

## LOCAL-GLOBAL COMPATIBILITY FOR $l = p$ , II.

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ABSTRACT. We prove the compatibility at places dividing  $l$  of the local and global Langlands correspondences for the  $l$ -adic Galois representations associated to regular algebraic essentially (conjugate) self-dual cuspidal automorphic representations of  $GL_n$  over an imaginary CM or totally real field. We prove this compatibility up to semisimplification in all cases, and up to Frobenius semisimplification in the case of Shin-regular weight.

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## INTRODUCTION.

Our main result is as follows (see Theorem 1.1 and Corollary 1.2).

**Theorem A.** *Let  $F$  be an imaginary CM field or totally real field, let  $(\Pi, \chi)$  be a regular, algebraic, essentially (conjugate) self-dual automorphic representation of  $GL_m(\mathbb{A}_F)$  and let  $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ . If  $v|l$  is a place of  $F$ , then*

$${}^{\iota}\mathrm{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})^{\mathrm{ss}} \cong \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2})^{\mathrm{ss}}.$$

Furthermore, if  $\Pi$  has Shin-regular weight, then

$${}^{\iota}\mathrm{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})^{\mathrm{F-ss}} \cong \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

Here  $r_{l,\iota}(\Pi)$  denotes the  $l$ -adic representation associated to  $\Pi$  (and  $\iota$ ); and  $\mathrm{WD}(r)$  denotes the Weil-Deligne representation attached to a de Rham  $l$ -adic representation  $r$  of the absolute Galois group of an  $l$ -adic field; and  $\mathrm{rec}$  denotes the local Langlands correspondence; and  $\mathrm{F-ss}$  denotes Frobenius semi-simplification. (See Section 1 for details on the terminology.) In fact, we prove a slight refinement of this result which gives some information about the monodromy operator in the case where  $\Pi$  does not have Shin-regular weight; see Section 1 for the details of this.

It is important in some applications to have this compatibility at places dividing  $l$ ; for example, our original motivation for considering this problem was to improve the applicability of the main results of [BLGGT10]; in that paper a variety of automorphy lifting theorems are proved via making highly ramified base changes, and one loses control of the level of the automorphic representations under consideration. This control can be recovered if one knows local-global compatibility, and this is important in applications to the weight part of Serre's conjecture (cf. [BLGG11a], [BLGG11b]).

The proof of Theorem A is surprisingly simple, and relies on a generalisation of a base change trick that we learned from the papers [Kis08] and [Ski09] (see the proof of Theorem 4.3 of [Kis08] and Section 2.2 of [Ski09]). The idea is as follows. Suppose that  $\Pi$  has Shin-regular weight. We wish to determine the Weil-Deligne representation  ${}^{\iota}\mathrm{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})^{\mathrm{F-ss}}$ . The monodromy may be computed after any finite base change, and in particular we may make a base change so that  $\Pi$  has Iwahori-fixed vectors, which is the situation covered by [BLGGT11]; so it suffices to compute the representation of the Weil group  $W_{F_v}$ . It is straightforward to check that in order to do so it is enough to compute the traces of the elements  $\sigma \in W_{F_v}$  of nonzero valuation (that is, those elements which map to a nonzero power of the Frobenius element in the Galois group of the residue field). This trace is then computed as follows: one makes a global base change to a CM field  $E/F$  such that there is a place  $w$  of  $E$  lying over  $v$  such that  $\mathrm{BC}_{E/F}(\Pi)_w$  has Iwahori-fixed vectors, and  $\sigma$  is an element of  $W_{E_w} \leq W_{F_v}$ . By the compatibility of base change with the local Langlands correspondence, the trace of  $\sigma$  on  ${}^{\iota}\mathrm{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})^{\mathrm{F-ss}}$  may then be computed over  $E$ , where the result follows from [BLGGT11].

The subtlety in this argument is that the field  $E/F$  need not be Galois, so one cannot immediately appeal to solvable base change. However, it will have solvable normal closure, so that by a standard descent argument due to Harris it is enough to know that for some prime  $l'$ , the global Galois representation  $r_{l',\iota'}(\Pi)$  is irreducible. Under the additional assumption that  $\Pi$  has extremely regular weight, the existence of such an  $l'$  is established in [BLGGT10]. Having thus established Theorem A in the case that  $\Pi$  has extremely regular and Shin-regular weight, we then pass to the

general case by means of an  $l$ -adic interpolation argument of Chenevier and Harris, [CH09] and [Che09]. The details are in Section 3.

**Notation and terminology.** We write all matrix transposes on the left; so  ${}^tA$  is the transpose of  $A$ . We let  $B_m \subset GL_m$  denote the Borel subgroup of upper triangular matrices and  $T_m \subset GL_m$  the diagonal torus. We let  $I_m$  denote the identity matrix in  $GL_m$ .

If  $M$  is a field, we let  $\overline{M}$  denote an algebraic closure of  $M$  and  $G_M$  the absolute Galois group  $\text{Gal}(\overline{M}/M)$ . Let  $\epsilon_l$  denote the  $l$ -adic cyclotomic character

Let  $p$  be a rational prime and  $K/\mathbb{Q}_p$  a finite extension. We let  $\mathcal{O}_K$  denote the ring of integers of  $K$ ,  $\wp_K$  the maximal ideal of  $\mathcal{O}_K$ ,  $k(\nu_K)$  the residue field  $\mathcal{O}_K/\wp_K$ ,  $\nu_K : K^\times \rightarrow \mathbb{Z}$  the canonical valuation and  $|\cdot|_K : K^\times \rightarrow \mathbb{Q}^\times$  the absolute value given by  $|x|_K = \#(k(\nu_K))^{-\nu_K(x)}$ . We let  $|\cdot|_K^{1/2} : K^\times \rightarrow \mathbb{R}_{>0}^\times$  denote the unique positive unramified square root of  $|\cdot|_K$ . If  $K$  is clear from the context, we will sometimes write  $|\cdot|$  for  $|\cdot|_K$ . We let  $\text{Frob}_K$  denote the geometric Frobenius element of  $G_{k(\nu_K)}$  and  $I_K$  the kernel of the natural surjection  $G_K \rightarrow G_{k(\nu_K)}$ . We will sometimes abbreviate  $\text{Frob}_{\mathbb{Q}_p}$  by  $\text{Frob}_p$ . We let  $W_K$  denote the preimage of  $\text{Frob}_K^{\mathbb{Z}}$  under the map  $G_K \rightarrow G_{k(\nu_K)}$ , endowed with a topology by decreeing that  $I_K \subset W_K$  with its usual topology is an open subgroup of  $W_K$ . We let  $\text{Art}_K : K^\times \xrightarrow{\sim} W_K^{\text{ab}}$  denote the local Artin map, normalized to take uniformizers to lifts of  $\text{Frob}_K$ .

