lambda v, v)$. Then

$$d\varphi\frac{\partial}{\partial\lambda} = (v,0), \qquad d\varphi\frac{\partial}{\partial v} = \left(\lambda\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right).$$

Thus $d\varphi$ is nonsingular everywhere.

Define $\theta : H \times S(H) \longrightarrow [0,+\infty) \times S(H)$ by $\theta(v_1, v_2) = (\|v_1\|, v_2)$. Then θ is continuous and $\theta\varphi = \text{Id}$. Thus φ is a smooth embedding. Then $\varphi^{-1} : \text{Im}\varphi \longrightarrow [0,+\infty) \times S(H)$ is also smooth.

Clearly, $\forall x \in L$, $(g(x), \tilde{g}(x)) \in \text{Im}\varphi$, and $\varphi^{-1}(g(x), \tilde{g}(x)) = (\|g(x)\|, \tilde{g}(x))$. Since φ^{-1}, $g(x)$ and $\tilde{g}(x)$ are smooth, then so is $\|g(x)\|$. □

Proof of Lemma 7.12. Define a matrix

$$A_1 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \frac{1}{\cos\theta} \end{pmatrix}$$

for $\theta \in [0, \frac{\pi}{2})$. Then A_1 is a symmetric positive matrix and $A_1(1,0)^T = (\cos\theta, \sin\theta)^T$, where $(*,*)^T$ is the transpose of $(*,*)$.

Suppose V_1 and V_2 are two vectors in a Hilbert space such that $\langle V_1, V_2\rangle > 0$, i.e., the angle between V_1 and V_2 is less than $\frac{\pi}{2}$. We define a symmetric positive operator $A(V_1, V_2)$ such that $A(V_1, V_2)V_1 = V_2$ as follows.

If V_1 and V_2 are colinear, then define $A(V_1, V_2) = \frac{\|V_2\|}{\|V_1\|}\text{Id}$. If V_1 and V_2 are not colinear, then they span a plane $V_1 \wedge V_2$. First, we define an operator $A_2(e_1, e_2)$ for $e_1 = \frac{V_1}{\|V_1\|}$ and $e_2 = \frac{V_2}{\|V_2\|}$. In $(V_1 \wedge V_2)^\perp$, $A_2(e_1, e_2)$ is the identity. In $V_1 \wedge V_2$, $A_2(e_1, e_2)$ is the above A_1 mapping e_1 to e_2. Define $A(V_1, V_2) = \frac{\|V_2\|}{\|V_1\|}A_2(e_1, e_2)$.

Thus, in general, $A(V_1, V_2) = \frac{\|V_2\|}{\|V_1\|}A_2(\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|})$ for $\langle V_1, V_2\rangle > 0$. Here, for $\|e_1\| = \|e_2\| = 1$ and $\langle e_1, e_2\rangle > 0$, we have

$$\begin{aligned} A_2(e_1,e_2)Y \;\; &= \;\; Y + \frac{\langle e_1,e_2\rangle\langle e_2,Y\rangle - (1+\langle e_1,e_2\rangle+\langle e_1,e_2\rangle^2)\langle e_1,Y\rangle}{1+\langle e_1,e_2\rangle}e_1 \\ &\quad + \frac{\langle e_1,e_2\rangle^2\langle e_1,Y\rangle + \langle e_2,Y\rangle}{\langle e_1,e_2\rangle(1+\langle e_1,e_2\rangle)}e_2. \end{aligned}$$

Then $A(V_1, V_2)$ smoothly depends on V_1 and V_2, and $A(V_1, V_1) = \mathrm{Id}$.

Let G_1 be the metric associated with X. Denote the gradient vector field of f with respect to G_1 by $\nabla_{G_1} f$. Then $\nabla_{G_1} f$ equals X near each critical point, and $\langle \nabla_{G_1} f, X \rangle = Xf > 0$ at each regular point. Define the operator $A(X, \nabla_{G_1} f)$ as above at each regular point. Define $A(X, \nabla_{G_1} f) = \mathrm{Id}$ at each critical point. Then $A(X, \nabla_{G_1} f)$ is smooth on M. Define a new metric G_2 such that $\langle *, * \rangle_{G_2} = \langle A(X, \nabla_{G_1} f) *, * \rangle_{G_1}$. Then $\nabla_{G_2} f = X$. □

Acknowledgements

I wish to thank Profs. Octav Cornea, Matthias Schwarz and Katrin Wehrheim, who told me which work had already been done. I wish to thank the referees of this paper who referred me to [1] and [2]. I wish to thank Prof. Ralph Cohen for suggesting to me this topic, his patient teaching, and his important referencing. I wish to thank Prof. Dan Burghelea for his explanation of [8] and for his generosity in sending me part of [7]. I'm indebted to Prof. John Milnor for his advice and suggesting [19], which motivated me to prove Theorem 3.7. All of Section 7 is a result of his influence. Most importantly, I am very grateful to my PhD advisor Prof. John Klein. It was he who helped me communicate with the above professors. I am also indebted to him for his direction, his patient educating, and his continuous encouragement.

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Mathematics Department of Wayne State University, Detroit, MI 48202, USA

E-mail address: dv6624@wayne.edu