Let  $\Omega$  be an algebraically closed field of characteristic 0. A Weil-Deligne representation of  $W_K$  over  $\Omega$  is a triple  $(V, r, N)$  where  $V$  is a finite dimensional vector space over  $\Omega$ ,  $r : W_K \rightarrow GL(V)$  is a representation with open kernel and  $N : V \rightarrow V$  is an endomorphism with  $r(\sigma)Nr(\sigma)^{-1} = |\text{Art}_K^{-1}(\sigma)|_K N$ . We say that  $(V, r, N)$  is Frobenius semisimple if  $r$  is semisimple. We let  $(V, r, N)^{\text{F-ss}}$  denote the Frobenius semisimplification of  $(V, r, N)$  (see for instance Section 1 of [TY07]) and we let  $(V, r, N)^{\text{ss}}$  denote  $(V, r^{\text{ss}}, 0)$ . If  $\Omega$  has the same cardinality as  $\mathbb{C}$ , we have the notions of a Weil-Deligne representation being *pure* or *pure of weight  $k$*  – see the paragraph before Lemma 1.4 of [TY07].

We will let  $\text{rec}_K$  be the local Langlands correspondence of [HT01], so that if  $\pi$  is an irreducible complex admissible representation of  $GL_n(K)$ , then  $\text{rec}_K(\pi)$  is a Weil-Deligne representation of the Weil group  $W_K$ . We will write  $\text{rec}$  for  $\text{rec}_K$  when the choice of  $K$  is clear. If  $\rho$  is a continuous representation of  $G_K$  over  $\overline{\mathbb{Q}}_l$  with  $l \neq p$  then we will write  $\text{WD}(\rho)$  for the corresponding Weil-Deligne representation of  $W_K$ . (See for instance Section 1 of [TY07].)

If  $m \geq 1$  is an integer, we let  $\text{Iw}_{m,K} \subset GL_m(\mathcal{O}_K)$  denote the subgroup of matrices which map to an upper triangular matrix in  $GL_m(k(\nu_K))$ . If  $\pi$  is an irreducible admissible supercuspidal representation of  $GL_m(K)$  and  $s \geq 1$  is an integer we let  $\text{Sp}_s(\pi)$  be the square integrable representation of  $GL_{mr}(K)$  defined for instance in Section I.3 of [HT01]. Similarly, if  $r : W_K \rightarrow GL_m(\Omega)$  is an irreducible representation with open kernel and  $\pi$  is the supercuspidal representation  $\text{rec}_K^{-1}(r)$ , we let  $\text{Sp}_s(r) = \text{rec}_K(\text{Sp}_s(\pi))$ . If  $K'/K$  is a finite extension and if  $\pi$  is an irreducible smooth representation of  $GL_n(K)$  we will write  $\text{BC}_{K'/K}(\pi)$  for the base change of  $\pi$  to  $K'$  which is characterized by  $\text{rec}_{K'}(\text{BC}_{K'/K}(\pi)) = \text{rec}_K(\pi)|_{W_{K'}}$ .

If  $\rho$  is a continuous de Rham representation of  $G_K$  over  $\overline{\mathbb{Q}}_p$  then we will write  $\text{WD}(\rho)$  for the corresponding Weil-Deligne representation of  $W_K$  (its construction, which is due to Fontaine, is recalled in Section 1 of [TY07]), and if  $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$  is a

continuous embedding of fields then we will write  $\text{HT}_\tau(\rho)$  for the multiset of Hodge-Tate numbers of  $\rho$  with respect to  $\tau$ . Thus  $\text{HT}_\tau(\rho)$  is a multiset of  $\dim \rho$  integers. In fact, if  $W$  is a de Rham representation of  $G_K$  over  $\overline{\mathbb{Q}}_p$  and if  $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$  then the multiset  $\text{HT}_\tau(W)$  contains  $i$  with multiplicity  $\dim_{\overline{\mathbb{Q}}_p} (W \otimes_{\tau, K} \widehat{K}(i))^{G_K}$ . Thus for example  $\text{HT}_\tau(\epsilon_l) = \{-1\}$ .

If  $F$  is a number field and  $v$  a prime of  $F$ , we will often denote  $\text{Frob}_{F_v}$ ,  $k(\nu_{F_v})$  and  $\text{Iw}_{m, F_v}$  by  $\text{Frob}_v$ ,  $k(v)$  and  $\text{Iw}_{m, v}$ . If  $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_p$  or  $\mathbb{C}$  is an embedding of fields, then we will write  $F_\sigma$  for the closure of the image of  $\sigma$ . If  $F'/F$  is a soluble, finite Galois extension and if  $\pi$  is a cuspidal automorphic representation of  $GL_m(\mathbb{A}_F)$  we will write  $\text{BC}_{F'/F}(\pi)$  for its base change to  $F'$ , an automorphic representation of  $GL_m(\mathbb{A}_{F'})$ . If  $R : G_F \rightarrow GL_m(\overline{\mathbb{Q}}_l)$  is a continuous representation, we say that  $R$  is *pure of weight  $w$*  if for all but finitely many primes  $v$  of  $F$ ,  $R$  is unramified at  $v$  and every eigenvalue of  $R(\text{Frob}_v)$  is a Weil  $(\#k(v))^w$ -number. (See Section 1 of [TY07].) If  $F$  is an imaginary CM field, we will denote its maximal totally real subfield by  $F^+$  and let  $c$  denote the non-trivial element of  $\text{Gal}(F/F^+)$ .

## 1. AUTOMORPHIC GALOIS REPRESENTATIONS

We recall some now-standard notation and terminology. Let  $F$  be an imaginary CM field or totally real field. Let  $F^+$  denote the maximal totally real subfield of  $F$ . By a *RAECSDC* or *RAESDC* (regular, algebraic, essentially (conjugate) self dual, cuspidal) automorphic representation of  $GL_m(\mathbb{A}_F)$  we mean a pair  $(\Pi, \chi)$  where

- $\Pi$  is a cuspidal automorphic representation of  $GL_m(\mathbb{A}_F)$  such that  $\Pi_\infty$  has the same infinitesimal character as some irreducible algebraic representation of the restriction of scalars from  $F$  to  $\mathbb{Q}$  of  $GL_m$ ,
- $\chi : \mathbb{A}_{F^+}^\times / (F^+)^\times \rightarrow \mathbb{C}^\times$  is a continuous character such that  $\chi_v(-1)$  is independent of  $v | \infty$ ,
- and  $\Pi^c \cong \Pi^\vee \otimes (\chi \circ \mathbf{N}_{F/F^+} \circ \det)$ .

If  $\chi$  is the trivial character we will often drop it from the notation and refer to  $\Pi$  as a *RACSDC* or *RASDC* (regular, algebraic, (conjugate) self dual, cuspidal) automorphic representation. We will say that  $(\Pi, \chi)$  has *level prime to  $l$*  (resp. *level potentially prime to  $l$* ) if for all  $v | l$  the representation  $\Pi_v$  is unramified (resp. becomes unramified after a finite base change).

If  $\Omega$  is an algebraically closed field of characteristic 0 we will write  $(\mathbb{Z}^m)^{\text{Hom}(F, \Omega), +}$  for the set of  $a = (a_{\tau, i}) \in (\mathbb{Z}^m)^{\text{Hom}(F, \Omega)}$  satisfying

$$a_{\tau, 1} \geq \cdots \geq a_{\tau, m}.$$

Let  $w \in \mathbb{Z}$ . If  $F$  is totally real or imaginary CM (resp. if  $\Omega = \mathbb{C}$ ) we will write  $(\mathbb{Z}^m)_w^{\text{Hom}(F, \Omega)}$  for the subset of elements  $a \in (\mathbb{Z}^m)^{\text{Hom}(F, \Omega)}$  with

$$a_{\tau, i} + a_{\tau \circ c, m+1-i} = w$$

(resp.

$$a_{\tau, i} + a_{c \circ \tau, m+1-i} = w.)$$

(These definitions are consistent when  $F$  is totally real or imaginary CM and  $\Omega = \mathbb{C}$ .) If  $F'/F$  is a finite extension we define  $a_{F'} \in (\mathbb{Z}^m)^{\text{Hom}(F', \Omega), +}$  by

$$(a_{F'})_{\tau, i} = a_{\tau|_F, i}.$$

Following [Shi10] we will be interested, *inter alia*, in the case that either  $m$  is odd; or that  $m$  is even and for some  $\tau \in \text{Hom}(F, \Omega)$  and for some odd integer  $i$  we have  $a_{\tau, i} > a_{\tau, i+1}$ . If either of these conditions hold then we will say that  $a$  is *Shin-regular*. (This is often referred to as ‘slightly regular’ in the literature. However as this notion is strictly stronger than ‘regularity’ we prefer the terminology ‘Shin-regular’.) Following [BLGGT10], we say that  $a$  is *extremely regular* if for some  $\tau$  the  $a_{\tau, i}$  have the following property: for any subsets  $H$  and  $H'$  of  $\{a_{\tau, i} + n - i\}_{i=1}^n$  of the same cardinality, if  $\sum_{h \in H} h = \sum_{h \in H'} h$  then  $H = H'$ .

If  $a \in (\mathbb{Z}^m)^{\text{Hom}(F, \mathbb{C}), +}$ , let  $\Xi_a$  denote the irreducible algebraic representation of  $GL_m^{\text{Hom}(F, \mathbb{C})}$  which is the tensor product over  $\tau$  of the irreducible representations of  $GL_n$  with highest weights  $a_\tau$ . We will say that a RAECSDC automorphic representation  $\Pi$  of  $GL_m(\mathbb{A}_F)$  has *weight*  $a$  if  $\Pi_\infty$  has the same infinitesimal character as  $\Xi_a^\vee$ . Note that in this case  $a$  must lie in  $(\mathbb{Z}^m)_w^{\text{Hom}(F, \mathbb{C})}$  for some  $w \in \mathbb{Z}$ .

We recall (see Theorem 1.2 of [BLGHT09]) that to a RAECSDC or RAESDC automorphic representation  $(\Pi, \chi)$  of  $GL_m(\mathbb{A}_F)$  and  $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$  we can associate a continuous semisimple representation

$$r_{l, \iota}(\Pi) : \text{Gal}(\overline{F}/F) \longrightarrow GL_m(\overline{\mathbb{Q}}_l)$$

with the properties described in Theorem 1.2 of [BLGHT09]. In particular

$$r_{l, \iota}(\Pi)^c \cong r_{l, \iota}(\Pi)^\vee \otimes \epsilon_l^{1-m} r_{l, \iota}(\chi)|_{G_F},$$

where  $r_{l, \iota}(\chi) : G_{F^+} \rightarrow \overline{\mathbb{Q}}_l^\times$  is as defined in Section 1 of [BLGHT09]. For  $v|l$  a place of  $F$ , the representation  $r_{l, \iota}(\Pi)|_{G_{F_v}}$  is de Rham and if  $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$  then

$$\text{HT}_\tau(r_{l, \iota}(\Pi)) = \{a_{\tau, 1} + m - 1, a_{\tau, 2} + m - 2, \dots, a_{\tau, m}\}.$$

If  $v \nmid l$ , then the main result of [Car10] states that

$${}_\iota \text{WD}(r_{l, \iota}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \cong \text{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

Let  $p$  be a prime number,  $K/\mathbb{Q}_p$  be a finite extension and let  $\Omega$  be an algebraically closed field of characteristic 0. Let  $\mathcal{J}$  denote the set of equivalence classes of irreducible representations of  $W_K$  over  $\Omega$  with open kernel, where  $s \sim s'$  if  $s \cong s' \otimes \chi \circ \det$  for some unramified character  $\chi : K^\times \rightarrow \Omega^\times$ . Let  $\rho = (V, r, N)$  be a Weil-Deligne representation of  $W_K$  over  $\Omega$ . We decompose

$$V \cong \bigoplus_{\sigma \in \mathcal{J}} V[\sigma]$$

where  $V[\sigma]$  is the largest  $W_K$ -submodule of  $V$  with all its irreducible subquotients lying in  $\sigma$ . Then each  $V[\sigma]$  is stable by  $N$  and  $\rho[\sigma] := (V[\sigma], r|_{V[\sigma]}, N|_{V[\sigma]})$  is a Weil-Deligne subrepresentation of  $(V, r, N)$ . For each  $\sigma \in \mathcal{J}$  with  $V[\sigma] \neq (0)$ , there is a unique decreasing sequence of integers  $m_1(\rho, \sigma) \geq \dots \geq m_n(\rho, \sigma) \geq 1$  with

$$\rho[\sigma]^{\text{F-ss}} \cong \bigoplus_{i=1}^{n(\rho, \sigma)} \text{Sp}_{m_i(\rho, \sigma)}(s_i)$$

$s_i \in \sigma$  for each  $i$ . If  $\rho'$  is another Weil-Deligne representation of  $W_K$  over  $\Omega$ , we say that

$$\rho \prec \rho'$$

if  $\rho^{\text{ss}} \cong (\rho')^{\text{ss}}$  and if for each  $\sigma \in \mathcal{J}$  we have

$$m_1(\rho, \sigma) + \dots + m_i(\rho, \sigma) \leq m_1(\rho', \sigma) + \dots + m_i(\rho', \sigma)$$

for each  $i \geq 1$ . The goal of this paper is to establish the following local-global compatibility result at places dividing  $l$ .

We now state our main theorem.

**Theorem 1.1.** *Let  $(\Pi, \chi)$  be a RAECSDC automorphic representation of  $GL_m(\mathbb{A}_F)$  and let  $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ . If  $v|l$  is a place of  $F$ , then*

$$\iota \text{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \prec \text{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

Furthermore, if  $\Pi$  has Shin-regular weight, then

$$\iota \text{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \cong \text{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

The following corollary follows immediately using base change as in Proposition 4.3.1 of [CHT08].

**Corollary 1.2.** *Let  $(\Pi, \chi)$  be a RAESDC automorphic representation of  $GL_m(\mathbb{A}_F)$  and let  $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ . If  $v|l$  is a place of  $F$ , then*

$$\iota \text{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \prec \text{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

Furthermore, if  $\Pi$  has Shin-regular weight, then

$$\iota \text{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \cong \text{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

## 2. THE EXTREMELY REGULAR, SHIN-REGULAR CASE

We start by treating the special case where, thanks to the irreducibility results of [BLGGT10], we can give a direct argument. We use an analogue of the trick of [Kis08] and [Ski09] (see the proof of Theorem 4.3 of [Kis08] and Section 2.2 of [Ski09]), but in a situation where we need to use a non-abelian, indeed non-Galois, base change. Because of this the argument makes essential use of the irreducibility results of [BLGGT10], and hence at present can only be made in the extremely regular case.

**Theorem 2.1.** *Let  $m \geq 2$  be an integer,  $l$  a rational prime and  $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ . Let  $F$  be an imaginary CM field and  $(\Pi, \chi)$  a RAECSDC automorphic representation of  $GL_m(\mathbb{A}_F)$ . If  $\Pi$  has extremely regular and Shin-regular weight and  $v|l$  is a place of  $F$ , then*

$$\iota \text{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \cong \text{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

*Proof.* We first reduce to the RACSDC case: using Lemma 4.1.4 of [CHT08] we choose an algebraic Hecke character  $\psi : \mathbb{A}_F^\times/F^\times \rightarrow \mathbb{C}^\times$  such that  $\psi \cdot (\psi \circ c) = \chi_F^{-1} \circ \mathbf{N}_{F/F^+}$ . Then  $\Pi \otimes \psi \circ \det$  is RACSDC and the theorem holds for  $\Pi$  if and only if it holds for  $\Pi \otimes \psi \circ \det$ . We may therefore assume that  $\Pi$  is RACSDC.

To prove the theorem, it suffices to establish the weaker result that

$$\iota \text{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})^{\text{ss}} \cong \text{rec}(\Pi_v \otimes |\det|^{(1-m)/2})^{\text{ss}}.$$

(Suppose this weaker result holds. By Proposition 1.1 of [BLGGT11], it suffices to prove that  $\text{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})$  is pure. This is established in Corollary 1.3 of [BLGGT11].)

To establish the weaker result, it suffices to show that

$$\text{tr}(\sigma | \iota \text{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})) = \text{tr}(\sigma | \text{rec}(\Pi_v \otimes |\det|^{(1-m)/2}))$$

for every  $\sigma \in W_{F_v}$  mapping to a non-zero power of  $\text{Frob}_v \in G_{k(v)}$ . (This follows from the proof of Lemma 1 of [Sai97].) Fix such an element  $\sigma \in W_{F_v}$ . We can and do choose a finite extension  $E_v/F_v$  inside  $\overline{L}_v$  such that

- $\sigma \in W_{E_v} \subset W_{F_v}$  and
- $\text{BC}_{E_v/F_v}(\Pi_v)^{\text{Iw}_{m,E_v}} \neq \{0\}$ .

(If we write  $\text{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}}) = (V, r, N)$ , we could take  $E_v$  to be the fixed field of the subgroup of  $W_{F_v}$  generated by  $\sigma$  and the kernel of  $r|_{I_{F_v}}$ .) Let  $E'_v/E_v$  denote the normal closure of  $E_v/F_v$ . Choose a finite CM soluble Galois extension  $F'/F$  such that for each place  $w|v$  of  $F'$ ,  $F'_w/F_v \cong E'_v/F_v$ . Let  $\Pi_{F'} = \text{BC}_{F'/F}(\Pi)$ . By Theorem 5.4.2 of [BLGGT10] we can and do choose a rational prime  $l'$  and  $\iota' : \mathbb{Q}_{l'} \xrightarrow{\sim} \mathbb{C}$  such that  $r_{l',\iota'}(\Pi_{F'})$  is irreducible. Choose a prime  $w|v$  of  $F'$  and an  $F_v$ -embedding  $F'_w \hookrightarrow \overline{F}_v$ . Let  $E = F' \cap E_v \subset F'_w$  be the fixed field of  $\text{Gal}(F'_w/E_v) \subset \text{Gal}(F'/F)$ . The inclusion  $E \hookrightarrow E_v$  determines a prime  $u$  of  $E$ . By Lemma 1.4 of [BLGHT09], there exists a RACSDC automorphic representation  $\Pi_E$  of  $GL_m(\mathbb{A}_E)$  with  $r_{l',\iota'}(\Pi_E) \cong r_{l',\iota'}(\Pi)|_{G_E}$  and hence  $r_{l,\iota}(\Pi_E) \cong r_{l,\iota}(\Pi)|_{G_E}$ . Since  $\Pi_{E,u}^{\text{Iw}_{m,u}} \neq \{0\}$ , Theorem 1.2 of [BLGGT11] implies that

$$\begin{aligned} \text{tr}(\sigma|\iota \text{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})) &= \text{tr}(\sigma|\iota \text{WD}(r_{l,\iota}(\Pi_E)|_{G_{E_u}})) \\ &= \text{tr}(\sigma|\text{rec}(\Pi_{E,u} \otimes |\det|^{(1-m)/2})) \\ &= \text{tr}(\sigma|\text{rec}(\Pi_v \otimes |\det|^{(1-m)/2})), \end{aligned}$$

and the result follows.  $\square$

### 3. THE GENERAL CASE

We will prove the next result using Theorem 2.1 and the methods of [Che09] and [BC09]. It establishes the first statement of Theorem 1.1.

**Theorem 3.1.** *Let  $m \geq 2$  be an integer,  $l$  a rational prime and  $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ . Let  $F$  be an imaginary CM field and  $(\Pi, \chi)$  a RAECSDC automorphic representation of  $GL_m(\mathbb{A}_F)$ . If  $v|l$  is a place of  $F$ , then*

$$\iota \text{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \prec \text{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

Before giving the proof, we first deduce the second statement of Theorem 1.1 as a corollary.

**Corollary 3.2.** *Let  $m \geq 2$  be an integer,  $l$  a rational prime and  $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ . Let  $F$  be an imaginary CM field and  $(\Pi, \chi)$  a RAECSDC automorphic representation of  $GL_m(\mathbb{A})$ . If  $\Pi$  has slightly regular weight and  $v|l$  is a place of  $F$ , then*

$$\iota \text{WD}(r_{l,\iota}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \cong \text{rec}(\Pi_{F,v} \otimes |\det|^{(1-m)/2}).$$

*Proof.* This follows immediately from Theorem 3.1 together with Corollary 1.3 of [BLGGT11] and Proposition 1.1 of [BLGGT11].  $\square$

*Proof of Theorem 3.1.* As in the proof of Theorem 2.1, we may assume that  $\Pi$  is RACSDC. Replacing  $F$  by a suitable finite soluble CM Galois extension in which  $v$  splits we may also assume that:

- $[F^+ : \mathbb{Q}]$  is even;
- $F/F^+$  is unramified at all finite places;
- all places of  $F^+$  dividing  $l$  are split in  $F$ ;
- if  $\Pi_w$  is ramified, then  $w|_{F^+}$  is split in  $F$ ;

- if  $w \neq v$  then  $\Pi_u^{\text{Iw}_{m,w}} \neq \{0\}$ .

Since  $[F^+ : \mathbb{Q}]$  is even, we can and do choose a unitary group  $U/F^+$  such that:

- $U \times_{F^+} F \cong GL_m/F$ ;
- $U \times_{F^+} F_u^+$  is quasi-split for each prime  $u$  of  $F^+$ ;
- $U(F_\sigma^+)$  is compact for each  $\sigma : F^+ \hookrightarrow \mathbb{R}$ .

(We write  $F_\sigma^+$  for the completion of  $F^+$  with respect to the absolute value induced by  $\sigma$ .) For each place  $u$  of  $F^+$  which splits in  $F$  and  $w|u$  a prime of  $F$ , we fix an isomorphism  $\iota_w : U(F_u^+) \xrightarrow{\sim} GL_m(F_w)$  such that  $\iota_{w^c} = {}^t \iota_w^{-c}$ . If  $\sigma : F^+ \hookrightarrow \mathbb{R}$  and  $\tilde{\sigma} : F \hookrightarrow \mathbb{C}$  extends  $\sigma$ , we fix an embedding  $\iota_{\tilde{\sigma}} : U(F_\sigma^+) \hookrightarrow GL_n(F_{\tilde{\sigma}})$  which identifies  $U(F_\sigma^+)$  with the set of all  $g$  with  ${}^t g^c \cdot g = 1_m$ . By Corollaire 5.3 and Théorème 5.4 of [Lab09], there exists an automorphic representation  $\pi_F$  of  $U(\mathbb{A}_{F^+})$  such that:

- if  $u$  is a prime of  $F^+$  which splits as  $ww^c$  in  $F$ , then  $\pi_{F,u} \cong \Pi_{F,w} \circ \iota_w$ ;
- if  $u$  is a prime of  $F^+$  which is inert in  $F$ , then  $\Pi_{F,u}$  is given by the local base change of  $\pi_{F,u}$  (see [Lab09]);
- if  $\sigma : F^+ \hookrightarrow \mathbb{R}$  and  $\tilde{\sigma} : F \hookrightarrow \mathbb{C}$  extends  $\sigma$ , then there is an irreducible algebraic representation  $W_{\tilde{\sigma}}$  of  $GL_m(F_{\tilde{\sigma}})$  such that  $\pi_{F,\sigma} \cong W_{\tilde{\sigma}}^\vee \circ \iota_{\tilde{\sigma}}$ . Moreover, if  $W_{\tilde{\sigma}}$  has highest weight  $a_{\tilde{\sigma}} = (a_{\tilde{\sigma},1}, \dots, a_{\tilde{\sigma},m})$ , then  $\Pi_F$  has weight  $a = (a_{\tilde{\sigma}})_{\tilde{\sigma}:F \hookrightarrow \mathbb{C}}$ .

We now follow the arguments of [Che09]. We have chosen to closely follow [Che09] even when we could somewhat simplify the argument in the case of interest to us, in order to ease comparison with that paper. We note however we take the prime  $p$  of [Che09] to be the prime  $l$  of this paper. Make the following definitions: let  $\tilde{S}_l$  (resp.  $\tilde{S}_v$ ) denote the set of primes of  $F$  dividing  $l$  but not lying above  $v$  (resp. lying above  $v$ ). Let  $\tilde{R}$  denote the set of primes  $w$  of  $F$  not dividing  $l$  and with  $\Pi_{F,w}$  ramified. Set  $\tilde{S} = \tilde{S}_v \cup \tilde{R}$ . Let  $S_l, S_v, R$  and  $S$  denote the sets of primes of  $F^+$  lying under  $\tilde{S}_l, \tilde{S}_v, \tilde{R}$  and  $\tilde{S}$  respectively. For each  $u \in S_l \cup S$ , fix a prime  $\tilde{u}$  of  $F$  dividing  $u$ . We will henceforth identify  $U(F_u^+)$  and  $GL_m(F_{\tilde{u}})$  via  $\iota_{\tilde{u}}$  for  $u \in S_l \cup S$ .

Fix embeddings  $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$  and  $\iota_l : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$  such that  $\iota \circ \iota_l = \iota_\infty$ . For  $u|l$  a prime of  $F^+$ , following [Che09], we let  $\Sigma(u) \subset \text{Hom}(F^+, \overline{\mathbb{Q}}_l)$  denote the set of embeddings inducing  $u$  and let  $\Sigma_\infty(u) = \iota \Sigma(u) \subset \text{Hom}(F^+, \mathbb{R})$ . Let  $W_\infty$  denote the representation  $\otimes_{u \in S_v, \sigma \in \Sigma_\infty(u)} \pi_{F,\sigma}$  of  $\prod_{u \in S_v, \sigma \in \Sigma_\infty(u)} U(F_\sigma^+)$ .

Let  $K^S = \prod_{u \notin S} K_u \subset U(\mathbb{A}_{F^+}^{\infty,S})$  be a compact open subgroup with

- $K_u = \text{Iw}_{m,\tilde{u}}$  if  $u \in S_l$ ;
- $K_u$  a hyperspecial maximal compact subgroup of  $U(F_u^+)$  otherwise.

Let  $\mathcal{H}^{S \cup S_l} = \mathbb{Z}[U(\mathbb{A}_{F^+}^{\infty,S \cup S_l}) // K^{S \cup S_p}]$  denote the commutative spherical Hecke algebra. For  $u \notin S \cup S_l$ , let  $e_u \in \mathcal{H}(U(F_u^+))$  be the idempotent corresponding to  $K_u$ .

Choose a finite Galois extension  $E/\mathbb{Q}$  in  $\overline{\mathbb{Q}}$  such that  $\Pi_{F,\tilde{u}}$  can be defined over  $E$  for each  $u \in S$ . For  $u \in S$ , let  $\mathcal{B}_u$  denote the subcategory of the category of smooth  $E$ -representations of  $GL_m(F_{\tilde{u}})$  determined by the supercuspidal support of  $\Pi_{F,\tilde{u}}$  (see Proposition-définition 2.8 of [Ber84]). Let  $\mathfrak{z}_u$  denote the center of the category  $\mathcal{B}_u$ . For  $u \in R$ , let  $e_u$  denote the idempotent in  $\mathcal{H}(GL_m(F_{\tilde{u}}))$  corresponding to  $\text{Iw}_{m,\tilde{u}}$ . For  $u \in S_v$ , choose an idempotent  $e_u$  in  $\mathcal{H}(GL_m(F_{\tilde{u}}))$  such that

- $b_u e_u = e_u$  where  $b_u \in \mathcal{H}(GL_m(F_{\tilde{u}}))$  is the projector to  $\mathcal{B}_u$ ;
- $e_u \Pi_{F,\tilde{u}} \neq \{0\}$ ;

- for every irreducible  $\pi \in \mathcal{B}_u \otimes_{E, \iota_\infty} \mathbb{C}$ , if  $e_u \pi \neq \{0\}$ , then

$$\text{rec}(\pi) \prec_I \text{rec}(\Pi_{F, \bar{u}}).$$

(We refer to Section 3.10 of [Che09] for the definition of  $\prec_I$  and to Section 3.6 of *op. cit.* for the fact that one can choose such an idempotent  $e_u$ .)

Extending  $E$  if necessary, we may assume that  $e_u$  is defined over  $E$  for each  $u \in S$  and we set  $e = \otimes'_{u \notin S_l} e_u$  and

$$\mathcal{H} = \mathcal{H}^{S \cup S_l} \otimes_{\mathbb{Z}} \left( \bigotimes_{u \in S} \mathfrak{z}_u \right).$$

Let  $L_E$  denote the Galois closure (over  $\mathbb{Q}_l$ ) of the closure of  $\iota_l(EF)$  in  $\overline{\mathbb{Q}_l}$ . Let  $T$  denote the diagonal maximal torus in  $\prod_{u \in S_l} GL_m(F_{\bar{u}})$  and let  $\mathcal{T} = \text{Hom}(T, \mathbb{G}_m^{\text{rig}})$  denote the rigid analytic space over  $\mathbb{Q}_l$  parametrizing continuous  $l$ -adic characters of  $T$ .

Let  $\mathcal{A}$  denote the set of automorphic representations  $\pi$  of  $U(\mathbb{A}_{F^+})$  for which  $e(\pi^\infty)^{K_{S_l}} \neq \{0\}$  and  $\otimes_{\sigma \in \Sigma_\infty(u), u \in S_l} \pi_\sigma \cong W_\infty$ . If  $\pi \in \mathcal{A}$ , then  $\mathcal{H}$  acts on  $e(\pi^\infty, S_l)$  through an  $E$ -algebra homomorphism  $\psi_{\mathbb{C}}(\pi) : \mathcal{H} \rightarrow \mathbb{C}$  (this follows from the fact that  $\pi_u^{K_u}$  is 1-dimensional for  $u \notin S \cup S_l$  while  $\mathfrak{z}_u$  acts on  $\pi_u$  through a character for  $u \in S$ ). We define  $\psi(\pi) : \mathcal{H} \rightarrow \overline{\mathbb{Q}_l}$  to be  $\iota^{-1} \circ \psi_{\mathbb{C}}(\pi)$ .

If  $\pi \in \mathcal{A}$ , we associate to it an algebraic character  $\kappa(\pi) \in \mathcal{T}(L_E)$  as in Section 1.4 of [Che09] (this character records the highest weights of the representations  $\pi_\sigma$  for  $\sigma \in \Sigma_\infty(u)$  and  $u \in S_l$ ). If  $u \in S_l$  and  $\pi_{\bar{u}}$  is an irreducible smooth representation of  $GL_m(F_{\bar{u}})$  with  $\pi_{\bar{u}}^{\text{Iw}, m, \bar{u}} \neq \{0\}$ , an *accessible refinement* of  $\pi_{\bar{u}}$  is an unramified character  $\chi_{\bar{u}} : T_m(F_{\bar{u}}) \rightarrow \mathbb{C}^\times$  such that  $\pi_{\bar{u}}$  embeds as a subrepresentation of  $n\text{-Ind}_{B_m(F_{\bar{u}})}^{GL_m(F_{\bar{u}})} \chi_{\bar{u}}$ . (Such a character always exists.) If  $\pi \in \mathcal{A}$ , then an accessible refinement of  $\pi$  is a character  $\chi = \prod_{u \in S_l} \chi_{\bar{u}} : T = \prod_{u \in S_l} T_m(F_{\bar{u}}) \rightarrow \overline{\mathbb{Q}_l}^\times$  where each  $\chi_{\bar{u}} : T_m(F_{\bar{u}}) \rightarrow \overline{\mathbb{Q}_l}^\times$  is unramified and  $\iota \chi_{\bar{u}}$  is an accessible refinement of  $\pi_{\bar{u}} \otimes |\det|^{(1-m)/2}$ . Given such a pair  $(\pi, \chi)$ , we associate to it the character  $\nu(\pi, \chi) \in \mathcal{T}(\overline{\mathbb{Q}_l})$  as in Section 1.4 of [Che09].

We let

$$\mathcal{Z} \subset \text{Hom}_E(\mathcal{H}, \overline{\mathbb{Q}_l}) \times \mathcal{T}(\overline{\mathbb{Q}_l})$$

denote the set of all pairs  $(\psi(\pi), \nu(\pi, \chi))$  where  $\pi \in \mathcal{A}$  and  $\chi$  is an accessible refinement of  $\pi$ .

By Théorème 1.6 of [Che09], the data  $(S_l, W_\infty, \mathcal{H}, e)$  determines a four-tuple  $(X, \psi, \nu, Z)$  where:

- $X$  is a reduced rigid analytic space over  $L_E$  which is equidimensional of dimension  $n \sum_{u \in S_l} [F_u^+ : \mathbb{Q}_l]$ ;
- $\psi : \mathcal{H} \rightarrow \mathcal{O}(X)$  is a ring homomorphism with  $\psi(\mathcal{H}^{S \cup S_l}) \subset \mathcal{O}(X)^{\leq 1}$ ;
- $\nu : X \rightarrow \mathcal{T}$  is a finite analytic morphism;
- $Z \subset X(\overline{\mathbb{Q}_l})$  is a Zariski-dense accumulation subset of  $X(\overline{\mathbb{Q}_l})$  such that the map

$$X(\overline{\mathbb{Q}_l}) \rightarrow \text{Hom}_E(\mathcal{H}, \overline{\mathbb{Q}_l}) \times \mathcal{T}(\overline{\mathbb{Q}_l})$$

which sends  $x \mapsto (h \mapsto (\psi(h)(x), \nu(x)))$  induces a bijection  $Z \xrightarrow{\sim} \mathcal{Z}$ . (We refer to Section 1.5 of [Che09] for the definition of ‘Zariski-dense accumulation’.) We henceforth identify  $Z$  and  $\mathcal{Z}$ .

If  $\pi \in \mathcal{A}$ , then by Corollaire 5.3 of [Lab09] there exists a partition  $m = m_1 + \dots + m_r$  of  $m$  and conjugate self-dual discrete automorphic representations  $\Pi_i$  of  $GL_{m_i}(\mathbb{A}_F)$  such that  $\Pi = \Pi_1 \boxplus \dots \boxplus \Pi_r$  is a strong base change of  $\pi$ . Let  $\Sigma = \tilde{S} \cup \tilde{S}_l$  and let  $F_\Sigma$  denote the maximal extension of  $F$  which is unramified outside  $\Sigma$ . Let  $G_{F,\Sigma} = \text{Gal}(F_\Sigma/F)$ . By Theorem 3.2.5 of [CH09] and the argument of Theorem 2.3 of [Gue09], there is a continuous semisimple representation  $r_{l,i}(\pi) : G_{F,\Sigma} \rightarrow GL_m(\overline{\mathbb{Q}}_l)$  with

$$\iota \text{WD}(r_{l,i}(\pi)|_{G_{F_w}})^{\text{ss}} \cong \text{rec}(\Pi_w \otimes |\det|^{(1-m)/2})^{\text{ss}}$$

for each prime  $w \nmid l$  of  $F$ . Moreover, there is a unique continuous  $m$ -dimensional pseudo-representation  $T : G_{F,\Sigma} \rightarrow \mathcal{O}(X)$  such that  $T_z = \text{tr}(r_{l,i}(\pi))$  for each  $z = (\psi(\pi), \nu(\pi, \chi)) \in Z$ . (Here, for any  $x \in X(\overline{\mathbb{Q}}_l)$ ,  $T_x$  denotes the composition of  $T$  with the evaluation map  $\mathcal{O}(X) \rightarrow \overline{\mathbb{Q}}_l; g \mapsto g(x)$ .) The existence of  $T$  follows from the proof of Proposition 7.1.1 of [Che04] together with Proposition 7.2.11 of [BC09] (which shows that  $\mathcal{O}(X)^{\leq 1}$  is compact, as  $\mathcal{T}$  is nested and  $\nu$  is finite) and the fact that  $\psi(\mathcal{H}^{S \cup S^t}) \subset \mathcal{O}(X)^{\leq 1}$ . By Theorem 1 of [Tay91], for any  $x \in X(\overline{\mathbb{Q}}_l)$ , there is a unique continuous semisimple representation  $r_x : G_{F,\Sigma} \rightarrow GL_m(\overline{\mathbb{Q}}_l)$  with  $T_x = \text{tr}(r_x)$ .

Now, let  $u \in S$ . By Proposition 3.11 of [Che09], there is a unique  $m$ -dimensional pseudo-character

$$T^{\mathcal{B}_u} : W_{F_{\bar{u}}} \rightarrow \mathfrak{z}_u$$

such that for each irreducible smooth representation  $\pi_{\bar{u}}$  of  $GL_m(F_{\bar{u}})$  in  $\mathcal{B}_u \otimes_{E, \iota_\infty} \mathbb{C}$ , if  $T_{\pi_{\bar{u}}}^{\mathcal{B}_u}$  denotes the composition of  $T^{\mathcal{B}_u}$  with the character  $\mathfrak{z}_u \rightarrow \mathbb{C}$  giving the action of  $\mathfrak{z}_u$  on  $\pi_{\bar{u}}$ , then

$$T_{\pi_{\bar{u}}}^{\mathcal{B}_u} = \text{tr}(\text{rec}(\pi_{\bar{u}} \otimes |\det|^{(1-m)/2})).$$

Let  $z_0 \in Z$  be a point corresponding to  $\pi_F$  together with the choice of some accessible refinement. Let  $Z^{\text{reg}} \subset Z$  denote the subset associated to pairs  $(\pi, \chi)$  where  $\pi_\infty$  is slightly regular and extremely regular. (If  $\tilde{\sigma} : F \hookrightarrow \mathbb{C}$  and  $\sigma := \tilde{\sigma}|_{F^+}$ , then  $\pi_\sigma \circ \iota_{\tilde{\sigma}}$  is the restriction of an irreducible algebraic representation of  $GL_m(F_{\tilde{\sigma}})$  of highest weight  $b_{\tilde{\sigma}}$ , say. We say  $\pi_\infty$  is Shin-regular or extremely regular if  $b := (b_{\tilde{\sigma}})_{\tilde{\sigma}}$  has the corresponding property.) Then  $Z^{\text{reg}}$  is a Zariski-dense accumulation subset of  $X(\overline{\mathbb{Q}}_l)$ . Choose an open affinoid  $\Omega \subset X$  such that  $z_0 \in \Omega$  and  $Z^{\text{reg}} \cap \Omega$  is Zariski-dense in  $\Omega$ . Let  $T_\Omega$  denote the restriction of  $T$  to  $\Omega$ . By Lemme 7.8.11 of [BC09], there exists a reduced, separated, quasi-compact rigid analytic space  $Y$  and a proper, generically finite, surjective morphism  $f : Y \rightarrow \Omega$  such that there exists an  $\mathcal{O}_Y$ -module  $M$  which is locally free of rank  $n$  and carries a continuous action of  $G_{F,\Sigma}$  whose trace is given by  $f^* T_\Omega$ .

By Proposition 3.16 of [Che09] (a result of Sen), the (generalized) Hodge-Tate weights of  $M_y|_{G_{F_{\bar{u}}}}$  are independent of  $y \in Y(\overline{\mathbb{Q}}_l)$ . (This follows from the quoted result and the fact that the Hodge-Tate weights of  $r_z|_{G_{F_{\bar{u}}}}$  are independent of  $z \in Z$ .) Moreover, by the improvement to Theorem C of [BC08] made in Corollary 3.19 of [Che09], there exists a finite Galois extension  $F'_{\bar{u}}/F_{\bar{u}}$  such that if  $F'_{\bar{u},0} \subset F'_{\bar{u}}$  denotes the maximal unramified extension of  $\mathbb{Q}_l$ , then the  $\mathcal{O}_Y \otimes_{\mathbb{Q}_l} F'_{\bar{u},0}$ -module

$$D_{\text{st}}^{F'_{\bar{u}}}(M) := (M \otimes_{\mathbb{Q}_l} B_{\text{st}})^{G_{F'_{\bar{u}}}}$$

is locally free of rank  $m$  and satisfies the following: if  $y \in Y(\overline{\mathbb{Q}}_l)$ , then the natural map  $D_{\text{st}}^{F'_{\bar{u}}}(M)_y \rightarrow D_{\text{st}}^{F'_{\bar{u}}}(M_y)$  is an isomorphism (and hence  $M_y|_{G_{F'_{\bar{u}}}}$  is semistable).

The diagonal action of  $G_{F_{\bar{u}}}$  on  $M \otimes_{\mathbb{Q}_l} B_{\text{st}}$  induces an  $\mathcal{O}_Y$ -linear,  $F'_{\bar{u},0}$ -semilinear action of  $G_{F_{\bar{u}}}$  on  $D_{\text{st}}^{F'_{\bar{u}}}(M)$ . We define an  $\mathcal{O}_Y \otimes_{\mathbb{Q}_l} F'_{\bar{u},0}$ -linear action  $r_{\bar{u}}$  of  $W_{F_{\bar{u}}} \subset G_{F_{\bar{u}}}$  on  $D_{\text{st}}^{F'_{\bar{u}}}(M)$  by letting  $g \in W_{F_{\bar{u}}}$  act as  $g \circ \varphi^{w(g)}$  where  $w(g) \in \mathbb{Z}$  is the power of  $\text{Frob}_l$  to which  $g$  maps in  $G_{F_{\bar{u}}}/I_{F_{\bar{u}}}$ . We have that  $N \circ r_{\bar{u}}(g) = l^{w(g)} r_{\bar{u}}(g) \circ N$  on  $D_{\text{st}}^{F'_{\bar{u}}}(M)$ . For each continuous embedding  $\tau : F'_{\bar{u},0} \hookrightarrow L_E$ , we let

$$\text{WD}_{\bar{u},\tau} = D_{\text{st}}^{F'_{\bar{u}}}(M) \otimes_{\mathcal{O}_Y \otimes_{\mathbb{Q}_p} F'_{\bar{u},1} \otimes \tau} \mathcal{O}_Y.$$

Then  $\text{WD}_{\bar{u},\tau}$  is locally free of rank  $m$  as an  $\mathcal{O}_Y$ -module and  $N \circ r_{\bar{u}}(g) = l^{w(g)} r_{\bar{u}}(g) \circ N$  on  $\text{WD}_{\bar{u},\tau}$ . Moreover,  $\varphi$  induces an isomorphism  $\text{WD}_{\bar{u},\tau \circ \text{Frob}_l} \xrightarrow{\sim} \text{WD}_{\bar{u},\tau}$  compatible with  $r_{\bar{u}}$  and  $N$ . We let  $\text{WD}_{\bar{u}}$  denote  $\text{WD}_{\bar{u},\tau}$  for some choice of  $\tau$ , regarded as a  $W_{F_{\bar{u}}}$ -module with an operator  $N$ . We note that for each  $y \in Y(\overline{\mathbb{Q}}_l)$ ,  $\text{WD}_{\bar{u},y}$  is the Weil-Deligne representation associated to  $M_y|_{G_{F_{\bar{u}}}}$ . It follows that  $N^m = 0$  on  $\text{WD}_{\bar{u}}$ . Let

$$T^{Y,\bar{u}} = \text{tr}(r_{\bar{u}}(\cdot)|\text{WD}_{\bar{u}}) : W_{F_{\bar{u}}} \rightarrow \mathcal{O}_Y.$$

We claim that

$$T^{Y,\bar{u}} = f^* \circ \psi \circ T^{\mathcal{B}_u}.$$

This is proved as follows: let  $y \in f^{-1}(Z^{\text{reg}} \cap \Omega)$  and let  $z = f(y)$ . Then  $z$  corresponds to a pair  $(\pi, \chi)$  where  $\pi \in \mathcal{A}$  is Shin-regular and extremely regular (and  $\chi$  is an accessible refinement of  $\pi$ ). Theorem 2.1 together with the regularity conditions satisfied by  $\pi$  and the construction of the representation  $r_{l,\iota}(\pi)$  in the proof of Theorem 2.3 of [Gue09] show that

$$\text{WD}(r_{l,\iota}(\pi)|_{G_{F_{\bar{u}}}})^{\text{F-ss}} \cong \iota^{-1} \text{rec}(\pi_u \circ \iota_u^{-1} \otimes |\det|^{(1-m)/2}).$$

Since  $M_y^{\text{ss}} \cong r_z = r_{l,\iota}(\pi)$ , we deduce that  $T^{Y,\bar{u}}(g)$  and  $f^*(\psi(T^{\mathcal{B}_u}(g)))$  agree on  $y \in Y(\overline{\mathbb{Q}}_l)$  for each  $g \in W_{F_{\bar{u}}}$ . The claimed result now follows from the Zariski-density of  $f^{-1}(Z^{\text{reg}} \cap \Omega)$  in  $Y$ .

We now choose some  $y_0 \in Y(\overline{\mathbb{Q}}_l)$  with  $f(y_0) = z_0$ . Since  $r_{l,\iota}(\Pi_F) = r_{l,\iota}(\pi_F) = r_{z_0} \cong M_{y_0}^{\text{ss}}$ , the result just proved shows that

$$\iota \text{WD}(r_{l,\iota}(\Pi_F)|_{G_{F_{\bar{u}}}})^{\text{ss}} \cong \text{rec}(\Pi_{F,\bar{u}} \otimes |\det|^{(1-m)/2}).$$

We deduce from this that

$$\iota \text{WD}(r_{l,\iota}(\Pi_F)|_{G_{F_{\bar{u}}}})^{\text{F-ss}} \prec \text{rec}(\Pi_{F,\bar{u}} \otimes |\det|^{(1-m)/2}),$$

as follows: By Lemma 3.14(ii) of [Che09], it suffices to show that

$$\iota \text{WD}(r_{l,\iota}(\Pi_F)|_{G_{F_{\bar{u}}}})^{\text{F-ss}} \prec_I \text{rec}(\Pi_{F,\bar{u}} \otimes |\det|^{(1-m)/2}).$$

(See the paragraph of [Che09] two before Lemma 3.14 for the definition of  $\prec_I$ .) For each  $y \in f^{-1}(Z^{\text{reg}} \cap \Omega)$  with  $z = f(y)$  corresponding to a pair  $(\pi, \chi)$ , we have

$$\iota \text{WD}(M_y^{\text{ss}}|_{G_{F_{\bar{u}}}})^{\text{F-ss}} \cong \text{rec}(\pi_u \circ \iota_u^{-1} \otimes |\det|^{(1-m)/2}) \prec_I \text{rec}(\Pi_{F,\bar{u}} \otimes |\det|^{(1-m)/2})$$

(where the last relation follows from the choice of idempotent  $e_u$ ). By the proof of Proposition 7.8.19(iii) of [BC09] and the Zariski-density of  $f^{-1}(Z^{\text{reg}} \cap \Omega)$  in  $Y$ , we have  $\iota \text{WD}(M_y^{\text{ss}}|_{G_{F_{\bar{u}}}})^{\text{F-ss}} \prec_I \text{rec}(\Pi_{F,\bar{u}} \otimes |\det|^{(1-m)/2})$  for all  $y \in Y(\overline{\mathbb{Q}}_l)$ . Taking  $y$  above  $z_0$  gives the required result.  $\square$

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